EXACT BOUNDS FOR THE FORM FACTOR OF K_{l3} DECAY

By L. Tataru*

University of Helsinki**

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Assuming the Callan-Treiman and Dashen, Li, Pagels, Weinstein relations, exact inequalities for the K_{l3} scalar form factor $f(t)$ and its derivatives have been studied. These inequalities are too stringent to fit with the experimental data. Only for a very large value of the propagator $\Delta(0)$ of the divergence of the strangeness changing current at zero momentum there is a better agreement. Various causes of this disagreement are analyzed.

1. Introduction

Recently, the problem of deriving bounds for the K_{l_3} parameters has been investigated by various authors [1, 2, 3, 4, 5]. The key feature is that analyticity of the strangeness current is used to exploit information above the K_{l_3} threshold in order to constrain the form factor in the experimental decay region. The numerical estimation of the bounds involves an estimation of $\Delta(0)$, and the problem has been solved by using the chiral $SW(3)$ model of Gell-Mann, Oakes and Renner [6].

However, in the above discussion, the Dashen, Li, Pagels, Weinstein [8] relation was not taken into account. We wish to emphasize that although this sum rule is based on chiral $SW(3)$ power series perturbation theory, it remains unchanged, as far as numerical results are concerned, when the non-analytic character of expansion of matrix elements in the symmetry breaking parameters is taken into account.

The purpose of this paper is to find some rigorous bounds for the K_{l_3} form factor valid in the decay region by using the Callan-Treiman relation [7] together with the Dashen, Li, Pagels, Weinstein relation.

In Section 2 we present our main result. This is a bound involving derivates of the divergence of the strangeness changing current $f(t)$ at points $t = 0$ and $t = m_K^2$. In Section 3 we calculate numerical values for the bounds for $f(t)$. For the sake of comparison with

* On leave of absence from the University of Craiova, Romania.

** Address: University of Helsinki, Research Institute for Theoretical Physics, Siltavuorenpenger 20 B, 00170 Helsinki 17, Finland.

the present data we consider the case where $f(m_K^2)$ is treated as a free parameter. In Section 4 the bounds for $\Lambda_0 = m_\pi f'(0)/f(0)$ are given, and a comparison with experimental data is made.

In Conclusion we discuss the implication of the results for the different hypotheses that have been made in the derivation. An Appendix is devoted to a proof of the general inequality for the K_{l_3} form factor which has been used in various particular cases in this paper.

2. *Derivation of the general bound*

In this section we shall give a very short summary of the general results concerning the K_{l_3} form factor obtained from the Cabibbo theory of the semi-leptonic interaction. Using these results we may write our main inequality for this form factor.

Let $V_\mu^{(4-i5)}$ be the strangeness changing current. Then the K_{l_3} form factor $f(t)$ is introduced and parametrized in the standard way

$$\langle \pi^0(p)|V_\mu^{(4-i5)}(0)|K^+(k)\rangle = -(8k_0p_0)^{-\frac{1}{2}}(2\pi)^3\,\{(p+k)_\mu f_+(t)+(p-k)_\mu f_-(t)\}, \tag{1}$$

$$f(t) = \frac{d(t)}{m_K^2-m_\pi^2} = f_+(t)+\frac{t}{m_K^2-m_\pi^2}f_-(t), \tag{2}$$

where p and k are the momenta of π^0 mesons and K^+ mesons respectively, and $t = (k-p)^2$.

The propagator $\Delta(t)$ is defined as follows

$$\Delta(t) = \frac{i}{2}\int d^4x\langle 0|T(\partial_\mu V_\mu^{(4-i5)}(x)\partial_\nu V_\nu^{(4+i5)}(0))\,|0\rangle \exp iqx, \tag{3}$$

and it is explicitly assumed to obey an unsubtracted Källen-Lehman representation

$$\Delta(t) = \int_{t_0}^{+\infty}\frac{\varrho(t')dt'}{t'-t}, \tag{4}$$

where $t_0 = (m_\pi+m_K)^2$ and

$$\varrho(t) = \tfrac{1}{2}\,(2\pi)^3\sum_n|\langle 0|\partial_\mu V_\mu^{(4-i5)}(0)\,|n\rangle|^2\delta^4(q-q_n). \tag{5}$$

Retaining only the $K\pi$ intermediate state form the summation over $|n\rangle$, and using the positivity of the spectral weight it is easy to show that

$$\frac{32\pi}{3}\,\frac{\Delta(t)}{(m_K^2-m_\pi^2)^2} \geqslant \frac{1}{2\pi}\int_{t_0}^{+\infty} dt'(t'-t)^{-1}t'^{-1}(t'-t_0)^{\frac{1}{2}}(t'-t_1)^{\frac{1}{2}}|f(t')|^2 \tag{6}$$

with $t_1 = (m_K-m_\pi)^2$.

The Eq. (6) represents our underlying relation. In order to obtain the bounds for $f(t)$ in the interval $(0, t_0)$ we should find a lower bound of the integral

$$I = \frac{1}{2\pi} \int_{t_0}^{+\infty} dt'(t'-t)^{-1} t'^{-1} (t'-t_0)^{\frac{1}{2}} (t'-t_1)^{\frac{1}{2}} |f(t')|^2 \tag{7}$$

which appears in Eq. (6). By changing the integration variables, this integral may be rewritten

$$I = \frac{1}{2\pi} \int_{1}^{+\infty} dx(x-x_0)^{-1} x^{-1} (x-1)^{\frac{1}{2}} (x-\omega^2)^{\frac{1}{2}} |f(x)|^2, \tag{8}$$

where

$$\omega = (m_K - m_\pi)/(m_K + m_\pi) = 0.571, \tag{9}$$

$$x_0 = t/(m_K + m_\pi)^2. \tag{10}$$

From Eqs. (6) and (A.21) we find

$$\frac{32\pi}{3} \frac{\Delta(t)}{(m_K - m_\pi)^2} \geqslant \sum_{l,m} (A^{-1})_{l,m} \gamma_l \gamma_m, \tag{11}$$

where

$$\gamma_i^{(p_i)} = \frac{d^{p_i}}{dz^{p_i}} (\psi(z) f(z))_{z=z_i} \tag{12}$$

and

$$\psi(z) = \frac{1}{\sqrt{2}} \frac{1-z}{\sqrt{1+z}} [1-z+(1+z)\sqrt{1-x_0}]^{-1} [1-z+(1+z)\sqrt{1-\omega^2}]^{\frac{1}{2}}, \tag{13}$$

where z is given by (A.2) and the matrix A is defined in the Appendix.

Now it is easy to recover from Eq. (11) the main results of Refs [1, 2, 3, 4, 5]. If we choose $N = 1$, $z_1 = z(t)$ and $n_1 = 0$ then Eq. (11) becomes

$$\frac{32\pi}{3} \frac{\Delta(t)}{(m_K^2 - m_\pi^2)^2} \geqslant [1-z^2(t)] \psi^2(z(t)) f^2(z(t)) \tag{14}$$

which represents the bound obtained by Radescu [4]. For $t = 0$ we may recover from Eq. (14) the main result of Refs [1, 3]. For $N = 2$, $n_1 = n$, $n_2 = 0$, $z_1 = 0$, $z_2 = z_2(m_K^2)$ it is easy to show that Eq. (14) is exactly the same as one used by Shih and Okubo [2].

3. *A bound for* $d'(m_K^2)$

Taking as given quantities $\Delta(0), f(0)$ and $f(m_K^2)$ and applying Eq. (11) for $N = 2$, $n_1 = 0$, $n_2 = 1$, $z_1 = 0$, $z_2 = z(m_K^2) = a$, and $t = 0$ we shall find here the best possible bound (under this given input) for $d'(m_K^2) = (m_K^2 - m_\pi^2) f'(m_K^2)$.

Many estimations for $\Delta(0)$ are given in Ref. [2]. For this reason it is convenient to write

$$\Delta^{\frac{1}{2}}(0) = 1.01 m_\pi f_\pi M, \tag{15}$$

where M is a parameter. If we take $M = 1$ we obtain the value ot $\Delta(0)$ evaluated by Mathur and Okubo, using the symmetry-breaking Hamiltonian introduced by Gell-Mann, Oakes and Renner. All evaluations from Ref. [2] give M smaller than 2.3. For $f(0) = f_+(0)$ we shall take two values $f_+(0) = 1$ and $f_+(0) = 0.81$, both being in agreement with the Ademollo-Gatto theorem. $f(m_K^2)$ is given by the Callan-Treiman relation

$$f(m_K^2) = 1.28 f_+(0). \tag{16}$$

At this point we want to remark that there have been some discussions connected with this value. Thus in a recent letter [12] it has been argued that the Kemmer-Duffin formalism, when used to describe pseudoscalar mesons, leads to a more satisfactoty theory of K_{l_3} form factor than does the conventional Klein-Gordon formalism. In particular, the theory provides the following value for the scalar form factor: $f(m_K^2) = 0.53 f_+(0)$. However, Deshpainde and Mc Namee [11] pointed to some difficulties connected with this formalism. Because of these difficulties we shall not use this result, but this possibility must be kept in mind.

Eq. (11) in this particular case becomes

$$\frac{32\pi}{3} \frac{\Delta(0)}{(m_K^2 - m_\pi^2)^2} \geqslant \frac{1}{a^4} \gamma_0^2 + \frac{1-a^4}{a^4} \gamma_1^2 + \frac{(1-a^2)^3}{a^2} \gamma_1'^2 - \\ -2 \frac{(1-a^2)^2}{a^3} \gamma_1 \gamma_1' + 2 \frac{(1-a^2)^2}{a^3} \gamma_1' \gamma_0 - 2 \frac{1-a^4}{a^4} \gamma_0 \gamma_1, \tag{17}$$

where

$$\gamma_0 = \psi(0) f_+(0), \quad \gamma_1 = \psi(a) f(m_K^2), \quad \gamma_1' = \left. \frac{d(f(z)\psi(z))}{dz} \right|_{z=a}, \quad a = z(m_K^2). \tag{18}$$

The numerical calculations of the bounds obtained for $d(m_K^2)$ from Eq. (17) are presented in Table I. We see that the value of $d(m_K^2) = 0.28$ obtained by Dashen, Li, Pagels, and Weinstein (DLPW) could be compatible with Eq. (17) only if $M > 1$. But in view of the

TABLE I

The value of the bounds for $d'(m_K^2)$ as function of $Mf_+(0) = 1$

M	1.2	1.4	1.6	1.8	2.0	2.3	4
$d'(m_K^2)_{\max}$	—	0.57946	0.62173	0.66565	0.689147	0.734096	0.963744
$d'(m_K^1)_{\min}$	—	0.40442	0.362155	0.318231	0.294738	0.249789	0.20140

M	1.2	1.4	1.6	1.8	2.0	2.3	4
$d'(m_K)_{\max}$	0.494961	0.535882	0.569960	0.608878	0.630497	0.673787	0.89
$d'(m_K)_{\min}$	0.341341	0.300421	0.266343	0.227424	0.205805	0.370543	0.060177

different evaluations of $\Delta(0)$ we may say that the DLPW result, in fact, agrees well with Eq. (17). At any rate, choosing

$$\Delta^{\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(m_K f_K - m_\pi f_\pi) \tag{19}$$

as in Ref. [2] we obtain $M = 2.3$ with very good agreement between DLPW result and Eq. (17).

If we assume that Eq. (19) is the correct form of $\Delta(0)$, then the DLPW value does not contradict Eq. (17). On the other hand, if this is really the case, then it follows that the parameters $a = \varepsilon_8/\sqrt{2}\varepsilon_0$ and $b = \langle 0|u_8|0\rangle/\sqrt{2}\langle 0|u_0|0\rangle$ (where ε_8, ε_0 u_0, u_8 are the quantities which appear in the symmetry-breaking Hamiltonian introduced by Gell-Mann, Oakes and Renner $H' = \varepsilon_0 u_0 + \varepsilon_8 u_8$) satisfy the relation $a \cong b$. This is rather hard to understand since on physical grounds we expect to have $a \cong -1$ and $b \cong 0$.

However, without assuming any specific form for the symmetry-breaking Hamiltonian we may estimate the value of $\Delta(0)$ by using kappa dominance. In this case

$$\Delta^{\frac{1}{2}}(0) = \frac{1}{\sqrt{2}} m_\kappa f_\kappa \cong 1.70 m_\pi f_\pi \tag{20}$$

i. e. $M = 1.7$ for which the DLPW relation still agrees with Eq. (17).

In conclusion we want to emphasize that the DLPW relation can be used together with the Callan-Treiman relation only if one chooses for $\Delta(0)$ values given by Eqs (19) or (20). Insofar as the theoretical calculations for $\Delta(0)$ are still somewhat uncertain, we might say that the DLPW relation could be compatible with the rigorous unitarity bound (17).

4. *Comparison with experiment*

In this section we wish to take into account the experimental K_{l_3} data. In order to do this we may write down a bound which contains the value of

$$f'(0) = \frac{4}{\omega} f_+(0) \frac{m_K^2 - m_\pi^2}{m_\pi^2} \left[\lambda_+ + \frac{m_\pi^2}{m_K^2 - m_\pi^2} \xi\right] \tag{21}$$

with

$$\lambda_+ = m_\pi^2 \frac{d}{dt} \log f_+(t)|_{t=0}, \qquad \xi = \frac{f_-(0)}{f_+(0)}. \tag{22}$$

This bound is obtained from Eq. (11) by choosing $N = 2$, $n_1 = 1$, $n_2 = 1$, $z_1 = 0$, $z_2 = $ a. In this case Eq. (11) becomes

$$\frac{32\pi}{3} \frac{\Delta(0)}{(m_K^2 - m_\pi^2)^2} \geqslant \frac{1}{a^4} \gamma_0'^2 + 2\left[\frac{2(1-a^2)}{a^5}(\gamma_0 - \gamma_1) + \frac{(1-a^2)^2}{a^4} \gamma_1'\right] \gamma_0' +$$
$$+ \frac{4a^4 - 7a^2 + 4}{a^6} \gamma_0^2 + \frac{(1-a^2)^3}{a^4} \gamma_1'^2 + \frac{(a^4 - 3a^2 + 4)(1-a^2)}{a^6} \gamma_1(\gamma_1 - 2\gamma_0) +$$
$$+ \frac{2(2-a^2)(1-a^2)^2}{a^5} \gamma_1'(\gamma_0 - \gamma_1), \tag{23}$$

where

$$\gamma_0' = \psi'(0)f_+(0)+\psi(0)f'(0). \tag{24}$$

At this point we shall take the DLPW relation for granted, and use Eq. (21) to obtain some bounds for

$$\Lambda_0 = \lambda_+ + \frac{m_\pi^2}{m_K^2 - m_\pi^2}\,\xi. \tag{25}$$

Using Eqs. (18), (22) and (23) it is easy to write down these bounds. The numerical calculations are presented in Table II. In order to see how well the experimental data are fitted we have taken three values for $f_+(0)$, disregarding the fact that $f_+(0) = 0.31$ is far from the value predicted by the Ademollo-Gatto theorem.

The present experimental data concerning the decay parameters are as follows: Chouvet and Gaillard [13] report

$$\lambda_+ = 0.045\pm 0.015,\ \xi = -0.85\pm 0.20,\ \Lambda_0 = -0.024\pm 0.015. \tag{26}$$

X2 Collaboration [14] with fixed λ_+

$$\lambda_+ = 0.029 \text{ as world average value},\ \xi = -0.65\pm 0.13,\ \Lambda_0 = -0.024\pm 0.011. \tag{27}$$

X2 Collaboration [14] with λ_+ free parameter

$$\lambda_+ = 0.060\pm 0.019,\ \xi = -1.0\pm 0.5,\ \Lambda_0 = -0.022\pm 0.019. \tag{28}$$

On comparison of the experimental and the theoretical results (see Table II) it turns out that our bounds are too wide to fit the present experimental data, using the present input. This fact might be roughly explained in the following way. In order to achieve the value at $t = m_K^2$ for both the function and its slope it is necessary that $f(t)$ should exhibit considerable curvature. On the other hand the unitarity condition expressed by Eq. (23) does not allow such a dip mechanism to explain the fall-off of the data for an accepted value of M. It can be seen that such a mechanism is permissible only for a very large value of M.

5. *Conclusion*

The purpose of this paper was twofold: firstly, to check the compatibility of the DLPW relation and the unitarity condition (11), secondly, to make a comparison, by means of Eq. (23), between the theoretical results and the experimental data. It has turned out that the DLPW relation may be used together with the Callan-Treiman relation and Eq. (23).

On the other hand, we can say that, from the results of the previous section, our bounds should not fit with the present experimental data. The main difficulty in reconciling our bounds with the experimental data comes from the Callan-Treiman relation (and probably the DLPW relation) which requires a dip mechanism to explain the fall-off of the data. On the other hand, the maximum value for $\Delta(0)$ allowed seems to be too small.

TABLE II

The values of bounds for Λ_0 as function of M

$f_+(0) = 1$

M	2.3	4	6	8	10	20	40	80	100	200
$\Lambda_0$max	0.01631	0.01817	0.01952	0.02079	0.02202	0.0281	0.0400	0.0639	0.07594	0.135482
$\Lambda_0$min	0.01612	0.01426	0.01291	0.01165	0.01041	0.0044	−0.0076	−0.0315	−0.0434	−0.103051

$f_+(0) = 0.85$

M	1.6	1.8	2.0	2.3	4	6	8	10	20	40	80	100	150
$\Lambda_0$max	0.0155	0.0158	0.0161	0.0164	0.0177	0.0192	0.0207	0.0221	0.0291	0.0432	0.0715	0.0853	0.120384
$\Lambda_0$min	0.0148	0.0145	0.0142	0.0139	0.0125	0.01108	0.00964	0.0082	0.0012	−0.0129	−0.0410	−0.0550	−0.090089

$F_+(0) = 0.31$

M	1.2	1.4	1.6	1.8	2.0	2.3	4	6	8	10	20	40	80	100
$\Lambda_0$max	0.0048	0.0055	0.0060	0.0065	0.0070	0.0077	0.0112	0.0331	0.0153	0.0229	0.042192	0.0807	0.1577	0.1961
$\Lambda_0$min	0.0028	0.0021	0.0015	0.0010	0.0005	−0.0001	−0.0036	−0.0115	−0.0076	−0.0154	−0.03465	−0.0732	−0.1501	−0.1886

However, in the present author's opinion, it is rather hard to believe that another theory may give a value for $\Delta(0)$ which is by a thousandfold larger. The use of a subtracted dispersion relation for $\Delta(0)$ has been proposed, but in spite of an additional constant which cannot be calculated theoretically, it seems improbable that this will reduce such a large modification.

If the experimental data remains unchanged, the Callan-Treiman relation is the only one to be considered. Indeed, as we have already mentioned, the Kemmer-Duffin formalism leads to a value for the scalar form factor in the point $t = m_K^2 : f(m_K^2) = 0.53 f_+(0)$, which agrees well with our bounds. However, since there are some difficulties connected with this formalism, it is rather difficult to use these results.

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APPENDIX

In this Appendix we derive a general bound for the propagator $\Delta(0)$. Let I be an integral of the form

$$I = \frac{1}{2\pi} \int_1^{+\infty} (x+a_1)^{\alpha_1}(x+a_2)^{\alpha_2} \dots (x+a_n)^{\alpha_n} |f(x)|^2 dx, \tag{A1}$$

where it is assumed that the integral exists and is positive. It is easy to see that an integral of this form appears in the Eq. (6). Changing the integration variables in (A1) by the conformal transformation

$$u = \sqrt{x-1} = i\,\frac{1-z}{1+z} \tag{A2}$$

(which maps the upper and lower sides of the right hand cut in z plane onto the unit circle $z = e^{i\theta}$) one may write

$$I = \frac{1}{2\pi} \int_0^{2\pi} p(\theta) |f(e^{i\theta})|^2 d\theta, \tag{A3}$$

where

$$p(\theta) = \tfrac{1}{2}\,|u|\,(u^2+1) \prod_{j=1}^{n} [1+a_j+u^2]^{\alpha_j}. \tag{A4}$$

If we consider the outer function $\psi(z)$ defined inside the unit circle $|z| < 1$ by

$$\psi(z) = \exp\left\{\frac{1}{4\pi} \int_0^{2\pi} d\theta\, \frac{e^{i\theta}+z}{e^{i\theta}-z} \log p(\theta)\right\}, \tag{A5}$$

then it is easy to verify that

$$p(\theta) = |\psi(e^{i\theta})|^2. \tag{A6}$$

The integral in Eq. (A3) may be calculated by using the Jensen-Poisson formula, and we have

$$\psi(z) = \sqrt{2}\,(1-z)^{\frac{1}{2}}(1+z)^{-\frac{3}{2}-\sum_{i=1}^{n}\alpha_i} \prod_{j=1}^{n} [1-z+(1+z)\sqrt{1+a_j}]^{\alpha_j}. \tag{A7}$$

Finally, Eq. (A3) may be written as

$$I = \frac{1}{2\pi}\int_0^{2\pi} |h(e^{i\theta})|^2 d\theta, \tag{A8}$$

where

$$h(z) = \psi(z)\, f(z). \tag{A9}$$

Now, the question we want to answer is the following: What is the best lower bound for I when the values of $h(z)$ and its derivatives are known at some points inside the unit circle. In order to solve this problem properly it is necessary to introduce some mathematical notions.

Let $\mathscr{H}^2$ be the Hilbert space of all analytic functions inside the unite circle $|z| < 1$ with boundary values almost everywhere on the boundary and let

$$(g, f) = \frac{1}{2\pi}\int_0^{2\pi} d\theta g^*(e^{i\theta}) f(e^{i\theta}) \tag{A10}$$

be the inner product of this Hilbert space. Let F be a subset of $\mathscr{H}^2$ consisting of all functions $h(z) \in \mathscr{H}^2$ which satisfy the conditions

$$h^{(p_i)}(z_i) = \left.\frac{d^{p_i}h(z)}{dz^{p_i}}\right|_{z=z_i} = \gamma_i^{(p_i)}, \quad i = \overline{1, N}, \quad p_i = \overline{0, n_i}, \tag{A11}$$

where $\gamma_i^{(p_i)}$ are some fixed numbers and z_i are N points inside the unit circle. The question presented earlier may now be solved by using the following theorem.

The functional $||h||^2 = (h, h)$ defined on F has an extremum (in F) if, and only if,

$$h(z) = h_0(z) = \sum_{i=1}^{N}\sum_{p_i=0}^{n_i} C_i^{(p_i)}\varphi_i^{(p_i)}(z), \tag{A12}$$

where

$$\varphi_i^{(p_i)}(z) = p_i!\,\frac{z^{p_i-1}}{(1-z_i z)^{p_i}} \tag{A13}$$

and $C_i^{(p_i)}$ are some constants chosen in such a way that $h_0(z)$ belongs to F.

Proof: First of all we notice that

$$(\varphi_i^{(p_i)}, h) = h^{(p_i)}(z_i). \tag{A14}$$

If $h(z)$ belongs to F and $h(z) \neq h_0(z)$ then the function

$$F(z) = h(z) - h_0(z) \not\equiv 0 \tag{A15}$$

will have the following properties

$$F^{(p_i)}(z_i) = 0, \qquad i = \overline{1, N}. \tag{A16}$$

Therefore we may write

$$||h||^2 = ||h_0 + F||^2 = ||h_0||^2 + ||F||^2 \geqslant ||h_0||^2. \tag{A17}$$

The coefficients $C_i^{(p_i)}$ and the value of $||h_0||^2$ are easily calculated by using Eq. (A12). Defining the matrix

$$A = \begin{pmatrix} \hat{A}_{11} & \dots & \hat{A}_{1N} \\ . & . & . \\ \hat{A}_{N1} & \dots & \hat{A}_{NN} \end{pmatrix}, \tag{A18}$$

where

$$\hat{A}_{ij} = (\langle \varphi_i^{(p_i)}, \varphi_j^{(p_j)} \rangle), \qquad p_i = \overline{0, n_i}, \qquad p_j = \overline{0, n_j}, \tag{A19}$$

one may write $||h_0||^2$ in the following form

$$||h_0||^2 = \sum_{l,m} (A^{-1})_{l,m} \gamma_l \gamma_m \tag{A20}$$

where $\vec{\gamma} = (\gamma_1^{(0)}, \dots, \gamma_1^{(n_1)}, \gamma_2^{(0)}, \dots, \gamma_N^{(n_N)})$ and $A.A^{-1} = I$. For all other functions $h(z)$ belonging to F we have

$$||h||^2 \geqslant \sum_{l,m} (A^{-1})_{l,m} \gamma_l \gamma_m. \tag{A21}$$

We want to emphasize that the bound (A21) is the best one possible for $||h||^2$ defined on F.

A particular case, very important in the applications, is one in which $n_1 = n_2 = \dots = n_N = 0$. Then the matrix A^{-1} has a simple form

$$(A^{-1})_{n,m} = \prod_{i \neq n} \prod_{j \neq m} \frac{(1 - z_i z_m)(1 - z_j z_n)}{(z_j - z_m)(z_i - z_n)} \times (1 - z_n z_m). \tag{A22}$$

It is important to note that all bounds obtained in previous works [1, 2, 3, 4, 5] are, in fact, special cases of Eq. (A21).

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