COHERENT SCATTERING ON DEUTERIUM: FORMULA INCLUDING SPINS

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A general formula for coherent scattering on deuterium, including all the complications due to isospin and spin, is derived from Glauber's theory. The advantage of using tensor transition amplitudes instead of the usual amplitudes with two spin projections as indices is stressed.

1. Introduction

There seems to be growing interest in high energy coherent scattering on nuclei [1, 2]. By coherent scattering we understand the scattering processes, where in both the initial and the final state the target nucleus is in its ground state. Simplified versions of Glauber's theory have been successfully used for a phenomenological analysis of the data [1]. The results, however, show a puzzling feature: the nucleus is unexpectedly transparent for newly produced particles. It is plausible that a careful study of coherent production processes on the smallest nuclei, could help to resolve the problem (*cf. e.g.* Ref. [3]).

In the present paper we consider high energy coherent scattering on deuterium:

$$x+\mathrm{d} \rightarrow y+\mathrm{d}, \tag{1.1}$$

where x and y are well defined but arbitrary particles or groups of particles. Deuterium is the smallest nucleus, where the puzzling rescattering terms appear. Their contribution is, however, small and therefore a particularly careful theoretical analysis is necessary in order not to misinterpret the experimental results. Our analysis is based on the ordered Glauber formula [4], but includes all the spin and isospin complications and also inelastic shadowing and longitudinal momentum transfers. A general formula for the amplitude of process (1.1) is derived. It should be remembered, however, that in t-region where the rescattering terms are important, Glauber's approximation may require corrections. Therefore caution is necessary when using the formula to fit data.

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Elastic πd scattering — which is a particular case of our problem for $x = y = \pi$ — has already in this approximation been described [5–7]. The calculations seem to have been elementary, but rather tedious. Here in the calculation for the general case we extensively use the theory of angular momentum. This makes the calculations much shorter and, we hope, the result more transparent.

2. Results required and necessary input

In order to describe process (1.1) for given x and y it is necessary to find the scattering amplitude as a function of the total energy, the momentum transfer $\boldsymbol{q}$, the spin states of x and y, the spin projection M of the initial deuteron and the spin projection M' of the final deuteron. For simplicity the usual assumption will be made that the same spin reference frame is used for the initial and for the final deuteron.

The amplitude will be denoted $A_{M'M}(\boldsymbol{q})$ and normalized so that the differential cross--section is

$$\frac{d\sigma(M \to M')}{d\Omega} = |A_{M'M}(\boldsymbol{q})|^2. \tag{2.1}$$

This formula is applicable for given spin states of x and y. In order to describe the experimental data it may be necessary to average over the spin states of x and/or to sum over the spin states of y.

Glauber's formula is quoted in Section 4. In this and the following section the necessary input is described.

In the framework of Glauber's theory the scattering amplitude A is expressed in terms of the wave function for the deuteron ground state and of the scattering amplitudes for scattering on single nucleons.

Since the deuteron is a parity plus, spin one object, with its two nucleons in the triplet spin state, the most general wave function can be written in the form [5–7]

$$\Psi(\boldsymbol{r}, \sigma) = \sum_L \sum_\sigma \langle L, M-\sigma; 1, \sigma | 1, M\rangle \frac{u_L(r)}{r} Y^L_{M-\sigma}(\hat{\boldsymbol{r}})\psi_\sigma. \tag{2.2}$$

Here $Y^L_m(\hat{\boldsymbol{r}})$ is the spherical harmonic having as arguments the spherical angles of $\hat{\boldsymbol{r}}$, ψ_σ is the spin wave function for the two nucleons with resultant spin projection σ, and the S and D radial wave functions $u_S(r)$ and $u_D(r)$ can be chosen real and are normalized by the condition

$$\int_0^\infty [u_S^2(r)+u_D^2(r)]dr = 1. \tag{2.3}$$

In order to calculate numerically the amplitude A, it is necessary to choose the two functions u_S and u_D. The wave functions appear in the form factors

$$S_{lL'L}(\boldsymbol{q}) = \int_0^\infty j_l(qr)u_{L'}(r)u_L(r)dr \tag{2.4}$$

and

$$S^{m}_{lL'L}(\boldsymbol{q}) = (4\pi)^{-1/2} \int r^{-2} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \varepsilon(z) Y^{l*}_{m}(\hat{\boldsymbol{r}}) u_{L'}(r) u_{L}(r) d^3 r, \tag{2.5}$$

where $\varepsilon(z) = z/|z|$. The form factors are needed only for the sets $(l, L', L) = (0, 0, 0)$, $(0, 2, 2)$, $(2, 0, 2)$, $(2, 2, 2)$ and $(4, 2, 2)$. One needs 5 formfactors (2.4) and 13 formfactors (2.5), since from the properties of spherical harmonics $m \leqslant l$ and

$$S^{m*}_{lL'L}(\boldsymbol{q}) = S^{-m}_{lL'L}(-\boldsymbol{q}) \cdot (-1)^m. \tag{2.6}$$

Note that when longitudinal momentum transfer q_l is negligible, all the formfactors (2.5) vanish.

Let us denote by $F^{xyN}(\boldsymbol{q})$ the amplitude for the process

$$x + \mathrm{N} \rightarrow y + \mathrm{N}', \tag{2.7}$$

where N, N′ are nucleons. According to Glauber's theory the scattering amplitude $A_{M'M}$ is a sum of two terms. In the single scattering term only the amplitudes F^{xyN} occur. In the double scattering term there are the amplitudes F^{xcN} and F^{cyN}. Here c can in principle be any particle, which could occur as an intermediate state of the incident particle between its impact as x on the first nucleon and its re-creation as y on the second. We derive the general formula for an arbitrary set of particles c, in practice usually, only $c = x$ and $c = y$ are included in the calculation. Thus in the usual approximation the necessary amplitudes for scattering on single nucleons are F^{xyN}, F^{xxN} and F^{yyN}. The representation of the amplitudes, which is suitable for our calculations, is described in the following section.

To summarize: the amplitudes $A_{M'M}(\boldsymbol{q})$ will be expressed in terms of the form factors (2.4) and (2.5) *i.e.* of the radial wave functions $u_S(r)$ and $u_D(r)$, and of the amplitudes for scattering on single nucleons F^{xyN}, F^{xcN} and F^{cyN}.

3. *Representation of the amplitudes for scattering on single nucleons*

Pauli spinors will be used to describe the spin orientation of the target nucleons. Thus the most general form of the scattering amplitude F^{xyN} is an arbitrary 2×2 matrix. The elements of this matrix are in general operators transforming the spin state of x into the spin state of y. It is convenient to write F in the form

$$F^{xyN}(\boldsymbol{q}) = a^{xyN}(\boldsymbol{q}) S^0 + b^{xyN}_0(\boldsymbol{q}) S^1_0 + q_{\mathrm{T}} [b^{xyN}_{+}(\boldsymbol{q}) S^1_{+} + b^{xyN}_{-}(\boldsymbol{q}) S^1_{-}], \tag{3.1}$$

where S^0 is the unit matrix and S^1_ν are the Pauli matrices in the spherical basis:

$$S^1_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S^1_{+} = \begin{pmatrix} 0 & -2^{1/2} \\ 0 & 0 \end{pmatrix}, \quad S^1_{-} = \begin{pmatrix} 0 & 0 \\ 2^{1/2} & 0 \end{pmatrix}; \tag{3.2}$$

q_{T} is the transverse component of the momentum transfer $\boldsymbol{q}$. In order to specify the amplitude F it is necessary to specify the four coefficients a, b_0, b_+, b_-. It will be assumed that the coefficients are written in a rotation invariant form. Thus if they contain tensors, these must be included in scalar or pseudoscalar products.

In order to illustrate the interpretation of formula (3.1), we present two examples.

These examples are comparatively simple, but they seem to contain all the main features of the problem, so that extensions to more complicated cases should present no new difficulties.

a) Elastic scattering of a pseudoscalar particle

The most general amplitude for the process $0^- \frac{1}{2}^+ \to 0^- \frac{1}{2}^+$ can be written in the form [5, 7]:

$$F(\boldsymbol{q}) = a^{\mathrm{el}}(\boldsymbol{q}) + \boldsymbol{\sigma} \cdot (\boldsymbol{q} \times \hat{\boldsymbol{k}}) b^{\mathrm{el}}(\boldsymbol{q}), \tag{3.3}$$

where $\hat{\boldsymbol{k}}$ is a unit vector parallel to the c.m.s. momentum of the incident particle and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector composed from the Pauli matrices. Our parameters depend on the spin reference frame chosen. We take the z axis parallel to $\hat{\boldsymbol{k}}$, the x axis parallel to $\boldsymbol{q}_{\mathrm{T}}$ — the transverse component of $\boldsymbol{q}$ — and the y axis normal to the reaction plane and oriented so as to make the x, y, z system right-handed. Then using

$$S_0^1 = \sigma_z, \quad S_\pm^1 = 2^{-1/2}(\mp\sigma_x - i\sigma_y) \tag{3.4}$$

and comparing with (3.1) we find $b_0 = 0$ and

$$a = a^{\mathrm{el}}, \quad b_\pm = -2^{-1/2} i b^{\mathrm{el}}. \tag{3.5}$$

b) Scattering of a $\frac{1}{2}^+$ particle

The most general amplitude for the process $1/2^+ \, 1/2^+ \to 1/2^+ \, 1/2^+$ can be written in the form

$$F = [c_1 + c_2 \boldsymbol{\sigma} \cdot (\boldsymbol{q} \times \hat{\boldsymbol{k}})] S^0 + \boldsymbol{\sigma} \cdot (c_3 \boldsymbol{q} + c_4 \hat{\boldsymbol{k}}) \sigma_z +$$
$$+ [c_5 + c_6 \boldsymbol{\sigma} \cdot (\boldsymbol{q} \times \hat{\boldsymbol{k}})] \sigma_y + \boldsymbol{\sigma} \cdot (c_7 \boldsymbol{q} + c_8 \hat{\boldsymbol{k}}) \sigma_x, \tag{3.6}$$

where in each term the matrix following the brackets acts on the Pauli spinor of the target nucleon. Using formula (3.4), rearranging and comparing with (3.1) we find

$$a(\boldsymbol{q}) = c_1(\boldsymbol{q}) + c_2(\boldsymbol{q}) \boldsymbol{\sigma} \cdot (\boldsymbol{q} \times \hat{\boldsymbol{k}}), \tag{3.7}$$

$$b_0(\boldsymbol{q}) = \boldsymbol{\sigma} \cdot (c_3 \boldsymbol{q} + c_4 \boldsymbol{k}), \tag{3.8}$$

$$b_\pm(\boldsymbol{q}) = 2^{1/2} i [c_5(\boldsymbol{q}) + c_6(\boldsymbol{q}) \boldsymbol{\sigma} \cdot (\boldsymbol{q} \times \hat{\boldsymbol{k}}) \pm i \boldsymbol{\sigma} \cdot (c_7 \boldsymbol{q} + c_8 \hat{\boldsymbol{k}})]. \tag{3.9}$$

Note that as required all the vectors are included in scalar (or pseudoscalar) products. In most of the practical cases the coefficients $c_i(\boldsymbol{q})$ are interrelated by various internal symmetries, but there is no point in going into it here.

4. *Glauber's formula*

Our starting point is the ordered form of Glauber's formula [4]. We generalize it including the phase factors containing the longitudinal momentum transfers. The transition amplitude for the process $x + \mathrm{d} \to y + \mathrm{d}$ is given by

$$A_{M'M}(\boldsymbol{q}) = \langle 1M' | F^{xyp}(\boldsymbol{q}) e^{i\boldsymbol{q} \cdot \boldsymbol{r}/2} + F^{xyn}(\boldsymbol{q}) e^{-i\boldsymbol{q} \cdot \boldsymbol{r}/2} | 1M \rangle +$$
$$+ \hat{I} \langle 1M' | \, [F^{cyp}(\boldsymbol{q}^+) F^{xcn}(\boldsymbol{q}^-) + F^{cyn}(\boldsymbol{q}^+) F^{xcp}(\boldsymbol{q}^-)] \, [1 + \varepsilon(z)] e^{i\boldsymbol{Q} \cdot \boldsymbol{r}} | 1M \rangle, \tag{4.1}$$

where for shortness we used the operator $\hat{I}$ defined as

$$\hat{I} = (4\pi p)^{-1} i \sum_c \eta(c) \int d^2 q'. \tag{4.2}$$

The coefficient $\eta(c)$ equals 1 for non charge exchange processes and -1 for charge exchange processes. The transverse vectors perpendicular to the momentum $\boldsymbol{p}$ of the incoming particle x are

$$\boldsymbol{q}^{\pm} = \boldsymbol{q}_{\mathrm{T}}/2 \pm \boldsymbol{q}', \tag{4.3}$$

where $\boldsymbol{q}_{\mathrm{T}}$ is the transverse component of the total momentum transfer $\boldsymbol{q}$. Vector $\boldsymbol{q}'$ is transverse and together with the longitudinal component

$$q_c = \tfrac{1}{2}(q_{cy} - q_{xc}) \tag{4.4}$$

it forms the vector $\boldsymbol{Q}$. By q_{cy} and q_{xc} we denote the longitudinal momentum transfers for the processes $c+\mathrm{N} \to y+\mathrm{N}$ and $x+\mathrm{N} \to c+\mathrm{N}$ respectively. At high energies

$$q_c \approx (m_x^2 + m_y^2 - 2m_c^2)/4p \tag{4.5}$$

and the longitudinal component q_l of the total momentum transfer $\boldsymbol{q}$ is

$$q_l \approx (m_y^2 - m_x^2)/2p. \tag{4.6}$$

Here m_i denotes the mass of particle i and p is the laboratory momentum of particle x.

The first and second term in (4.1) are known as the single and double scattering contributions. In the double scattering term the amplitudes for scattering on single nucleons do not commute in general and therefore the order in which they occur is important for the further calculations.

5. *Amplitudes for scattering on deuterium*

In this section our results are given. Most of the proofs are placed in the Appendix, which the reader not interested in the technical details of the derivation is advised to skip.

The amplitude $A_{M'M}(q)$ is given by the formula

$$A_{M'M}(\boldsymbol{q}) = \sum_{j=0}^{2} \langle 1, M; j, M'-M | 1, M' \rangle A^j_{M'-M}(\boldsymbol{q}), \tag{5.1}$$

where $\langle 1, M; j, M'-M | 1, M' \rangle$ denote Clebsch-Gordan coefficients. This formula introduces the tensor amplitudes $A^j_\varrho(\boldsymbol{q})$. We found that these amplitudes are given by simpler formulae and are easier both to transform (when the spin reference frame is rotated) and to interpret than the amplitudes $A_{M'M}(\boldsymbol{q})$. For these reasons most of our results are formulated in terms of the tensor amplitudes. The ordinary amplitudes, whenever necessary, can be found using formula (5.1).

The tensor amplitudes are given by the formula

$$A^j_\varrho(q) = \sum_{l,s} \Big[A^j_\varrho(l, s) B(l, s, j) + \sum_{m=-l}^{l} A^j_{\varrho m}(l, s) B_m(l, s, j)\Big]. \tag{5.2}$$

TABLE I

l	s	j	$B(l, s, j)$	l	s	j	$B(l, s, j)$
0	0	0	$S_{0SS}+S_{0DD}$	2	0	2	$2S_{2SD}-\frac{1}{\sqrt{2}}S_{2DD}$
2	2	0	$\frac{2}{\sqrt{5}}S_{2SD}-\frac{1}{\sqrt{10}}S_{2DD}$	0	2	2	$S_{0SS}+\frac{1}{10}S_{0DD}$
0	1	1	$S_{0SS}-\frac{1}{2}S_{0DD}$	2	2	2	$\sqrt{7/5}S_{2SD}+\frac{1}{\sqrt{70}}S_{2DD}$
2	1	1	$-S_{2SD}-\frac{1}{\sqrt{2}}S_{2DD}$	4	2	2	$\frac{9}{5}\sqrt{2/7}S_{4DD}$

The functions $B(l, s, j)$ depend on the wave function of the deuteron, but neither on the scattering amplitudes F nor on the spin reference frame. The coefficients $B_m(l, s, j)$ also do not depend on the scattering amplitudes, but do depend on the reference frame. Expressions in terms of the form factors (2.4) for the non-zero functions $B(l, s, j)$ are collected in Table I. The argument of each of the form factors is understood to be equal $\boldsymbol{q}/2$.

The factors $A_\varrho^j(l, s)$ and $A_{\varrho m}^j(l, s)$ depend on the choice of the spin reference frame and on the amplitudes F. They do not depend on the deuteron wave functions. The factors $A_\varrho^j(l, s)$ are given by the formula

$$A_\varrho^j(l, s) = \sum_{m,\sigma} \langle l, m; s, \sigma | j, \varrho\rangle \big[\sqrt{4\pi}\, i^l Y_m^{l*}(\hat{\boldsymbol{q}}) \langle 1 \| S^s \| 1\rangle b_\sigma^s + \\ + \hat{I} \sum_{s_1,s_2} \sqrt{4\pi}\, Y_m^{l*}(\hat{\boldsymbol{Q}}) \langle 1 \| S^s(s_1, s_2) \| 1\rangle b_\sigma^s(s_1, s_2)\big]. \tag{5.3}$$

The coefficients b_σ^s and $b_\sigma^s(s_1, s_2)$ are given by the formulae (5.6–5.16). The reduced matrix elements $\langle 1 \| S^s \| 1\rangle$ and $\langle 1 \| S^s(s_1, s_2) \| 1\rangle$ are given in Table II. The operator $\hat{I}$ (*cf.* formula

TABLE II

Operator	$\langle 1 \| S \| 1\rangle$
$S^0, S^0\,(0, 0)$	1
S^1	$\sqrt{2}$
$S^1(1, 0), S^1(0, 1)$	$\sqrt{2}$
$S^0\,(1, 1)$	$-1/\sqrt{3}$
$S^1\,(1, 1)$	0
$S^2\,(1, 1)$	$2\sqrt{5/3}$

(5.5)) contains the factor (4.2), where the integration over d^2q' is understood to act also on the following coefficients $B(l, s, j)$ and $B_m(l, s, j)$ (*cf.* (5.2)). Moreover the operator $\hat{I}$ replaces (before the integration) the argument $\boldsymbol{q}/2$ of B or B_m by $\boldsymbol{Q}$ and indicates that the coefficients $b_\sigma^s(s_1, s_2)$ should be calculated for the operator

$$F^{cyn}(\boldsymbol{q}^+)F^{xcp}(\boldsymbol{q}^-)+F^{cyp}(\boldsymbol{q}^+)F^{xcn}(\boldsymbol{q}^-). \tag{5.4}$$

The functions $B_m(l, s, j)$ are obtained from the tabulated functions $B(l, s, j)$ by replacing each form factor $S_{lL'L}$ by the form factor $S^m_{lL'L}$. The coefficients $A^j_{\varrho m}(l, s)$ are given by

$$A^j_{\varrho m}(l, s) = \sum_\sigma \langle l, m; s, \sigma | j, \varrho\rangle \langle 1 \| S^s(s_1, s_2) \| 1\rangle \hat{I} b^s_\sigma(s_1, s_2). \tag{5.5}$$

Formula (5.2) is the main result of this paper.

The relevant non-zero coefficients b^s_σ and $b^s_\sigma(s_1, s_2)$ are:

$$b^0_0(\boldsymbol{q}) = a^{xy}(\boldsymbol{q}), \quad b^1_0(\boldsymbol{q}) = b^{xy}_0(\boldsymbol{q}), \quad b^1_{\pm 1}(\boldsymbol{q}) = q_\mathrm{T} b^{xy}(\boldsymbol{q}), \tag{5.6}$$

where the superscript xy denotes the summation over the proton and neutron scattering amplitudes. Thus, for instance, in the notation from Section 3

$$a^{xy} = a^{xyn}(\boldsymbol{q}) + a^{xyp}(\boldsymbol{q}). \tag{5.7}$$

Further

$$b^0_0(0, 0) = aa, \; b^0_1(0, 1) = ab_0, \; b^0_1(1, 0) = b_0 a. \tag{5.8}$$

In these formulae and in the following ones the interpretation of the products is given by the definition of the operator $\hat{I}$ thus *e.g.*

$$aa = a^{cyn}(\boldsymbol{q}^+)a^{xcp}(\boldsymbol{q}^-) + a^{cyp}(\boldsymbol{q}^+)a^{xcn}(\boldsymbol{q}^-). \tag{5.9}$$

The remaining formulae are:

$$b^1_{\pm 1}(0, 1) = ab_\pm[q_\mathrm{T}/2 - q' \exp(\pm i\alpha)], \tag{5.10}$$

$$b^1_\pm(1, 0) = b_\pm a[q_\mathrm{T}/2 + q' \exp(\pm i\alpha)], \tag{5.11}$$

$$b^0_0(1, 1) = 3^{-1/2}[b_+ b_- C_+(q_\mathrm{T}, q', \alpha) + b_- b_+ C_-(q_\mathrm{T}, q', \alpha) - b_0 b_0], \tag{5.12}$$

$$b^2_{\pm 1}(1, 1) = 2^{-1/2}\{b_0 b_\pm[q_\mathrm{T}/2 - q' \exp(\pm i\alpha)] + b_\pm b_0[q_\mathrm{T}/2 + q' \exp(\pm i\alpha)]\}, \tag{5.13}$$

$$b^2_0(1, 1) = 6^{-1/2}[b_+ b_- C_+(q_\mathrm{T}, q', \alpha) + b_- b_+ C_-(q_\mathrm{T}, q', \alpha) + 2b_0 b_0], \tag{5.14}$$

$$b^2_{\pm 2}(1, 1) = b_\pm b_\pm[q^2_\mathrm{T}/4 - q'^2 \exp(\pm 2i\alpha)], \tag{5.15}$$

$$C_\pm(q_\mathrm{T}, q', \alpha) = q^2_\mathrm{T}/4 - q'^2 \pm i q_\mathrm{T} q' \sin\alpha. \tag{5.16}$$

where α is the angle between $\boldsymbol{q}'$ and $\boldsymbol{q}_\mathrm{T}$ measured from $\boldsymbol{q}'$ to $\boldsymbol{q}_\mathrm{T}$. These formulae are valid for the spin reference frame x, y, z defined in example (a) from Section 3. If amplitudes in a different spin reference frame x', y', z' are needed, it is enough to replace in formula (5.1) the tensor amplitudes $A^j_\varrho(q)$ by the transformed tensor amplitudes

$$A^j_\varrho(\boldsymbol{q})' = \sum_\mu D^j_{\mu\varrho}(\alpha, \beta, \gamma) A^j_\mu(\boldsymbol{q}). \tag{5.17}$$

Here D^j are the rotation matrices and the Euler angles α, β, γ correspond to the rotation converting the $x\,y\,z$ frame into the $x'\,y'\,z'$ frame.

Once the amplitudes are known, the differential cross-sections are fixed by (2.1).

In particular for an experiment with unpolarized deuterons

$$\frac{d\sigma}{d\Omega} = \frac{1}{3} \sum_{M,M'} |A_{MM'}(\boldsymbol{q})|^2 = \sum_j \frac{1}{2j+1} \sum_\varrho |A^j_\varrho(\boldsymbol{q})|^2, \tag{5.18}$$

where the second equality follows from (4.1) and the orthonormality of the Clebsch--Gordan coefficients.

6. *Conclusions*

In order to calculate in Glauber's ordered approximation the amplitudes for a coherent process $x+\mathrm{d} \to y+\mathrm{d}$ it is necessary to use as input data:

a) The two radial wave functions of the deuteron (2.2) or equivalently the five form factors (2.4) and moreover if required the thirteen form factors (2.5);

b) A set of amplitudes for scattering on single nucleons. This set should include at least the amplitudes for the processes $x\mathrm{N} \to x\mathrm{N}$, $x\mathrm{N} \to y\mathrm{N}$, $y\mathrm{N} \to y\mathrm{N}$, for N being a proton and for N being a neutron.

The amplitudes are given in terms of the input data by formula (5.1) and the following definitions. The differential cross-section for unpolarized deuterons is given by formula (5.18). Both formulae include all the complications due to spin and isospin as well as the effects of the longitudinal momentum transfer, and can accomodate arbitrary intermediate states (inelastic screening).

It seems to be often advantageous to use the tensor amplitudes $A^j_\varrho(\boldsymbol{q})$ instead of the standard amplitudes $A^j_{M'M}(\boldsymbol{q})$. Formula (5.2) expressing the tensor amplitudes in terms of the data is simpler than the formula corresponding to (5.1). The transformation law is given by (5.17), while the corresponding formula for the amplitudes $A_{M'M}$ is bilinear in the rotation matrices and in general expresses each transformed component as a sum of nine terms. The formula expressing the unpolarized cross-section in terms of the tensor amplitudes is not more complicated than that with the standard amplitudes (5.18).

APPENDIX

Derivation of formula (5.1)

In this appendix formula (5.1) is derived from formula (4.1). The necessary theorems from the theory of angular momentum can be found in any textbook on the subject (*cf. e.g.* [8]). In order to facilitate reading the derivation is devided into four steps.

a) Orbital and spin parts of the transition amplitude

Each of the products of operators in formula (4.1) contains an orbital part (exponential multiplied by 1 or by $\varepsilon(z)$), which does not depend on spins. Further there are the spin parts $F(\boldsymbol{q})$ or $F(\boldsymbol{q}^+)\,F(\boldsymbol{q}^-)$, which do not depend on $\boldsymbol{r}$. The orbital parts and the spin parts will be separately expanded into series of tensor operators.

Some care is necessary, when handling the spin parts, because, according to our definition of the amplitudes F, in the product $F(\boldsymbol{q}^+)\,F(\boldsymbol{q}^-)$ the first term corresponds to a spin reference frame with the x-axis parallel to $\boldsymbol{q}^+$ and the other to a frame with the x-axis parallel to $\boldsymbol{q}^-$. In the following derivation all the operators will be transformed

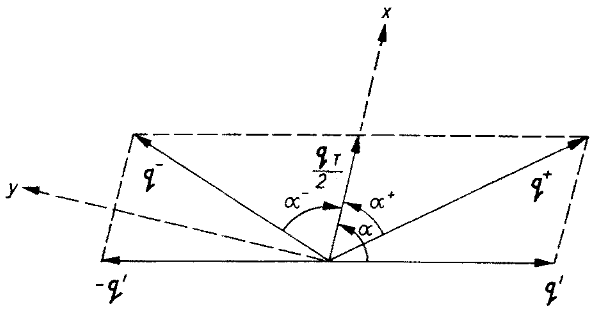

Fig. 1.

to a common spin reference frame with the x-axis parallel to $\boldsymbol{q}_\mathrm{T}$. This leaves the operators S^0 and S^1_0 unchanged, while the other two operators are transformed according to

$$q^\nu S^1_{\pm 1} \rightarrow q^\nu \exp{(\pm i\alpha^\nu)} S^1_{\pm 1} = [q_\mathrm{T}/2 + \nu q' \exp{(\pm i\alpha)}] S^1_{\pm}. \tag{A1}$$

The angles $\alpha, \alpha^+, \alpha^-$ are defined in Fig. 1. The second equality in (A1) is a trigonometric identity ($\nu = \pm$).

b) Expansion of spin parts in series of tensor operators

The single scattering operator in (4.1) is a sum of two terms. Changing the integration variable in the second term from $\boldsymbol{r}$ to $-\boldsymbol{r}$, one finds that the operator factorises into an orbital part, and a spin part (*cf.* (3.1))

$$F^{xyp}(\boldsymbol{q}) + F^{xyn}(\boldsymbol{q}) = \sum_{\mathrm{N}=\mathrm{n,p}} \sum_{s,\sigma} b^s_\sigma(\mathrm{N}) S^s_\sigma(\mathrm{N}). \tag{A2}$$

Here the notation for the operators defined by (3.2) has been supplemented by the argument N indicating whether the operator acts on the proton or the neutron spin. Since the operators S^s_σ are tensor operators — for this reason they were chosen as basis in Section 3 — formula (A2) gives the required expansion for the spin part of the single scattering operator. The deuteron wave function is symmetric in the proton and neutron spins. Consequently the matrix element of S^s_σ does not depend on the argument N. Putting in all the terms of (A2) N = p we obtain an equivalent expansion:

$$F^{xyp}(\boldsymbol{q}) + F^{xyn}(\boldsymbol{q}) \sim \sum_{s,\sigma} b^s_\sigma S^s_\sigma(\mathrm{p}). \tag{A3}$$

The coefficients b^s_σ, obtained by comparison with (A2) and (3.1) are given by formula (5.6).

The double scattering amplitude for each intermediate particle c contains the spin operator $F^{cyp}(\boldsymbol{q}^+)F^{xcn}(\boldsymbol{q}^-) + F^{cyn}(\boldsymbol{q}^+)F^{xcp}(\boldsymbol{q}^-)$ which is a bilinear form in the operators

S^s_σ(p) and S^s_σ(n). Aftei transforming according to (A1) all the operators to a common spin reference frame one can, using the standard Clebsch-Gordan expansion for each operator product, expand the bilinear form into the tensor operators:

$$S^s_\sigma(s_1, s_2) = \sum_{\sigma_1, \sigma_2} \langle s_1, \sigma_1; s_2, \sigma_2 | s, \sigma \rangle S^{s_1}_{\sigma_1} S^{s_2}_{\sigma_2}. \tag{A4}$$

The coefficients $b^s_\sigma(s_1, s_2)$ of the final expansion

$$\sum b^s_\sigma(s_1, s_2) S^s_\sigma(s_1, s_2) \tag{A5}$$

are given by the foımulae (5.8)–(5.16).

For further work the reduced matrix elements of the tensoı operators occurring in this section are necessary. Among the many alternative definitions of the reduced matrix elements we choose the definition by the Eckaıt-Wigner theorem written in the form

$$\langle S''\sigma'' | S^s_\sigma | S'\sigma' \rangle = \langle S', \sigma'; S, \sigma | S''\sigma'' \rangle \langle S'' \| S^\sigma \| S' \rangle. \tag{A6}$$

We need the reduced matrix elements for the case when $|S'\sigma'\rangle$ and $|S''\sigma''\rangle$ are spin states of the deuteron ($S' = S'' = 1$). Substituting *e.g.* $|11\rangle$ for both, we find the reduce matrix elements given in Table II.

c) Expansion of the orbital parts in series of tensor operators

Formula (4.1) contains two kinds of orbital parts: exponential and expcnential multiplied by $\varepsilon(z)$. Each can be expanded into spherical harmonics:

$$\exp(i\boldsymbol{Q}\cdot\boldsymbol{r}) = \sum_{l,m} a^l_m(Q, r)\,(4\pi)^{1/2} Y^l_m(\hat{\boldsymbol{r}}), \tag{A7}$$

$$\exp(i\boldsymbol{Q}\cdot\boldsymbol{r})\varepsilon(z) = \sum_{l,m} d^l_m(Q, r)\,(4\pi)^{1/2} Y^l_m(\hat{\boldsymbol{r}}). \tag{A8}$$

Here the coefficients a^l_m and d^l_m depend on the length of $\boldsymbol{r}$ and on $\boldsymbol{Q}$ while the spherical harmonics $Y^l_m(\hat{\boldsymbol{r}})$ are tensor operators depending only on the spherical angles of $\boldsymbol{r}$. The coefficients a^l_m are known from the familiar expansion of the plane wave into spherical waves:

$$a^l_m(Q, r) = (4\pi)^{1/2} i^l Y^l_m(\hat{\boldsymbol{Q}})^* j_l(Qr), \tag{A9}$$

where $j_l(Qr)$ are the spherical Bessel functions of the first kind. Here the dependence on r and on the orientation of $\boldsymbol{Q}$ factorizes. No such factorization occurs for

$$d^l_m(Q, r) = (4\pi)^{-1/2} \int \exp(i\boldsymbol{Q}\cdot\boldsymbol{r})\varepsilon(z) Y^{l*}_m(\hat{\boldsymbol{r}}) d\Omega_r. \tag{A10}$$

The reduced matrix elements of the spherical harmonics are given by the formula

$$(4\pi)^{1/2} \langle L' \| Y^l(\hat{\boldsymbol{r}}) \| L \rangle = \sqrt{\frac{(2l+1)(2L+1)}{2L'+1}}\, \langle L, 0; l, 0 | L', 0 \rangle. \tag{A11}$$

Using the notation from this and the preceding Section it is possible to rewrite the operator from (4.1) in the form

$$\sum a^l_m(\boldsymbol{q}/2, r)b^s_\sigma(4\pi)^{1/2}Y^l_m(\hat{\boldsymbol{r}})S^s_\sigma(\mathrm{p}) + \sum \hat{l}[a^l_m(\boldsymbol{Q}, r) + \\ + d^l_m(\boldsymbol{Q}, r)]\,(4\pi)^{1/2}Y^l_m(\hat{\boldsymbol{r}})S^s_\sigma(s_1, s_2). \tag{A12}$$

The evaluation of the matrix element given by Eq. (4.1) involves the r-integration of the coefficients a^l_m and d^l_m, which leads to the form factors (2.4) and (2.5). Note that the r-independent factor in a^l_m is written explicitly in (5.3) and is not included in the form factor. Further there is the integration over the orientations of $\boldsymbol{r}$ and the summation over spins. This can be considerably simplified as shown in the following step.

d) Evaluation of the transition amplitude

As seen from (4.1) and (A12) the amplitude is a linear combination of matrix elements $\langle 1, M'|C^l_m(r)T^l_m(\hat{r})S^s_\sigma|1, M\rangle$. Here C^l_m denotes a coefficient a^l_m or d^l_m, $T^l_m = (4\pi)^{1/2}Y^l_m$ and S^s_σ is one of the spin operators.

Using the Eckart-Wigner theorem (A6) for the spin part and the orbital part separately we find from (2.2)

$$\langle 1, M'|C^l_mT^l_mS^s_\sigma|1, M\rangle = \sum \langle L', M'-\sigma_f; 1, \sigma_f|1, M'\rangle \langle L, M-\sigma_i; 1, \sigma_i|1, M\rangle \times \\ \times \langle L, M-\sigma_i; l, m|L', M'-\sigma_f\rangle \langle 1, \sigma_i; s, \sigma|1, \sigma_f\rangle \langle L'\|C^l_mT^l\|L\rangle \langle 1\|S^s\|1\rangle. \tag{A13}$$

Using the definition of the 9-j symbols and the properties of the Clebsch-Gordan coefficients it is possible to rewrite this formula in a more compact form

$$\langle 1, M'|C^l_mT^l_mS^s_\sigma|1, M\rangle = \sum_{j=0}^{2} B(l, s, j; C^l_m)\,\langle l, m; s, \sigma|j, M'-M\rangle \times \\ \times \langle 1, M; j, M'-M|1, M'\rangle \langle 1\|S^s\|1\rangle, \tag{A14}$$

where

$$B(l, s, j, C^l_m) = 3\sqrt{2j+1}\sum_{L,L'}\sqrt{2L'+1}\begin{Bmatrix} L & 1 & 1 \\ l & s & j \\ L' & 1 & 1 \end{Bmatrix}\int_0^\infty u_{L'}(r)u_L(r)C^l_m(r)dr \times \\ \times \langle L'\|T^l(\hat{r})\|L\rangle. \tag{A15}$$

Using the symbols $B(l, s, j)$ and $B_m(l, s, j)$ defined in Section 5, and substituting (A12) and later (A14) and (A11) into (4.1) we get (5.1).

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