Vol. B5 (1974) ACTA PHYSICA POLONICA No 4

SOLUTIONS OF THE EINSTEIN FIELD EQUATIONS FOR A ROTATING PERFECT FLUID

PART 1. PRESENTATION OF THE FLOW-STATIONARY AND VORTEX-HOMOGENEOUS SOLUTIONS

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(Received June 29, 1973)

The equations of isentropic rotational motion of a perfect fluid are investigated with use of the Darboux theorem. It is shown that, together with the equation of continuity, they ensure the existence of four scalar functions which constitute a dynamically distinguished set of coordinates. If in this system of coordinates the metric tensor is constant along the lines tangent to velocity and vorticity fields, then the field equations with $T_{ij} = (\epsilon+p)u_iu_j-pg_{ij}$ can be completely integrated. The resulting metrics divide into 3 families, first of which contains 6 types of new solutions with non-zero pressure. All of them are given explicitly in terms of hypergeometric or confluent hypergeometric functions, type IV being the only one containing entirely elementary functions. The second family contains only the solution of Gödel, and the third one — only the solution of Lanczos.

Introduction

Soon after the creation of the general relativity theory people started trying to solve the Einstein field equations for rotating matter. The problem was interesting both from theoretical and observational points of view because nobody knew how to describe the rotational motion in the formalism of general relativity while many stars and galaxies exhibited visible rotation. Today even the possibility of rotation of the Universe at large is admitted (see *e. g.* [1]).

However, for quite a long time models of rotating matter were constructed under very special assumptions. The development followed two lines. The first line was originated by Lense and Thirring [2] in 1918 and was based on the "slow rotation" approximation. The disadvantages of approximate solutions in comparison with exact ones are rather obvious, however there are more physical objections against this particular method. I shall

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present them in Part 3 of the paper. The second line started with the paper of Lanczos [3] in 1924 and contained models of dust with or without cosmological term. There are no physical indications that the assumption $p = 0$ is wrong in cosmology, but it is unsatisfactory from the mathematical point of view as it is just an escape from the difficulties connected with introduction of pressure into the solutions.

It was not till 1967 that Trümper [4] clearly stated the problem of searching for solutions with pressure different from zero. However he has just written down the field equations and stopped after arriving at some general statements. A few authors went further, but they abandoned the problem when the equations were simplified and nearly-integrated (*i. e.* there remained only one or two equations to be solved), giving just some special cases of solutions which were mathematically simple (*e. g.* Stewart and Ellis [5], Wainwright [6]). Until 1972 there were just two complete solutions with non-zero pressure given by Wahlquist [7] in 1968 and Herlt [8] in 1972 (I do not take into account such models as that of Raval and Vaidya [9] representing a fluid with anisotropic pressure because they are rather artificial).

One sees therefore that models of rotating matter are still needed. The aim of the present paper is to supply new solutions of the Einstein field equations with rotating perfect fluid as a source. The method of description of the isentropic rotational motion of a perfect fluid introduced by Plebański [10] is used here. Under the assumptions which are clearly stated in Chapter 1e the field equations are completely integrated. The first part of the paper deals with the procedure of solving the field equations and presents the complete set of solutions. Among them appear as special cases those of Lanczos [3] with $\Lambda = 0$, Gödel [11] and Raval-Vaidya [9] in the case of isotropic pressure. The metrics of the first family (according to the classification given in Chapter 4) are new.

The second and third part of the paper will we published separately.

In the second part the discussion of geometrical and physical properties of the new solutions will be given. It turns out that they are not quite realistic as models of the Universe or interior of stars. However they may be of some importance for relativistic hydrodynamics. Also they may constitute a base for more realistic non-stationary cosmological models.

The third part of the paper will be a review article where all exact solutions of the field equations for a rotating perfect fluid or dust, found up to the middle of 1973, are hoped to be included.

To make this paper readable independently of [10] I give all the necessary considerations in Chapter 1. It should be emphasized that all the theorems and statements given there are not global. They hold only in simply connected neighbourhoods.

1. General considerations and statement of the problem

a) The equations of isentropic motion of a perfect fluid

Throughout the paper we shall use the signature $(+\ ---)$. The energy-momentum tensor of a perfect fluid has the form:

$$T^{\alpha\beta} = (\epsilon + p)u^{\alpha}u^{\beta} - pg^{\alpha\beta}, \tag{1.1}$$

where ϵ is the energy density in the rest-frame, p is the pressure and u^α — the velocity field of matter, $(\epsilon+p)$ is the enthalpy density.

The equations of motion follow from the relationship:

$$T^\beta_{\alpha;\beta} = 0. \tag{1.2}$$

Let $\mathscr{H}$ denote the enthalpy per one particle of the fluid. Then:

$$\mathscr{H} = (\epsilon+p)/n, \tag{1.3}$$

where n is the density of the number of particles. Independently of (1.2) the conservation of the total number of particles is postulated:

$$(nu^\alpha)_{;\alpha} = 0. \tag{1.4}$$

By virtue of (1.3) and (1.4) the equations of motion can be written in the following way:

$$nu^\beta(\mathscr{H}u_\alpha)_{;\beta} - p_{,\alpha} = 0. \tag{1.5}$$

The enthalpy fulfils the following thermodynamical identity:

$$d\mathscr{H} = \frac{1}{n}\,dp + TdS, \tag{1.6}$$

where T is temperature and S — the entropy. This equation may be considered to be the definition of temperature and entropy in general relativity. Namely, the equation of state $F(\mathscr{H}, p, n) = 0$ guarantees that only two of the functions $\mathscr{H}, n, p$ are independent. Then the form $(d\mathscr{H} - 1/n\,dp)$ certainly has an integrating factor which we denote by $1/T$, and call its inverse the temperature. Consequently, the form $1/T\,(d\mathscr{H} - 1/n\,dp)$ is a total differential of a function S which we call entropy.

With the help of (1.6) we obtain in (1.5):

$$n[u^\beta(\mathscr{H}u_\alpha)_{;\beta} - \mathscr{H}_{,\alpha} + TS_{,\alpha}] = 0. \tag{1.7}$$

Now the identities $u^\alpha u_\alpha = 1$ and $u^\beta u_{\beta;\alpha} = 0$ allow us to write (1.7) as:

$$[(\mathscr{H}u_\alpha)_{,\beta} - (\mathscr{H}u_\beta)_{,\alpha}]u^\beta + TS_{,\alpha} = 0. \tag{1.8}$$

We have written partial derivatives instead of covariant ones because in torsionless Riemannian space the connection terms cancel out.

When $S_{,\alpha} = 0$ we call the motion isentropic. We shall confine ourselves to such motions only. Then (1.3) and (1.6) imply:

$$d[(\epsilon+p)/n] = (dp)/n \tag{1.9}$$

which we can write in the form

$$d[(\epsilon+p)/\varrho] = (dp)/\varrho, \tag{1.10}$$

where ϱ is the density of the rest-mass, $\varrho = m_0 n$, m_0 is the mean mass of a particle. Hence we see that $d\epsilon = [(\epsilon+p)/\varrho]d\varrho$ and therefore $\epsilon = \epsilon(\varrho)$ and $p = p(\varrho)$. Thus the isentropic motion is characterized by the condition:

$$\epsilon = \epsilon(p), \ \ \varrho = \varrho(p). \tag{1.11}$$

If so, then (1.10) is an ordinary differential equation, which can be easily integrated to give:

$$\epsilon + p = \varrho c^2 \left(H_0 + \frac{1}{c^2} \int\limits_0^p \frac{dp}{\varrho(p)} \right) \stackrel{\text{def}}{=} \varrho c^2 H, \tag{1.12}$$

where $H_0 = \text{const}$. Then (1.8) may be rewritten as:

$$[(Hu_\alpha)_{,\beta} - (Hu_\beta)_{,\alpha}]u^\beta = 0. \tag{1.13}$$

These are the equations of isentropic motion of a perfect fluid. In the case of dust ($p = 0$) we have $\epsilon = \varrho c^2$ and $H = H_0 = 1$ in (1.13). Then we abandon the assumption $\varrho = \varrho(p)$, as the equations of motion follow immediately from (1.2).

b) Two useful theorems

Let:

$$\omega = V_\alpha dx^\alpha \tag{1.14}$$

denote a differential form of the 1st order on some region of a Riemannian space, where V_α is a continuously differentiable vector field. We define:

If $\underbrace{d\omega \wedge \ldots \wedge d\omega}_{l \text{ times}} \neq 0$ *but* $\omega \wedge \underbrace{d\omega \wedge \ldots \wedge d\omega}_{l \text{ times}} = 0$

then ω *is of class* $2l$.

If $\omega \wedge \underbrace{d\omega \wedge \ldots \wedge d\omega}_{l \text{ times}} \neq 0$ *but* $d\omega \wedge \underbrace{d\omega \wedge \ldots \wedge d\omega}_{l \text{ times}} = 0$

then ω *is of class* $(2l+1)$.

It is obvious that the class of a form may be equal at most to the dimension of space on which it is defined. The following theorem holds:

Theorem 1 (Darboux)

ω is of class $2l$ if and only if there exists the set of independent functions (ξ_i, η_i), $i = 1, \ldots, l$ such that $\omega = \sum_{i=1}^{l} \eta_i \, d\xi_i$; ω is of class $(2l+1)$ if and only if there exists the set of independent functions (τ, ξ_i, η_i), $i = 1, \ldots, l$ such that $\omega = d\tau + \sum_{i=1}^{l} \eta_i d\xi_i$.

It is easy to verify that these conditions are sufficient. The proof that they are necessary can be found in [12].

If the dimension of a manifold is even: $n = 2m$, then for every antisymmetric tensor $F_{\alpha\beta}$ the following form can be defined:

$$\operatorname{Pf}(F_{\alpha\beta}) = (2^m m!)^{-1} \epsilon^{\alpha_1 \ldots \alpha_{2m}} F_{\alpha_1\alpha_2} \ldots F_{\alpha_{2m-1}\alpha_{2m}}, \tag{1.15}$$

where $\epsilon^{\alpha_1 \dots \alpha_{2m}}$ is the totally skew-symmetric Levi-Civita symbol. This form is called the pfaffian of the tensor $F_{\alpha\beta}$. We have:

Theorem 2

$$[\mathrm{Pf}(F_{\alpha\beta})]^2 = \det(F_{\alpha\beta}).$$

The proof of this theorem can be found in [13] and [14]. Now we shall apply the theorems 1 and 2 to an investigation of the equations (1.13).

c) The invariant hypersurfaces defined by the equations of motion

Let us define:

$$\omega = Hu_\alpha dx^\alpha, \tag{1.16}$$

$$F_{\alpha\beta} = (Hu_\alpha)_{,\beta} - (Hu_\beta)_{,\alpha}. \tag{1.17}$$

We see that:

$$F_{\alpha\beta} dx^\alpha \wedge dx^\beta = -2d\omega. \tag{1.18}$$

In the equations (1.13) $F_{\alpha\beta}$ when acting on the vector u^α, has the eigenvalue 0. Hence det $(F_{\alpha\beta}) = 0$ and, by virtue of theorem 2, $\mathrm{Pf}(F_{\alpha\beta}) = 0$ which means that $F_{[\alpha\beta}F_{\gamma\delta]} = 0$. This, together with (1.18), implies that $d\omega \wedge d\omega = 0$. We see then that the class of ω is at most 3. Theorem 1 implies now that there exist three independent functions τ, ξ, η such that $\omega = d\tau + \eta d\xi$, *i. e.*

$$Hu_\alpha = \tau_{,\alpha} + \eta\xi_{,\alpha}. \tag{1.19}$$

From (1.19) and (1.17) we see that:

$$F_{\alpha\beta} = \xi_{,\alpha}\eta_{,\beta} - \xi_{,\beta}\eta_{,\alpha}. \tag{1.20}$$

When $F_{\alpha\beta} = 0$ we call the motion irrotational. When $F_{\alpha\beta} \neq 0$ we call it rotational.

To distinguish rotational and irrotational motions we can use as well the vorticity vector defined as follows:

$$w^\alpha = -(-g)^{-1/2}\epsilon^{\alpha\beta\gamma\delta}u_\beta u_{\gamma,\delta}. \tag{1.21}$$

The differentiation between rotational and irrotational motions based on w^α agrees with that in Newtonian mechanics. Imagine an observer who is at rest relative to the matter at a point p. Then let us take a local inertial frame at p (*i. e.* such that $u^\alpha = \delta^\alpha_0$ and $g_{\alpha\beta}(p) = \mathrm{Diag}\,(+1, -1, -1, -1)$) and calculate the components of w^α in this frame. It turns out that

$$w^\alpha = \left(0, -\frac{1}{c}\boldsymbol{W}\right), \tag{1.22}$$

where $\boldsymbol{W} = \mathrm{rot}\,\boldsymbol{v}$ is the Newtonian vorticity vector. Thus w^α is a relativistic generalization of the vector $\boldsymbol{W}$. We have

Theorem 3

$$(F_{\alpha\beta} = 0) \Leftrightarrow (w^{\alpha} = 0).$$

The proof is left to the reader. In consequence of this theorem we are allowed to consider $F_{\alpha\beta}$ to be the angular velocity tensor. But there is the definition of the angular velocity tensor given by Ehlers in [15] and [16]:

$$\Omega_{\alpha\beta} = u_{[\alpha;\beta]} - u_{[\alpha;|\varrho|}u^{\varrho}u_{\beta]} \tag{1.23}$$

(ϱ between vertical strokes is not included into antisymmetrization). With the help of (1.17) and (1.13) it is easy to show that:

$$F_{\alpha\beta} = 2H\Omega_{\alpha\beta}, \tag{1.24}$$

so our differentiation between rotational and irrotational motions agrees with that of Ehlers.

From now on we shall deal with rotating matter only, so we assume:

$$F_{\alpha\beta} \neq 0. \tag{1.25}$$

The functions τ, ξ, η given by (1.19) are not unique. They are determined with the following arbitrariness:

$$\tau \to \tau^* = \tau + S(\xi, \eta),\ \xi \to \xi^*(\xi, \eta),\ \eta \to \eta^*(\xi, \eta), \tag{1.26}$$

where S, ξ^*, η^* are two-argument functions with the condition that:

$$\eta\, d\xi - \eta^* d\xi^* = dS. \tag{1.27}$$

Again the proof is left to the reader. From (1.13), (1.17), (1.20) and (1.25) we conclude that:

$$\xi_{,\alpha}u^{\alpha} = \eta_{,\alpha}u^{\alpha} = 0. \tag{1.28}$$

By virtue of (1.28) and the equation of continuity $[(-g)^{1/2}\varrho u^{\alpha}]_{;\alpha} = 0$ we can define a function ζ:

$$(-g)^{1/2}\varrho u^{\alpha} = \epsilon^{\alpha\beta\gamma\delta}\xi_{,\beta}\eta_{,\gamma}\zeta_{,\delta}. \tag{1.29}$$

For the proof see [10]. It is easy to show that ζ is defined exact to the transformation:

$$\zeta \to \zeta^* = \zeta + \Omega(\xi, \eta), \tag{1.30}$$

where Ω is an arbitrary two-argument function.

By contraction of (1.19) with (1.29) we obtain the third equation:

$$g = -\varrho^{-2}H^{-2}\left[\frac{\partial(\tau, \xi, \eta, \zeta)}{\partial(x^0, x^1, x^2, x^3)}\right]^2. \tag{1.31}$$

It is easy to verify that if (1.19) and (1.29) are assumed then the equations of motion (1.13) and the equation of continuity are just identities.

The functions (τ, ξ, η, ζ) define a set of geometrically distinguished coordinates.

d) Geometrically distinguished coordinates and the metric tensor

Let us execute the following coordinate transformation:

$$(x^{0'}, x^{1'}, x^{2'}, x^{3'}) = (\tau, \xi, \eta, \zeta). \tag{1.32}$$

By virtue of (1.31) this transformation is not singular. We have chosen τ to be the time coordinate as gradient τ is the only one having non-zero projection on the u^α direction; see (1.19) and (1.29). In these coordinates (1.31) reduces to:

$$g = -\varrho^{-2}H^{-2} \tag{1.33}$$

while (1.19) and (1.29), with the help of (1.33), simplify to:

$$u^\alpha = H[1, 0, 0, 0], \tag{1.34}$$

$$u_\alpha = H^{-1}[1, x^2, 0, 0]. \tag{1.35}$$

The coordinates (1.32), as is seen from (1.26)–(1.27), can be transformed as follows:

$$x^0 = x^{0'} - S(x^{1'}, x^{2'}),\ x^1 = F(x^{1'}, x^{2'}),\ x^2 = G(x^{1'}, x^{2'}),\ x^3 = x^{3'} + T(x^{1'}, x^{2'}), \tag{1.36}$$

where T is arbitrary while S, F, G obey the equations:

$$GF_{,1'} - x^{2'} = S_{,1'},\ GF_{,2'} = S_{,2'}, \tag{1.37}$$

with the integrability condition:

$$F_{,1'}G_{,2'} - F_{,2'}G_{,1'} = 1. \tag{1.38}$$

The vorticity vector (1.21) in the coordinate system (1.32) assumes the form:

$$w^\alpha = [0, 0, 0, \varrho H^{-1}]. \tag{1.39}$$

Since $u_\alpha = g_{\alpha\varrho}u^\varrho$, the equations (1.34)–(1.35) yield some information about the metric tensor:

$$g_{00} = H^{-2},\ g_{01} = x^2 H^{-2},\ g_{02} = g_{03} = 0. \tag{1.40}$$

Our approach cannot give a satisfactory model of the interior of a star. First, the fluid inside a star certainly is not perfect bacause it is viscous and heat-conducting. Second, the particles of the fluid have different masses, so one may doubt whether ϱ in (1.10) is a well defined quantity. And third, in the course of nuclear reactions the rest--mass of particles is constantly converted into energy. Therefore (1.4), where n should be understood as the number of particles minus the number of antiparticles, is not equivalent to the equation $(\varrho u^\alpha)_{;\alpha} = 0$. So we should rather have in mind the applications of our results to cosmological models.

e) Statement of the problem

The functions in (1.36) depend only on two variables: x^1 and x^2. Therefore the idea arises: if the metric tensor would also depend only on x^1 and x^2 (in the coordinate system used) then may be the transformations (1.36)–(1.38) would allow us to simplify the metric further. So we assume:

$$\frac{\partial}{\partial x^0} g_{\alpha\beta} = \frac{\partial}{\partial x^3} g_{\alpha\beta} = 0. \tag{1.41}$$

These conditions are covariant with the transformations (1.36)–(1.38). From (1.34) and (1.39) we see that they are equivalent to:

$$\partial_u g_{\alpha\beta} = \partial_w g_{\alpha\beta} = 0, \tag{1.42}$$

where $\partial_u = u^\alpha \, \partial/\partial x^\alpha$, $\partial_w = w^\alpha \, \partial/\partial x^\alpha$.

The first of (1.42) means that our metrics have the timelike Killing vector collinear with the velocity field of matter. This is why we call them flow-stationary. This assumption implies that the expansion and shear of the velocity field vanish. The second of (1.42) means that the space-time is homogeneous in the direction of the vorticity vector. This property we call vortex-homogeneity.

We emphasize that (1.41) is the only simplifying assumption which we make throughout the paper. From now on we maintain full generality.

2. *First integrals of the field equations and their consequences*

a) Specialization of the coordinate system

We see that one of the functions F, G in (1.36) is arbitrary and once it is fixed, the other is determined by (1.38). The function S is then fixed up to a constant by (1.37). Together with T we have got two arbitrary funcıions in (1.36). We expect therefore that by a suitable choice of these two functions we will be able to put two more components of the metric tensor equal to zero.[1]

The set of equations $g_{1'2'} = 0$ and $g_{1'3'} = 0$ does not require any limitations on the initial metric (1.40). It can be represented in the form:

$$T_{,1'} = -(g_{13}/g_{33})\, F_{,1'} - (g_{23}/g_{33})\, G_{,1'}, \tag{2.1}$$

$$(g_{11}g_{33} - G^2 g_{00}g_{33} - g_{13}^2)\, F_{,1'}F_{,2'} + (g_{12}g_{33} - g_{13}g_{23})\,(F_{,1'}G_{,2'} + F_{,2'}G_{,1'}) + \\ + (g_{22}g_{33} - g_{23}^2)\, G_{,1'}G_{,2'} = 0. \tag{2.2}$$

We know that $g_{33} \neq 0$ because w^α is a spacelike vector different from 0 (by virtue of (1.25) and theorem 3) and (1.39) implies:

$$0 > g_{\alpha\beta} w^\alpha w^\beta = \varrho^2 H^{-2} g_{33}. \tag{2.3}$$

(1.38) and (2.2) constitute the set of equations for F and G. It can be shown that, irrespective of the form of the metric components, this set does have solutions. To show

[1] However for a general metric (1.40)–(1.41) these components cannot be chosen arbitrarily. The reader is asked to verify that the set of equations $g_{1'3'} = 0$, $g_{2'3'} = 0$ requires an integrability condition for the function T which is equivalent to an additional equation $(g_{13}/g_{33})_{,2} = (g_{23}/g_{33})_{,1}$.

this one should solve the set algebraically for partial derivatives of one of the functions, *e. g.* G, and substitute the resulting expressions in the integrability condition $G_{,1'2'} - G_{,2'1'} = 0$. One obtains then a well defined second order partial differential equation for one function F.

So we have shown that under the assumptions (1.41) there exist such coordinates x^1 and x^2 that the metric given by (1.33) and (1.40) has the additional properties:

$$g_{12} = g_{13} = 0. \tag{2.4}$$

These coordinates are not unique. When (2.4) are fed back in (2.1) and (2.2) we obtain the equations for the transformations (1.36)–(1.38) preserving all the properties (1.33), (1.40) and (2.4):

$$(g_{11} - G^2 g_{00}) g_{33} F_{,1'} F_{,2'} + (g_{22} g_{33} - g_{23}^2)\, G_{,1'} G_{,2'} = 0, \tag{2.5}$$

$$T_{,1'} = -(g_{23}/g_{33}) G_{,1'}. \tag{2.6}$$

From now on there is no arbitrary function at our disposal. All the functions in (1.36) are submitted to the equations (1.37), (1.38), (2.5) and (2.6).

b) The fundamental first integrals of the field equations

Owing to (2.4) a few of the field equations become quite simple and can be integrated. We give them below. To arrive at them one should use (1.33) which means that $\varrho^2[(x^2)^2 H^{-2} - g_{11}](g_{22}g_{33} - g_{23}^2) = 1$:

$$0 = R^0_3 = \tfrac{1}{2}\,\varrho H[\varrho x^2 H^{-1}(g_{33} g_{23,1} - g_{23} g_{33,1})]_{,2}, \tag{2.7}$$

$$0 = R^1_3 = -\tfrac{1}{2}\,\varrho H[\varrho H^{-1}(g_{33} g_{23,1} - g_{23} g_{33,1})]_{,2}, \tag{2.8}$$

$$0 = R^1_0 = \tfrac{1}{2}\,\varrho H(\varrho H^{-3} g_{33})_{,2}, \tag{2.9}$$

$$0 = R^2_0 = -\tfrac{1}{2}\,\varrho H(\varrho H^{-3} g_{33})_{,1}. \tag{2.10}$$

The first two of them imply that:

$$g_{23} = K(x^2)\, g_{33}, \tag{2.11}$$

where K is an arbitrary function of one variable. The other two yield:

$$g_{33} = G\varrho^{-1} H^3, \tag{2.12}$$

where $G = \text{const.}$ (2.3) implies that:

$$G < 0. \tag{2.13}$$

(2.11) and (2.12) are the first integrals we have been searching for.

c) Further specialization of coordinates

Execute the following coordinate transformation:

$$x^0 = x^{0'} + x^{1'} x^{2'},\ x^1 = x^{2'},\ x^2 = -x^{1'},\ x^3 = x^{3'} - \int K(x^2)\, dx^2, \tag{2.14}$$

where K is the function from (2.11). One verifies easily that (2.14) is of the form (1.36) and fulfils the equations (1.37), (1.38), (2.5) and (2.6). In addition it yields:

$$g_{2'3'} = 0. \tag{2.15}$$

So from now on we have the following metric (see (1.40) and (2.4)):

$$[g_{\alpha\beta}] = \begin{pmatrix} H^{-2} & x^2 H^{-2} & 0 & 0 \\ x^2 H^{-2} & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & G\varrho^{-1} H^3 \end{pmatrix}. \tag{2.16}$$

The coordinates in which the metric tensor has the form (2.16) with (1.33), are still not unique. They are determined up to the transformations (1.36)–(1.38) where $T = \text{const}$ and:

$$(g_{11} - G^2 g_{00}) F_{,1'} F_{,2'} + g_{22} G_{,1'} G_{,2'} = 0. \tag{2.17}$$

These coordinates will still be used in the paper, with some specializations in Chapters 4–5, 6 and 7.

The coordinate transformations, preserving our knowledge about the metric tensor on every step of integration of the field equations shall be called "admissible transformations". At present all the transformations (1.36)–(1.38) with (2.17) and $T = \text{const}$ are admissible. Two special cases will prove to be important later, so we investigate them separately.

d) Transformations preserving the x^1 and x^2 lines and transformations interchanging x^1 and x^2

The transformations (1.36)–(1.38) in which $F = F(x^1)$, $G = G(x^2)$ preserve the lines x^1 and x^2. From (1.38) and (2.17) we see that: $(F_{,2'} = 0) \Leftrightarrow (G_{,1'} = 0)$. For such transformations the equations (1.37), (1.38) and (2.17) are easily integrated to give:

$$x^0 = x^{0'} - \alpha\gamma x^{1'} + \delta,\ x^1 = \alpha x^{1'} + \beta,\ x^2 = \alpha^{-1} x^{2'} + \gamma,\ x^3 = x^{3'} + \varepsilon, \tag{2.18}$$

where $\alpha, \ldots, \varepsilon$ are arbitrary constants, $\alpha \neq 0$.

The transformations (1.36)–(1.38) in which $F = F(x^2)$, $G = G(x^1)$ interchange the lines x^1 and x^2. One sees again that: $(F_{,1'} = 0) \Leftrightarrow (G_{,2'} = 0)$. This time integration of (1.37), (1.38) and (2.17) yields:

$$x^0 = x^{0'} + x^{1'} x^{2'} - \alpha\gamma x^{2'} + \delta,\ x^1 = \alpha x^{2'} + \beta,\ x^2 = -\alpha^{-1} x^{1'} + \gamma,\ x^3 = x^{3'} + \varepsilon. \tag{2.19}$$

3. The complete set of the field equations and classification of the solutions

a) The base of differential forms

We shall use the method of computing the Riemann tensor given in [17], [18], [19] (and many others). The signature requirements imply $[(x^2)^2 H^{-2} - g_{11}] > 0$, $g_{22} < 0$. Therefore we introduce new functions h and k:

$$g_{11} \stackrel{\text{def}}{=} (x^2)^2 H^{-2} - h, \tag{3.1}$$

$$g_{22} \stackrel{\text{def}}{=} -k, \tag{3.2}$$

where $h > 0$, $k > 0$. (1.33), (2.16), (3.1) and (3.2) imply:

$$-G\varrho hk H^3 = 1. \tag{3.3}$$

Now, substituting (3.1) and (3.2) in (2.16) we obtain:

$$ds^2 = H^{-2}(dx^0 + x^2 dx^1)^2 - (h^{1/2} dx^1)^2 - (k^{1/2} dx^2)^2 - [(-G\varrho^{-1} H^3)^{1/2} dx^3]^2. \tag{3.4}$$

Consequently, we have the following base of differential forms:

$$e^0 = H^{-1}(dx^0 + x^2 dx^1), \qquad e^2 = k^{1/2} dx^2,$$
$$e^1 = h^{1/2} dx^1, \qquad e^3 = (-G\varrho^{-1} H^3)^{1/2} dx^3. \tag{3.5}$$

In this base the metric (3.4) assumes the form $ds^2 = \eta_{ij} e^i e^j$, $i, j = 0, 1, 2, 3$, where $\eta_{ij} = \text{Diag}\,(+1, -1, -1, -1)$. Proceeding in the standard way we can compute the forms of connection and curvature, and then find the scalar components of the Ricci tensor R^i_j.

b) The right-hand side of the field equations

We shall prefer a form of the field equations different from the standard one. First, we have to translate them into scalar components, *i. e.* substitute $T^i_j = e^i_\alpha e^\beta_j T^\alpha_\beta$ instead of T^α_β where e^i_α are the coefficients of the forms $e^i = e^i_\alpha dx^\alpha$ given by (3.5), and e^α_j are given by $e^i_\alpha e^\alpha_j = \delta^i_j$. Next we change to the form:

$$R^i_j = \frac{\kappa}{c^2}\left(T^i_j - \frac{1}{2}\delta^i_j T\right) + \Lambda \delta^i_j, \tag{3.6}$$

where $T \stackrel{\text{def}}{=} T^i_i$ and $\kappa \stackrel{\text{def}}{=} 8\pi k/c^2$. After easy computation we find that, in consequence of (1.1), (1.12), (1.34) and (1.35), the equations (3.6) assume the form:

$$R^0_0 = \kappa(\tfrac{1}{2}\varrho H + p/c^2) + \Lambda,$$
$$R^1_1 = R^2_2 = R^3_3 = \kappa(-\tfrac{1}{2}\varrho H + p/c^2) + \Lambda$$
$$\text{Other } R^i_j = 0. \tag{3.7}$$

c) The complete set of the field equations

Substituting the explicit expressions for R^i_j in (3.7) we find:

$$R^0_0 = (2hH)^{-1}(h^{-1}h_{,1}H_{,1} - k^{-1}k_{,1}H_{,1} + H^{-1}H^2_{,1} - 2H_{,11} + \varrho^{-1}\varrho_{,1}H_{,1}) +$$
$$+ (2kH)^{-1}(-h^{-1}h_{,2}H_{,2} + k^{-1}k_{,2}H_{,2} + H^{-1}H^2_{,2} - 2H_{,22} + \varrho^{-1}\varrho_{,2}H_{,2}) +$$
$$+ (2hkH^2)^{-1} = \kappa(\tfrac{1}{2}\varrho H + p/c^2) + \Lambda, \tag{3.8}$$

$$R^1_1 = (4hk)^{-1}[-h^{-1}h^2_{,2}+2h_{,22}-k^{-1}h_{,2}k_{,2}-h_{,2}(\varrho^{-1}\varrho_{,2}-H^{-1}H_{,2})]+$$
$$+(4h^2)^{-1}[-hk^{-2}k^2_{,1}+2hk^{-1}k_{,11}-k^{-1}h_{,1}k_{,1}+h_{,1}(\varrho^{-1}\varrho_{,1}-H^{-1}H_{,1})]+$$
$$+(4h)^{-1}(3\varrho^{-2}\varrho^2_{,1}-2\varrho^{-1}\varrho_{,11}+11H^{-2}H^2_{,1}+2H^{-1}H_{,11}-6\varrho^{-1}\varrho_{,1}H^{-1}H_{,1})+$$
$$-(2hkH^2)^{-1} = \kappa(-\tfrac{1}{2}\varrho H+p/c^2)+\Lambda, \tag{3.9}$$

$$R^1_2 = (16h^3k)^{-1/2}h_{,2}(\varrho^{-1}\varrho_{,1}-H^{-1}H_{,1})+(16hk^3)^{-1/2}k_{,1}(\varrho^{-1}\varrho_{,2}-H^{-1}H_{,2})+$$
$$+(16hk)^{-1/2}(3\varrho^{-2}\varrho_{,1}\varrho_{,2}-2\varrho^{-1}\varrho_{,12}+11H^{-2}H_{,1}H_{,2}+2H^{-1}H_{,12}+$$
$$-3\varrho^{-1}\varrho_{,1}H^{-1}H_{,2}-3\varrho^{-1}\varrho_{,2}H^{-1}H_{,1}) = 0, \tag{3.10}$$

$$R^2_2 = (4hk)^{-1}[-k^{-1}k^2_{,1}+2k_{,11}-h^{-1}h_{,1}k_{,1}-k_{,1}(\varrho^{-1}\varrho_{,1}-H^{-1}H_{,1})]+$$
$$+(4k^2)^{-1}[-h^{-2}kh^2_{,2}+2h^{-1}kh_{,22}-h^{-1}h_{,2}k_{,2}+k_{,2}(\varrho^{-1}\varrho_{,2}-H^{-1}H_{,2})]+$$
$$+(4k)^{-1}(3\varrho^{-2}\varrho^2_{,2}-2\varrho^{-1}\varrho_{,22}+11H^{-2}H^2_{,2}+2H^{-1}H_{,22}-6\varrho^{-1}\varrho_{,2}H^{-1}H_{,2})+$$
$$-(2hkH^2)^{-1} = \kappa(-\tfrac{1}{2}\varrho H+p/c^2)+\Lambda, \tag{3.11}$$

$$R^3_3 = (4h)^{-1}(\varrho^{-1}\varrho_{,1}-H^{-1}H_{,1})(2H^{-1}H_{,1}+h^{-1}h_{,1}-k^{-1}k_{,1})+$$
$$+(4k)^{-1}(\varrho^{-1}\varrho_{,2}-H^{-1}H_{,2})(2H^{-1}H_{,2}-h^{-1}h_{,2}+k^{-1}k_{,2})+$$
$$+(4h)^{-1}(3\varrho^{-1}\varrho^2_{,1}-2\varrho^{-1}\varrho_{,11}+3H^{-2}H^2_{,1}+6H^{-1}H_{,11}-6\varrho^{-1}\varrho_{,1}H^{-1}H_{,1})+$$
$$+(4k)^{-1}(3\varrho^{-2}\varrho^2_{,2}-2\varrho^{-1}\varrho_{,22}+3H^{-2}H^2_{,2}+6H^{-1}H_{,22}-6\varrho^{-1}\varrho_{,2}H^{-1}H_{,2}) =$$
$$= \kappa(-\tfrac{1}{2}\varrho H+p/c^2)+\Lambda. \tag{3.12}$$

The equation $R^2_1 = 0$ is identical with (3.10), and all the others are fulfilled identically. To see this one should use the following identity, resulting from (3.3):

$$\varrho^{-1}\varrho_{,i}+h^{-1}h_{,i}+k^{-1}k_{,i}+3H^{-1}H_{,i} = 0,\ i = 1,2. \tag{3.13}$$

d) Classification of the solutions

We divide the solutions of the equations (3.8)–(3.12) into 3 families characterized by the following conditions:

$$\varrho_{,2} \neq 0 \neq \varrho_{,2} \text{ family Ia,} \tag{3.14}$$
$$\varrho_{,1} \neq 0 = \varrho_{,2} \text{ family Ib,} \tag{3.15}$$
$$\varrho_{,1} = 0 \neq \varrho_{,2} \text{ family Ic,} \tag{3.16}$$
$$\varrho_{,1} = 0 = \varrho_{,2} \text{ family II,} \tag{3.17}$$
$$\varrho_{,\alpha} \neq 0,\ H = 1,\ p = 0 \text{ family III.} \tag{3.18}$$

A comment about family III is needed. All the time we dealt with a perfect fluid in isentropic motion and therefore H was a function of ϱ (see (1.11) and (1.12)). Hence

$H_{,i} = (dH/d\varrho)\, \varrho_{,i}$ and in families I–II the partial derivatives of H vanish or do not vanish only together with the corresponding derivatives of ϱ. In particular in family II $H = \text{const.}$

However we have noticed after (1.13) that in the case of dust the equations of motion can be obtained from (1.13) by substituting $H = 1$ and abandoning the assumption $\varrho = \varrho(p)$. The whole argument from (1.13) up to (3.13) can be thus repeated for dust because we have never used the fact that $H = H(\varrho)$.

The differentiation between the families I, II and III is clearly invariant, while the criterion of splitting up the family I is not. We show in the next chapter that families Ia, Ib and Ic can be transformed into one another by admissible coordinate transformations, thus they are geometrically identical.

4. *Invariant classification*

a) The idea of the proof

Families Ia and Ib are easily seen to be identical. Transformations (2.19) interchange them. Transformations (1.36)–(1.38) with (2.17) and $T = \text{const}$ lead from the family Ib or Ic to Ia if the functions F, G, S depend both on x^1 and x^2. The only difficult point is to prove that an arbitrary solution of the family Ia can be reduced to a solution of the family Ic by means of the (1.36)–(1.38), (2.17) transformations.

If we have $\varrho = \varrho(x^1, x^2)$ before the transformation and $\varrho = \varrho(x^{2'})$ after the transformation, with $\varrho_{,2'} \neq 0$, then $x^{2'} = v(\varrho(x^1, x^2))$ where $v(\varrho)$ is the function reciprocal to $\varrho(x^{2'})$. So we should verify whether there exist such functions $s(x^1, x^2)$, $u(x^1, x^2)$ and $v(\varrho)$ that the transformation:

$$x^{0'} = x^0 - s(x^1, x^2), \quad x^{1'} = u(x^1, x^2), \quad x^{2'} = v(\varrho(x^1, x^2)), \quad x^{3'} = x^3, \tag{4.1}$$

is consistent with the equations (1.37), (1.38) and (2.17). These equations concern the transformation inverse to (4.1). Notice that (1.38) means that the Jacobian of (1.36) is equal to 1. Since the matrix $[x^{\alpha}_{,\alpha'}]$ must be the inverse of $[x^{\alpha'}_{,\alpha}]$ we find easily the following equalities:

$$F_{,1'} = v'\varrho_{,2}, \quad F_{,2'} = -u_{,2}, \quad G_{,1'} = -v'\varrho_{,1}, \quad G_{,2'} = u_{,1}, \tag{4.2}$$

where $v' = dv/d\varrho$. When we substitute (4.2) and (3.1)–(3.2) in (1.38) and (2.17) we obtain

$$[(G\varrho kH^3)^{-1}u_{,2}\varrho_{,2} - ku_{,1}\varrho_{,1}]v' = 0,$$

$$(\varrho_{,2}u_{,1} - \varrho_{,1}u_{,2})v' = 1. \tag{4.3}$$

If we solve the set (4.3) algebraically for $u_{,1}$ and $u_{,2}$ and then substitute the results in the integrability condition $u_{,12} - u_{,21} = 0$ we obtain:

$$(v')^{-1}v'' + (G\varrho k^2H^3\varrho_{,1}^2 - \varrho_{,2}^2)^{-2}[(G\varrho k^2H^3\varrho_{,1}^2 + \varrho_{,2}^2)(G\varrho k^2H^3\varrho_{,11} + \varrho_{,22}) +$$

$$+ 2G\varrho k^2H^3\varrho_{,1}\varrho_{,2}(k^{-1}k_{,1}\varrho_{,2} - k^{-1}k_{,2}\varrho_{,1} - 2\varrho_{,12})] = 0. \tag{4.4}$$

This is an equation for the function $v(\varrho)$. It does make sense if the term $(\ )^{-2}\cdot[\ ]$ is a function of ϱ. Using the equations (3.9)–(3.11) one can prove that this is really the case. The proof is given in Appendix A.

b) Invariant classification

The conclusion of this chapter is that there are the following physically different families of solutions of the field equations (3.8)–(3.12):

Family I in which we can choose coordinates so that $\varrho = \varrho(x^2)$ and $H = H(x^2)$;

Family II in which $\varrho = \text{const}$, $p = \text{const}$ and $H = 1+p/(c^2\varrho) = \text{const}$;

Family III in which $p = 0$, $H = 1$, $\varrho \neq \text{const}$.

5. *The first family of solutions*

a) The group of admissible transformations

The functions ϱ, H, p are scalars and so $\varrho_{,\alpha}$ is a covariant vector. The condition $\varrho_{,1} = 0$ is not covariant, it is preserved by those transformations (1.36) for which $G_{,1'} = 0$. In Chapter 2d we have shown that they are given by (2.18). Therefore this is the group of admissible transformations for family I.

b) Algebraic form of the metric tensor and the fundamental differential equation

Let us substitute $\varrho_{,1} = H_{,1} = 0$ in (3.10). We obtain:

$$(4h^{1/2}k^{3/2})^{-1}k_{,1}(\varrho^{-1}\varrho_{,2}-H^{-1}H_{,2}) = 0. \tag{5.1}$$

If $\varrho^{-1}\varrho_{,2}-H^{-1}H_{,2} = 0$ then $H = \text{const}\cdot\varrho$. If we substitute this in (3.9) and (3.11) and subtract them, we get $-2H_{,2}^2/(kH^2) = 0$, and consequently $H = \text{const}$. Such solutions do not belong to the first family. Therefore (5.1) implies:

$$k_{,1} = 0. \tag{5.2}$$

If $\varrho_{,1} = H_{,1} = k_{,1} = 0$ then (3.13) yields at once:

$$h_{,1} = 0. \tag{5.3}$$

Hence all the components of the metric tensor are functions of one variable x^2 and (3.8)–(3.12) reduce to the set of ordinary differential equations. We shall integrate them. Subtract (3.9) from (3.8) and substitute $k = -1/(G\varrho hH^3)$ from (3.3). The resulting differential equation after integration yields:

$$h = W(x^2)/(GH^2), \tag{5.4}$$

where:

$$W(x^2) = (G+\kappa)(x^2)^2+B'x^2+E', \qquad B', E' = \text{const}. \tag{5.5}$$

Substituting (5.4) in (3.3) we obtain:

$$k = -(W\varrho H)^{-1}. \tag{5.6}$$

Now subtract (3.12) from (3.9) and use (5.4) and (5.6) in the resulting equation. After integration we get:

$$\varrho = D\frac{H^5}{W}\exp\left[\int\frac{G(x^2+C)}{W}\,dx^2\right], \qquad C, D = \text{const.} \tag{5.7}$$

(The symbol of an integral whenever used in the paper denotes just a primary function without any new constant.) If we execute the transformation (2.18) with $\gamma = -C$ then C in (5.7) vanishes and the constants B', E' in (5.5) change their values. Their new values shall be denoted by B and E without primes, whereas in (5.7) we have

$$\varrho = D\frac{H^5}{W}\exp\left(\int\frac{Gx^2}{W}\,dx^2\right). \tag{5.8}$$

From now on, however, the group of admissible transformations has one parameter less and is given by (2.18) with $\gamma = 0$.

Finally subtract (3.11) from (3.9) and use (5.4), (5.6) and (5.8). After multiplying by $(-2k)$ we obtain

$$4H^{-1}H_{,22}+8H^{-2}H_{,2}^2-4W^{-1}W_{,2}H^{-1}H_{,2}+4Gx^2W^{-1}H^{-1}H_{,2}-W^{-1}W_{,22}+$$
$$+W^{-2}W_{,2}^2-Gx^2W^{-2}W_{,2}+GW^{-1} = 0. \tag{5.9}$$

It is a differential equation for the function H which we call "the fundamental equation". We can substitute:

$$H = u^{1/3}. \tag{5.10}$$

Then, if we multiply (5.9) by $\frac{3}{4}u$, we obtain a linear homogeneous equation for u:

$$u_{,22}-\frac{W_{,2}-Gx^2}{W}u_{,2}+\frac{3}{4}\left(-\frac{W_{,22}}{W}+\frac{W_{,2}^2}{W^2}-\frac{Gx^2W_{,2}}{W^2}+\frac{G}{W}\right)u = 0. \tag{5.11}$$

Each solution of (5.11) determines algebraically through (5.10), (5.8), (5.6), (5.4) and (3.4) all the components of the metric tensor. It has the following form:

$$ds^2 = H^{-2}\{(dx^0)^2+2x^2dx^0dx^1+[(x^2)^2-W(x^2)/G](dx^1)^2\}+$$
$$+(W\varrho H)^{-1}(dx^2)^2+G\varrho^{-1}H^3(dx^3)^2. \tag{5.12}$$

c) Consistence of the field equations

To obtain (5.4), (5.8) and (5.9) we subtracted the equations (3.8), (3.9), (3.11) and (3.12) from one another, so in each case the terms with pressure on the right-hand side cancelled out. We have used three such equations and in the fourth one the pressure must appear. Since (5.4), (5.6), (5.8) and (5.9) completely determine the metric tensor, one might expect that the fourth equation of the set (3.8)–(3.12) just defines the pressure. But (1.12) implies $p_{,2} = c^2\varrho H_{,2}$ and hence:

$$p/c^2 = \int \varrho H_{,2}dx^2+p_0/c^2. \tag{5.13}$$

Therefore we must check whether the equations (3.8)–(3.12) define the pressure consistently with (5.13). It turns out to be true. The proof is sketched in Appendix B.

d) Conditions for the physical reasonability of the solutions

The solutions of (3.8)–(3.12) to be physically reasonable have to obey the following conditions:

1. *All the components of the metric tensor and the physical scalars (ϱ, p, H, etc.) must be real functions.*

2. *ϱ, p and H must be positive or vanish.*

3. *In the whole region of space-time on which the metric tensor is defined* (except possibly for some sets of measure 0, where a singularity appears) *the signature of the metric must be* $(+\;-\;-\;-)$.

We investigate these conditions.

1. Since we deal with real functions and real variables this condition is, in general, fulfilled automatically. However, in the solutions of type I (see classification in Chapters $5f-5k$) we introduce complex numbers artificially to simplify the computations. As (5.11) is a linear equation we can always take real combinations of the solutions.

2. We ensure that $H > 0$ when we substitute $|H|$ instead of H everywhere. We are allowed to do it, for (5.9) may be rewritten in the form:

$$2H^{-2}(H^2)_{,22}+H^{-4}(H^2)^2_{,2}-2W^{-1}W_{,2}H^{-2}(H^2)_{,2}+2Gx^2W^{-1}H^{-2}(H^2)_{,2}+$$
$$-WW_{,22}+W^{-2}W^2_{,2}-Gx^2W^{-2}W_{,2}+GW^{-1} = 0.$$

Here, and also in the metric (5.12), H appears only in even powers, so the substitution of $|H|$ instead of H changes nothing at all. If $H > 0$ then $\varrho > 0$, in consequence of (5.8), means:

$$D/W > 0. \tag{5.14}$$

It depends on the type of the polynomial W whether this inequality holds. If W has two real roots, (5.14) may be fulfilled only in some range of values of x^2. Outside this range ϱ would be negative. In this case we have to match some exterior solution to our interior one so that the complete metric has no singularities. This is done in Chapter 9 (part two of the paper).

Since $H = H(x^2)$ we can introduce a new variable $x^2 = x^2(H)$ in (5.13) and then:

$$p/c^2 = \int\varrho(H)dH+p_0/c^2. \tag{5.15}$$

As we have ensured that $\varrho > 0$, the integral of ϱ is also positive. Thus p is positive if the constant p_0 is not too much below 0. This may always be assured by including a fraction of p_0 into the cosmological constant.

So the condition 2 requires: substitute $|H|$ instead of H everywhere, and bound the range of value of x^2 so that (5.14) holds.

3. This condition means that $g_{00} > 0$ (which is fulfilled automatically), $g_{00}g_{11}+g^2_{01} < 0$,

$g_{22} < 0$ and $g_{33} < 0$. The inequality for g_{33} is ensured by (2.13) and the condition 2. $g_{22} < 0$ implies, by virtue of the condition 2:

$$W(x^2) < 0. \tag{5.16}$$

Together with (5.14) this means that:

$$D < 0. \tag{5.17}$$

The inequality $g_{00}g_{11} - g_{01}^2 < 0$ is now fulfilled automatically.

e) Classification of the first family into types

(5.11) has various types of singularities corresponding to W having two complex roots, two real roots, one double root and so on. In each case the solutions of (5.11) and the results of integration in (5.8) are different. There are six such cases which give rise to the classification of the first family into 6 types. We give the corresponding solutions of (5.11) in the following sections. Sometimes a new constant is introduced for convenience:

$$a \overset{\text{def}}{=} G/(G+\kappa). \tag{5.18}$$

The reader should have (5.8), (5.12) and (5.18) in mind when studying the tables. The constants M, N are always real and never vanish simultaneously. When hypergeometric functions appear, we omit the special cases corresponding to integer values of parameters. If needed, they can be found with the help of the references given. Some interesting special cases will be discussed in Chapter 13 (part two of the paper).

TABLE I

Type I solutions

$$H = |M(u+u^*) - iN(u-u^*)|^{1/3}$$

$$u = \left(\frac{x^2-b}{c'-K}\right)^{\beta} \left(\frac{x^2-c'}{b-K}\right)^{\gamma} F\left(\alpha+\beta+\gamma, \alpha'+\beta+\gamma, 1+\beta-\beta', \frac{x^2-b}{c'-b}\right)$$

$$K = K^* = \text{const}, \qquad a = \text{const} > 1$$

$$\left.\begin{matrix}\alpha\\ \alpha'\end{matrix}\right\} = \tfrac{1}{2}[a-3 \pm (a^2-3a+3)^{1/2}]$$

$$\left.\begin{matrix}\beta\\ \beta'\end{matrix}\right\} = \frac{1}{2(b-c')}\{-(a-2)b-2c' \mp [a^2b^2+(b-c')(b-c'-ab)]^{1/2}\}$$

$$\left.\begin{matrix}\gamma\\ \gamma'\end{matrix}\right\} = \frac{1}{2(b-c')}\{2b+(a-2)c' \pm [a^2c'^2+(b-c')(b-c'+ac')]^{1/2}\} = \left\{\begin{matrix}\beta^*\\ \beta'^*\end{matrix}\right.$$

$$\int \frac{Gx^2}{W}\,dx^2 = \frac{a}{2}\ln[(x^2-\operatorname{Re} b)^2+(\operatorname{Im} b)^2] + a\frac{\operatorname{Re} b}{\operatorname{Im} b}\operatorname{arc\,tg}\left(\frac{x^2-\operatorname{Re} b}{\operatorname{Im} b}\right)$$

f) Type I solutions

$$G+\kappa \neq 0, \quad \Delta \overset{\text{def}}{=} B^2-4E(G+\kappa) < 0. \tag{5.19}$$

W has two complex roots $x^2 = b$ and $x^2 = c' = b^*$, and may be represented in the form:

$$W = (G+\kappa)(x^2-b)(x^2-c'). \tag{5.20}$$

$G < -\kappa$ by virtue of (5.16). For definiteness we assume Im $(b) > 0$. Here we can follow the standard methods of solving the Riemann equation [20], sometimes called the equation of Gauss [21] or Papperitz [22]. The solution of (5.11) depends on the hypergeometric function of a complex variable and complex parameters. We omit the special case when $u = u^*$.

g) Type II solutions

$$G+\kappa \neq 0, \ \Delta > 0. \tag{5.21}$$

W has two real roots $x^2 = b$ and $x^2 = c' > b$, and may be represented in the form (5.20). This time however the sign of W depends on the value of x^2. If we want the condition (5.16) to hold for $b < x^2 < c'$ then $G > -\kappa$, if we want it to hold for $x^2 < b$ and $x^2 > c'$ then

TABLE II

Type II solutions

$$H = |Mu + Nu_1|^{1/3}$$

u is given in Table I.

$K = \text{const} \neq b, c'; \quad a < 0 \quad \text{or} \quad a > 1.$

u is the second linearly independent solution of the Riemann equation for u. The formulas for α, α', β, β', γ, γ' are identical with those of Table I. Now no analogue of the equations $\gamma = \beta^*$ and $\gamma' = \beta'^*$ holds.

$$\int \frac{Gx^2}{W} dx^2 = \frac{a}{b-c'}[b \ln|x^2-b| - c' \ln|x^2-c'|].$$

When $a < 0$ the solution has the proper signature for $b < x^2 < c'$; when $a > 1$—for $x^2 < b$ and $x^2 > c'$.

$G < -\kappa$. In each case we have to match some empty-space metric to avoid the singularities $x^2 = b$ and $x^2 = c'$. Here we can repeat the same reasoning which we used for type I. The only difference is that now all the numbers are real.

h) Type III solutions

$$G+\kappa \neq 0, \ \Delta = 0, \ W(0) \neq 0. \tag{5.22}$$

W has one real root $x^2 = b \neq 0$ and is of the form:

$$W = (G+\kappa)(x^2-b)^2. \tag{5.23}$$

TABLE III

Type III solutions

$$H = |Mu_1 + Nu_2|^{1/3}$$

$$u_i = (x^2-b)^{q_i} F\left(\frac{3}{2} - q_i, 4-a-2q_i, \frac{ab}{x^2-b}\right), \quad i = 1, 2.$$

$$a = \text{const} > 1$$

$$\left.\begin{matrix} q_1 \\ q_2 \end{matrix}\right\} = \tfrac{1}{2}[3-a \pm (a^2-3a+3)^{1/2}]$$

$$\int \frac{Gx^2}{W}\, dx^2 = a \ln |x^2-b| - \frac{ab}{x^2-b}$$

Again (5.16) implies $G < -\kappa$. Here the hypergeometric function is substituted by the confluent hypergeometric function.

i) Type IV solutions

$$G+\kappa \neq 0, \ \Delta = 0, \ W(0) = 0. \tag{5.24}$$

Now W has the form:

$$W = (G+\kappa)(x^2)^2, \ G < -\kappa. \tag{5.25}$$

In this case (5.11) is integrated at once. The solution is a limiting case $b \to 0$ of type III solutions.

TABLE IV

Type IV solutions

$$H = |M(x^2)^{q_1} + N(x^2)^{q_2}|^{1/3}$$

$a > 1$, q_1 and q_2 are given in Table III

$$\varrho = -(D/\kappa)(a-1)|x^2|^{a-2} H^5, \quad D = \text{const} < 0$$

j) Type V solutions

$$G+\kappa = 0, \ B \neq 0. \tag{5.26}$$

W degenerates to (Bx^2+E). Now it is convenient to introduce E_0 by $E_0 = EB^2/\kappa$. If we execute the admissible transformation (2.18) with $\gamma = 0$ and $\alpha = B^{-1} \cdot \kappa$ then we see from (5.12) that (Bx^2+E) changes to $\kappa(x^2+E_0)$ and D changes to $D' = \kappa^2 B^{-2} D$. So, with no loss in generality, we can assume $B = \kappa$. Then:

$$W = \kappa(x^2+E_0) \tag{5.27}$$

but now only those transformations (2.18) are admissible, for which $\gamma = 0$ and $\alpha = 1$. We will see in Chapter 8 (part two) that this is the symmetry group for type V solutions.

Again the confluent hypergeometric function appears. (5.16) here means that:

$$x^2 < -E_0 \tag{5.28}$$

so we will have to match some empty-space metric to our one in order to avoid the singularity $x^2 = E_0$.

TABLE V

Type V solutions

$$H = |Mu_1 + Nu_2|^{1/3}$$

$$u_i = [\exp(x^2+E_0)](-x^2-E_0)^{q_i} F(q_i+E_0-1, 2q_i+E_0-1, -x^2-E_0) \qquad i = 1, 2.$$

$$\left.\begin{matrix} q_1 \\ q_2 \end{matrix}\right\} = \tfrac{1}{2}[2-E_0 \pm (E_0^2-E_0+1)^{1/2}]$$

$$\varrho = -(D/\kappa)\, e^{-x^2}(-x^2-E_0)^{E_0-1} H^5, \qquad D = \text{const} < 0$$

The signature is proper in the region $x^2 < -E_0$.

k) Type VI solutions

$$G+\kappa = B = 0. \tag{5.29}$$

Here W degenerates to a constant; and in consequence of (5.16):

$$W = E = \text{const} < 0. \tag{5.30}$$

TABLE VI

Type VI solutions

$$H = |Mu_1+Nu_2|^{1/3}$$

$$u_1 = F\left(\frac{3}{8}, \frac{1}{2}, \frac{\kappa}{2E}(x^2)^2\right)$$

$$u_2 = x^2 F\left(\frac{7}{8}, \frac{3}{2}, \frac{\kappa}{2E}(x^2)^2\right)$$

$$\varrho = \frac{D}{E}\exp\left[-\frac{\kappa}{2E}(x^2)^2\right], \qquad D, E = \text{const} < 0$$

6. *The second family of solutions*

a) Reduction of the problem to one partial differential equation

The second family is defined by $\varrho = \text{const}$, $p = \text{const}$, $H = 1+p/(c^2\varrho)$ (p not necessarily different from 0). Substituting this in (3.12) we find immediately:

$$\Lambda = \frac{1}{2}\kappa\left(\varrho - \frac{p}{c^2}\right). \tag{6.1}$$

In consequence of (6.1) and (3.3), (3.8) yields:

$$G = -2\kappa. \tag{6.2}$$

(3.10) is fulfilled identically. Substituting (6.1), (6.2), (3.3) and (3.13) in (3.9) or (3.11) we obtain an equation which can be written in the following form:

$$(1/u)_{,22}+u_{,11}-(2\kappa\varrho/H)^{1/2} = 0, \tag{6.3}$$

where $k \overset{\text{def}}{=} (2\kappa\varrho H^3)^{-\frac{1}{4}}u$ and $h = (2\kappa\varrho H^3)^{-\frac{1}{4}}u^{-1}$. So the set of the field equations (3.8)–(3.12) reduces here to one equation (6.3). We prove in Appendix C that this guarantees the existence of such a set of coordinates in which $u = u(x^2)$, $u_{,1} = 0$ [2].

b) The group of admissible transformations

All the physical scalars as ϱ, p, H are constant. The function u is not a scalar. Therefore the argument from Chapter 5a would not work here. The full group of admissible transformations is given by (1.38)–(2.17) with $g_{\alpha\beta}(x^2)$ and is larger than (2.18), which constitute a subgroup of admissible transformations. We will see it in Chapter 8b (part two).

c) The explicit form of the solution

When $u = u(x^2)$, (6.3) is integrated at once to give:

$$u^{-1} = (\kappa\varrho/2H)^{1/2}(x^2)^2+Ax^2+B, \quad A, B = \text{const.} \tag{6.4}$$

If we make now the transformation (2.18) with $\gamma = -A\cdot[H/(2\kappa\varrho)]^{1/2}$ then A in (6.4) will vanish. So we can assume $A = 0$, but then (2.18) are admissible only with $\gamma = 0$ (still this is only a subgroup of admissible transformations).

It seems that we have obtained three "types" of metrics according as $B > 0$, $B = 0$, $B < 0$. We can write them as follows:

$$ds_\varepsilon^2 = H^{-2}\{(dx^0+x^2dx^1)^2-\tfrac{1}{2}[(x^2)^2+\varepsilon a^2](dx^1)^2\}+$$
$$-(\kappa\varrho H[(x^2)^2+\varepsilon a^2]\}^{-1}(dx^2)^2-2\kappa\varrho^{-1}H^3(dx^3)^2, \tag{6.5}$$

where $a = \text{const} \neq 0$, $\varepsilon = +1, 0, -1$. However all these "types" represent the same solution in three different coordinate systems. We prove it by direct verification.

Take (6.5) with $\varepsilon = -1$ and execute the transformation:

$$x^0 = 2t-\sqrt{2}\varphi,\ x^1 = (\sqrt{2}/a)\varphi,\ x^2 = a\,\text{ch}\,(2r),\ x^3 = z. \tag{6.6}$$

The result is:

$$ds_{-1}^2 = 4H^{-2}dt^2+8\sqrt{2}H^{-2}\,\text{sh}^2\,r dt d\varphi+4H^{-2}(\text{sh}^4\,r-\text{sh}^2\,r)d\varphi^2+$$
$$-4(\kappa\varrho H)^{-1}dr^2-2\kappa\varrho^{-1}H^3dz^2. \tag{6.7}$$

---

[2] This is the reason why we need not solve the equation (6.3). Nevertheless it may be interesting from the purely mathematical point of view. It does not seem difficult, but in spite of many attempts (not only mine) no method of searching for its general solution has been found.

Now take (6.5) with $\varepsilon = 0$ and execute the transformation:

$$x^0 = 2t - \sqrt{2}\varphi + (2\sqrt{2}/K)\operatorname{arc\,tg}\left[e^{-2r}(1-\cos K\varphi)/\sin K\varphi\right],$$

$$x^1 = \frac{\sqrt{2}}{K}\cdot\frac{\sin K\varphi\,\operatorname{sh} 2r}{\operatorname{ch} 2r + \cos K\varphi\,\operatorname{sh} 2r},$$

$$x^2 = \operatorname{ch} 2r + \cos K\varphi\,\operatorname{sh} 2r,$$

$$x^3 = z,\ K = (\kappa\varrho/H)^{1/2}. \tag{6.8}$$

After a tedious calculation it appears that ds_0^2 transformed this way is identical with ds_{-1}^2 given by (6.7).

The coincidence of ds_0^2 and ds_{+1}^2 may be shown in the same way, but in (6.6) (ch $2r$) should be substituted by (sh $2r$), whereas in (6.8) (sh $2r$) and (ch $2r$) should be interchanged, (sin $K\varphi$) should be substituted by (sh $K\varphi$) and (cos $K\varphi$) by (ch $K\varphi$).

So there is no loss in generality if we put $\varepsilon = 0$ in (6.5). Finally, we have obtained the following unique solution:

$$ds^2 = H^{-2}\left[(dx^0 + x^2dx^1)^2 - \tfrac{1}{2}(x^2)^2(dx^1)^2\right] - \left[\kappa\varrho H(x^2)^2\right]^{-1}(dx^2)^2 - 2\kappa\varrho^{-1}H^3(dx^3)^2,$$

$$H = 1 + p/(c^2\varrho), \quad \Lambda = \frac{1}{2}\kappa\left(\varrho - \frac{p}{c^2}\right), \quad \varrho, p = \text{const.} \tag{6.9}$$

Notice that when $p = c^2\varrho$ then $\Lambda = 0$. When $p = 0$, Λ is necessarily different from 0.

The metric (6.9) has been found for the first time by Raval and Vaidya [9]. It is a generalization to the case of constant but non-zero pressure of the Gödel solution [11]. If $p = 0$ then $H = 1$ and (6.9) is precisely the metric of Gödel in a different coordinate system. The transformation (6.8) is taken from Gödel's paper. The three metrics (6.5) were known to be identical to Ellis [19] and Wainwright [6].

Compare (6.9) for $p = 0$ and $p \neq 0$. The difference between those cases consists in a different interpretation of constants appearing in the Ricci tensor. We can assume $p = 0$ and interpret the coefficient of $(u_\alpha u_\beta)$ as ϱc^2 or we can assume $p \neq 0$ and interpret the same quantity as $(\varrho c^2 + p)$. Thus these cases are different for qualitative physical reasons, but geometrically they do not differ, as a simple coordinate transformation yields $H = 1$ in the metric of (6.9) (with $\varrho' \overset{\text{def}}{=} \varrho H$). Notice finally that (6.9) is a limiting case of the type IV solution from the 1-st family. If we substitute $a = 2$ and $M = 0$ in Table IV then we obtain (6.9) with $H = N^{1/3}$ and $\varrho = -(D/\kappa)N^{5/3}$.

7. The third family of solutions

This family contains dust. We notice in the conclusion of Appendix A that here also a transformation like (1.36)–(1.38), (2.17) may be carried out, which yields $\varrho = \varrho(x^2)$. This property is preserved by the transformations (2.18).

If $\varrho = \varrho(x^2)$ and $H = 1$ then (3.10) and (3.13) show immediately that:

$$h_{,1} = k_{,1} = 0. \tag{7.1}$$

So again the field equations reduce to ordinary differential equations.

Using (3.3) with $H = 1$ in (3.8) we obtain:

$$-\tfrac{1}{2}(G+\kappa)\varrho = \Lambda. \qquad (7.2)$$

If $\Lambda \neq 0$ then $G \neq -\kappa$ and $\varrho =$ const which is the case of the second family. Therefore (7.2) implies:

$$G = -\kappa,\ \Lambda = 0. \qquad (7.3)$$

Now all the other field equations (3.9)–(3.12) are easily integrated with the help of (3.3), (3.13), (7.1) and (7.3). The result is:

$$ds^2 = (dx^0+x^2dx^1)^2-\left(\frac{x^2}{b'}+d\right)(dx^1)^2-\left[\kappa a'\left(\frac{x^2}{b'}+d\right)\right]^{-1} e^{-b'x^2}(dx^2)^2 - $$

$$-\frac{\kappa}{a'}e^{-b'x^2}(dx^3)^2, \qquad (7.4)$$

$$\varrho = a'e^{b'x^2}, \qquad (7.5)$$

with a', b', $d =$ const, $a' > 0 \neq b'$. If we use the transformation (2.18) with $\alpha = -b'$, $\gamma = -b'd$ and denote $a \overset{\text{def}}{=} a' \exp(-b'^2 d)$ then in (7.4) and (7.5) we obtain

$$ds^2 = (dx^0)^2+2x^2dx^0dx^1+x^2(x^2+1)(dx^1)^2+(\kappa a x^2)^{-1}e^{x^2}(dx^2)^2-\kappa a^{-1}e^{x^2}(dx^3)^2,$$

$$\varrho = ae^{-x^2}, \quad a = \text{const} > 0, \quad \Lambda = 0. \qquad (7.6)$$

The metric has the proper signature in the region $x^2 < 0$.

This solution has been found by Lanczos [3], and then rediscovered by van Stockum [23] and Wright [24]. Lanczos and Wright have found also the generalization of (7.6) to the case $\Lambda \neq 0$, but this generalization did not appear among out metrics because it does not fulfil the second assumption (1.41). We prove it in part three.

Notice finally that (7.6) is a limiting case of the type V solutions from the first family. If we execute in Table V the transformation $x^0 = x^{0'}+E_0x^1$, $x^2 = x^{2'}-E_0$, $x^1 = x^{1'}$, $x^3 = x^{3'}$ and then substitute $M = \frac{3}{2}$, $N = 1$ we obtain (7.6) as a limit $E_0 \to 1$, with $D = -\kappa a e^{-1}$.

APPENDIX A

The proof that (4.4) makes sense

From (3.3) we find $h = -(G\varrho k H^3)^{-1}$. We substitute this and $H_{,i} = (dH/d\varrho)\varrho_{,i}$ in (3.9) and (3.11). Then we subtract (3.11) from (3.9) and write the result in the form:

$$G\varrho k^2H^3\varrho_{,11}+\varrho_{,22} = -(G\varrho k^2H^3\varrho_{,1}^2+\varrho_{,2}^2)\left(\frac{H'}{H}-\frac{1}{\varrho}\right)^{-1}\times$$

$$\times\left(\frac{1}{\varrho^2}-4\frac{H'}{H\varrho}+7\frac{H'^2}{H^2}+\frac{H''}{H}\right). \qquad (A1)$$

Notice that $H'/H-\varrho^{-1} \neq 0$, since in the contrary we have $H = \text{const}\cdot\varrho$ and from (3.10) we obtain $2(hk)^{-1/2}\varrho^{-2}\varrho_{,1}\,\varrho_{,2} = 0$ which, in the case of the family Ia, is a contradiction.

Now we substitute $h = -(G\varrho k H^3)^{-1}$ and $H_{,i} = H' \cdot \varrho_{,i}$ in (3.10). We write the result in the form

$$k^{-1}k_{,1}\varrho_{,2} - k^{-1}k_{,2}\varrho_{,1} - 2\varrho_{,12} =$$

$$= 2\varrho_{,1}\varrho_{,2}\left(\frac{H'}{H} - \frac{1}{\varrho}\right)^{-1}\left(\frac{1}{\varrho^2} - 4\frac{H'}{H\varrho} + 7\frac{H'^2}{H^2} + \frac{H''}{H}\right). \tag{A2}$$

The left-hand sides of (A1) and (A2) are both present inside the square bracket in (4.4). Let us substitute then (A1) and (A2) in (4.4). We obtain the following equation:

$$\frac{v''}{v'} - \left(\frac{H'}{H} - \frac{1}{\varrho}\right)^{-1}\left(\frac{1}{\varrho^2} - 4\frac{H'}{H\varrho} + 7\frac{H'^2}{H^2} + \frac{H''}{H}\right) = 0. \tag{A3}$$

This is a well-defined ordinary differential equation for the function $v(\varrho)$.

The above argument is true when $H = 1$, *i.e.* in the case of the family III. Then (4.1)–(4.2) are also consistent with (1.37), (1.38), (2.17), and (A3) reduces to

$$\frac{v''}{v'} + \frac{1}{\varrho} = 0. \tag{A4}$$

We make use of this result in Chapter 7.

APPENDIX B

Consistence of the field equations

Substitute (5.4), (5.6) and (5.8) to any of the equations (3.8), (3.9), (3.11) or (3.12). The result is:

$$D\frac{H^4}{W}\left[\exp\left(\int\frac{Gx^2}{W}dx^2\right)\right]\left(-\frac{1}{2}GH^2 + W_{,2}HH_{,2} - WH_{,2}^2 + WHH_{,22}\right) =$$

$$= \frac{1}{2}\kappa D\frac{H^6}{W}\exp\left(\int\frac{Gx^2}{W}dx^2\right) + \kappa\frac{p}{c^2} + \Lambda. \tag{B1}$$

Now find $H_{,22}$ from (5.9) and substitute it in (B1). Then (B1) contains at most the first derivatives of H. Differentiate that expression with respect to x^2 and use the fact that $p_{,2} = c^2\varrho H_{,2}$. If we substitute again $H_{,22}$ from (5.9) in the resulting equation, we obtain the identity. This means that (5.13) and (B1) define the same function p, exact to an additive constant which may be compensated by p_0.

APPENDIX C

Specialization of coordinates in the family II

We have shown in (6.1)–(6.3) that the metric has the form:

$$ds^2 = H^{-2}(dx^0 + x^2dx^1)^2 - h(dx^1)^2 - k(dx^2)^2 - 2\kappa\varrho^{-1}H^3(dx^3)^2 \tag{C1}$$

with h and k given after (6.3). The group of admissible transformations is given by (1.36)–

–(1.38) with $T = \text{const}$ and (2.17). After such a transformation $1/u$ changes to $1/\tilde{u}$. We want $1/\tilde{u}$ to have the property that:

$$\frac{1}{\tilde{u}} \overset{\text{def}}{=} \frac{1}{u} F^2_{,1'} + uG^2_{,1'} = v(x^{2'}), \qquad v_{,1'} = 0. \tag{C2}$$

The equations (1.38) and (2.17) concern the transformation $x^\alpha \to x^{\alpha'}$. Similarly as in Chapter 4 it will be more convenient to consider the inverse transformation:

$$x^{0'} = x^0 - s(x^1, x^2),\ x^{1'} = w(x^1, x^2),\ x^{2'} = t(x^1, x^2),\ x^{3'} = x^3. \tag{C3}$$

The same argument as that used after (4.1) leads to the following relations:

$$F_{,1'} = t_{,2}, F_{,2'} = -w_{,2}\ G_{,1'} = -t_{,1}, G_{,2'} = w_{,1}. \tag{C4}$$

Thus the set of equations (1.38), (2.17) and (C3) is equivalent to:

$$\frac{1}{u} t_{,2}w_{,2} + ut_{,1}w_{,1} = 0, \tag{C5}$$

$$t_{,2}\, w_{,1} - t_{,1}\, w_{,2} = 1, \tag{C6}$$

$$\frac{1}{u} t^2_{,2} + ut^2_{,1} = v(t). \tag{C7}$$

If this set has solutions, there exists a function $v(t)$, *i. e.* there exists such an admissible transformation after which $1/\tilde{u} = 1/\tilde{u}(x^{2'})$, $u_{,1'} = 0$.

All we demand from $v(t)$ is for it to be a function of one variable. Therefore an arbitrary constant may be added to v, and so (C7) is equivalent to two equations obtained by differentiation of (C7) with respect to x^1 and x^2 respectively:

$$\left(t^2_{,1} - \frac{t^2_{,2}}{u^2}\right) u_{,1} + 2\frac{t_{,2}t_{,12}}{u} + 2ut_{,1}t_{,11} - v'(t)t_{,1} = 0,$$

$$\left(t^2_{,1} - \frac{t^2_{,2}}{u^2}\right) u_{,2} + 2\frac{t_{,2}t_{,22}}{u} + 2ut_{,1}t_{,12} - v'(t)t_{,2} = 0. \tag{C8}$$

Now, the set (C5)–(C6) may be solved algebraically for $w_{,1}$ and $w_{,2}$. If we substitute the result in the integrability condition $w_{,12} - w_{,21} = 0$ we obtain:

$$(u^2t^2_{,1} + t^2_{,2})^{-2}[(t^2_{,2} - u^2t^2_{,1})(t_{,22} - u^2t_{,11}) +$$
$$+ 2ut_{,1}t_{,2}(u_{,2}t_{,1} - u_{,1}t_{,2} + 2ut_{,12})] = 0. \tag{C9}$$

The set (C8)–(C9) can be solved algebraically for $t_{,11}$, $t_{,12}$ and $t_{,22}$. The conditions of integrability of the obtained set of equations are $t_{,112} - t_{,121} = 0$ and $t_{,122} - t_{,221} = 0$. Both reduce to:

$$(1/u)_{,22} + u_{,11} = v''(t). \tag{C10}$$

However, in virtue of (6.3) this is a well defined equation for v: $v''(t) = (2\kappa\varrho/H)^{1/2} =$ $=$ const. Thus we have shown that $v(t)$ with the property (C7) exists, and this completes the proof.

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