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# MODIFICATION OF THE QUANTUM-MECHANICAL EQUATIONS FOR THE SYSTEM OF CHARGED DIRAC PARTICLES BY INCLUDING ADDITIONAL TENSOR TERMS OF THE PAULI TYPE. PART I

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A new modified quasirelativistic equation (different from that of Breit) for $N$ charged Dirac particles in the external stationary electromagnetic field is proposed. This equation is an amplified quantum-mechanical Bethe-Salpeter equation obtained by adding (in a semi-phenomenological manner) terms which take into account radiative corrections. The application of this approximate equations is limited to third order terms in the fine structure constant $\alpha$.

It is well known that the most general relativistic description of a system of interacting charged Dirac particles is given by quantum-electrodynamics. However, its practical application is very tedious and leads to many, well-known, difficulties (connected with the mass and charge renormalization procedure). In theoretical spectroscopy of atomic systems the quantum-mechanical description based on the quasirelativistic equation (approximately to second order $\alpha = e^2/\hbar c$ or, simply, $c^{-1}$) is more simple. For a long time, the Breit equation [1], [2] (deduced after some approximations from quantum-electrodynamics) was treated as the only possible form of the quasirelativistic two-fermion equation. In this role it has been widely used, despite of known difficulties. Somewhat different equations have also been proposed (see, *e.g.*, Brown and Ravenhall [3], Barker and Glover [4], Hanus and Janyszek [5]–[8]). All these equations are based on some simplifications and additional assumptions. Further modification of such equations allows one to introduce the radiative corrections in a relatively simple, phenomenological way. This procedure has been known for the case of one charged Dirac particle in an external electromagnetic field. Pauli [9] was the first (see, Bethe and Salpeter [10], p. 136) who introduced into the ordinary Dirac equation additional interaction terms of tensor type (which are relativistic and gauge invariant). This equation gives a good account of radiative corrections,

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especially of the anomalous electron magnetic moment, if coupling constants are taken according to the results of quantum electrodynamics (for nucleons only from experimental data). It is known that the radiative corrections are of the order $\alpha^3$. This fact suggests that in the extended quasirelativistic equation also other terms of the same order should be taken into account. For one particle this problem was recently investigated by Hanus and Mrugała [11], [12].

The idea of Pauli was generalized to the two-particle problem (see Breit and Meyerott [13]). Chraplyvy [14] has extended these considerations including into the Breit equation tensor terms giving an account of the interaction of additional magnetic and "electric" moments with the external field and the field produced by other particles. Barker and Glover [4] have introduced these tensor terms into the quantum-mechanical Bethe-Salpeter equation. Considerations presented in papers [4], [13] and [14] are, in fact, restricted to terms of the order $c^{-2}$ and do not take into account all possible terms proportional to $c^{-3}$. Further investigations of the extended Breit equations including terms of the third order have been recently carried by Hegstrom [15], [16]. In these papers Hegstrom has calculated the Zeeman-spliting for the hydrogen atom (treated as a two-particle case) and his result is very good agreement with experimental data. However, he considers only those terms, which have immediate influence on the Zeeman-spliting.

Although the Breit equation has wide applications, it is not free of some deficiences such as the neglection of the hole theory postulate in its derivation (see, Bethe [17]). The problem of its reduction to the positive energy states leads also to many difficulties and controversions. This problem was discussed in detail in the quoted papers [5]-[8]. Because of the above-mentioned difficulties we proposed another quasirelativistic equation, different from the Breit equation in the term describing the interaction between the particles; it is a generalized quantum-mechanical Bethe-Salpeter equation in the presence of an external electromagnetic field for a system of $N$ Dirac particles. The reduction of this equation leads to a correct effective Hamiltonian in a simple and unambiguous way without any additional assumptions.

Regarding this fact, it seems to be justified to make a further generalization of this equation by including additional tensor terms (of the Pauli type), containing radiative corrections. As the starting point of our considerations, we take the generalized Bethe--Salpeter equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi, \quad H = \sum_{K} H_K + \frac{1}{2}\sum_{K,L}{}' W_{K,L}, \quad (K, L = I, II, \ldots, N), \tag{1}$$

where

$$H_K = \varrho_{3,K} m_K c^2 + c\varrho_{1,K}\sigma_K\left(p_K - \frac{e_K}{c}A_K^{\mathrm{ex}}\right) + e_K\Phi_K^{\mathrm{ex}}, \tag{2}$$

$$W_{K,L} = \tfrac{1}{4}\left[\lambda_K + \lambda_L, V_{K,L}\right]_+. \tag{3}$$

$\lambda_K$ stands for the sign kinetic energy operator depending on external magnetic field (it has eigenvalues $\pm 1$).

$$\lambda_K = T_K \cdot [(T_K)^2]^{-1/2}, \qquad T_K = \varrho_{3,K} m_K c^2 + c\varrho_{1,K}\boldsymbol{\sigma}_K\left(\boldsymbol{p}_K - \frac{e_K}{c} A_K^{\text{ex}}\right), \tag{4}$$

$V_{K,L}$ denoting the Breit interaction operator for particles $K, L$:

$$V_{K,L} = \frac{e_K e_L}{r_{K,L}} - \frac{1}{2}\varrho_{1,K}\varrho_{1,L} e_K e_L J_{K,L}, \tag{5}$$

$$J_{K,L} = \frac{\boldsymbol{\sigma}_K \boldsymbol{\sigma}_L}{r_{K,L}} + \frac{(\boldsymbol{\sigma}_K \boldsymbol{r}_{K,L})(\boldsymbol{\sigma}_L \boldsymbol{r}_{K,L})}{r_{K,L}^3}, \tag{6}$$

$$\boldsymbol{r}_{K,L} = \boldsymbol{r}_K - \boldsymbol{r}_L, \qquad r_{K,L} = |\boldsymbol{r}_{K,L}|. \tag{7}$$

The symbols used are the same as in [5]–[8]. The choice of the eigenvalues $\lambda_K$ and $\lambda_L$ with opposite signs gives rise to vanishing mutual interactions of particles (in accordance with the assumptions of the hole theory).

We shall now discuss a possibility of modifying this equation. We start from the extended Dirac equation ([11], [12])

$$i\hbar\frac{\partial\Psi}{\partial t} = H_D'\Psi, \qquad H_D' = H_D - \varrho_3 g^{(1)}\frac{e\hbar}{2mc}\boldsymbol{\sigma}\boldsymbol{H} - \varrho_2 g^{(1)}\frac{e\hbar}{2mc}\boldsymbol{\sigma}\boldsymbol{E} +$$

$$- g^{(2)}\frac{4\pi e\hbar^2}{m^2c^2}\left(\varrho - \frac{1}{c}\varrho_1\boldsymbol{\sigma}\boldsymbol{j}\right), \tag{8}$$

where

$$H_D = \varrho_3 mc^2 + c\varrho_1\boldsymbol{\sigma}\left(\boldsymbol{p} - \frac{e}{c}A\right) + e\Phi, \tag{9}$$

$$\boldsymbol{H} = \operatorname{rot} A, \boldsymbol{E} = -\operatorname{grad}\Phi, \tag{10}$$

$$\Delta A = -\frac{4\pi}{c}\boldsymbol{j}, \qquad \Delta\Phi = -4\pi\varrho \tag{11}$$

$\varrho$ and $\boldsymbol{j}$ are charge and current densities related to the source of external field. The dimensionless constants $g^{(1)}$ and $g^{(2)}$ are of the order $c^{-1}$ for the electron (according to the results of quantum-electrodynamics) and of the order of unity for nucleons. The first and second additional terms in (8) are of the same type as the interaction of additional magnetic and "electric" moment with external field, the last one being an additional contribution to the scalar and vector potentials. In this way we obtain simple generalized one particle Hamiltonian

$$H_K' = H_K - \varrho_{3,K} g_K^{(1)}\frac{e_K\hbar}{2m_K c}\boldsymbol{\sigma}_K\boldsymbol{H}_K^{\text{ex}} - \varrho_{2,K} g_K^{(1)}\frac{e_K\hbar}{2m_K c}\boldsymbol{\sigma}_K\boldsymbol{E}_K^{\text{ex}} +$$

$$- g_K^{(2)}\frac{4\pi e_K\hbar^2}{m_K^2c^2}\left(\varrho_K^{\text{ex}} - \frac{1}{c}\varrho_{1,K}\boldsymbol{\sigma}_K\boldsymbol{j}_K^{\text{ex}}\right), \tag{12}$$

where

$$H_K^{\text{ex}} = \text{rot}_K A_K^{\text{ex}}, \quad E = -\text{grad}_K \Phi_K^{\text{ex}}, \tag{13}$$

$$\Delta_K A_K^{\text{ex}} = -\frac{4\pi}{c} j_K^{\text{ex}}, \quad \Delta_K \Phi_K^{\text{ex}} = -4\pi\varrho_K^{\text{ex}}; \tag{14}$$

$\varrho_K^{\text{ex}}$, $j_K^{\text{ex}}$ are charge and current density of the external field in the place of the $K$-th particle.

Modification of the mutual interaction operator $W_{K,L}$ requires a justification of its form also in the third order. This term contains the Breit operator $V_{K,L}$, which has been obtained from quantum electrodynamics with accuracy up to $c^{-2}$. Further investigation along this way of possible higher order contributions to $V_{K,L}$ is impossible since the effective potential energy operator does not exist (see, Akhiezer and Berestetski [18] p. 451). Nevertheless, the presented form of the operator $V_{K,L}$, being of order $c^{-3}$ is confirmed by classical correspondence. The classical approximate expression for the interaction between particles (see, Landau and Lifshitz [19] p. 221, 270) is

$$V_{K,L}^{\text{cl}} = \frac{e_K e_L}{r_{K,L}} - \frac{e_K e_L}{2 r_{K,L} c^2}\left\{ v_K v_L + \frac{(v_K \cdot r_{K,L})(v_L \cdot r_{K,L})}{r_{K,L}^3} \right\}. \tag{15}$$

The approximate form of this expression is correct to the order $c^{-3}$, since in this order there are no contributions to the mutual interaction (it appears only in the fourth order in $c^{-1}$). It is useful to replace the velocities $v_K$ by operators $c\alpha_K = c\varrho_{1,K}\sigma_K$ in order to obtain the operator $V_{K,L}$, which may be treated as approximate to $c^{-3}$ (accordingly to the presented statements). This gives a possibility of obtaining a consistent description of radiative corrections. The operator $\frac{1}{2}W_{K,L}$ is amplified by including into $\frac{1}{2}V_{K,L}$ terms of analogical structure as those to $H_K$ and by taking next the anticommutator $\lambda_K + \lambda_L$ with this operator

$$W'_{K,L} = \frac{1}{4}\left[\frac{1}{2} V_{K,L} - \varrho_{3,K} g_K^{(1)} \frac{e_K \hbar}{2m_K c} \sigma_K H_{K,L} - \varrho_{2,K} g_K^{(1)} \frac{e_K \hbar}{2m_K c} \sigma_K E_{K,L} + \right.$$
$$\left. - g_K^{(2)} \frac{4\pi e_K \hbar^2}{m_K^2 c^2}\left(\varrho_{K,L} - \frac{1}{c}\varrho_{1,K}\sigma_K j_{K,L}\right), \ \lambda_K + \lambda_L\right]_+ , \tag{16}$$

(the interaction operator of the $K$-th and $L$-th particles is $W'_{K,L} + W'_{L,K}$) where

$$H_{K,L} = \text{rot}_K A_{K,L}, \quad E_{K,L} = -\text{grad}_K \Phi_{K,L}, \tag{17}$$

$$\Delta_K A_{K,L} = -\frac{4\pi}{c} j_{K,L}, \quad \Delta_K \Phi_{K,L} = -4\pi\varrho_{K,L}, \tag{18}$$

$A_{K,L}$ and $\Phi_{K,L}$ stands for the generalized potential operator for the case of particle interactions

$$A_{K,L} = \varrho_{1,L}\frac{e_L}{2}\left\{\frac{\sigma_L}{r_{K,L}} + \frac{(\sigma_L \cdot r_{K,L}) r_{K,L}}{r_{K,L}^3}\right\} + \frac{1}{2}\varrho_{3,L}\frac{e_L \hbar}{2m_L c} g_L^{(1)}\left(\sigma_L \times \frac{r_{K,L}}{r_{K,L}^3}\right), \tag{19}$$

$$\Phi_{K,L} = \frac{e_L}{r_{K,L}} + \frac{1}{2}\varrho_{2,L}\frac{e_L\hbar}{2m_Lc}g_L^{(1)}\left(\sigma_L \times \frac{r_{K,L}}{r_{K,L}^3}\right). \tag{20}$$

The form of the first term (19) is justified by the correspondence between classical particle interactions (see, (15)) expressed in the form

$$V_{K,L}^{\mathrm{cl}} = \frac{e_K e_L}{r_{K,L}} - \frac{e_K}{c} v_K \cdot A_{K,L}^{\mathrm{cl}}$$

and the interaction operator $V_{K,L}$ (a more extensive discussion may be found in [8]). The second terms (19), (20) are introduced in a phenomenological way as contributions to the potential from additional magnetic and "electric" moments of $L$-th particle $g_L^{(1)}e_L\hbar\sigma_L/2m_Lc$. Such an operator generalization was already used in the quoted paper [14]. $\varrho_{K,L}$ and $j_{K,L}$ are charge and current densities of the $L$-th particle in place of the $K$-th one.

The wave equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H'\Psi, \quad H' = \sum_K H'_K + {\sum_{K,L}}' W'_{K,L} \tag{21}$$

is gauge invariant. Hence, the obtained Hamiltonian contains all the terms which give an account of radiative corrections to the order $c^{-3}$. Beside the additional magnetic and "electric" moments it contains also terms with charge and current densities. For the electron these terms are of the order $c^{-3}$ and $c^{-4}$, the last may be omitted. For the proton they are of the order $c^{-2}$ and $c^{-3}$. Following the idea of the present paper we have assumed as the starting point the generalized Bethe-Salpeter equation (different from that of Breit) with the mutual interaction operator $\frac{1}{4}[V_{K,L}\lambda_K + \lambda_L]_+$. Including into this anticommutator additional tensor terms we have obtained a modified interaction operator (in accordance with the assumptions of the hole theory). In absence of external field and without restricting ourselves to two particles only, neglecting terms with charge and current densities, we obtain a modified quantum-mechanical Bethe-Salpeter equation introduced in the quoted papers of Barker and Glover [4]. Further justification of the correctness of the so-obtained equation (21) requires an investigation of its reduced form (to the subspace of the positive energy states) and, particularly, of the terms of the so-obtained effective Hamiltonian. This will be discussed in detail in Part II of this paper.

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