VACUUM POLARIZATION TENSOR AND CURRENT OPERATOR IN QUANTUM ELECTRODYNAMICS WITH COMPENSATING CURRENT

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The photon propagator in the compensating current dependent quantum electrodynamics, formulated previously, is investigated. In particular, it is shown that the vacuum polarization tensor can be defined in the same way as in conventional quantum electrodynamics. Since the theory is fully gauge-invariant, this tensor is a transverse quantity before renormalization and therefore there is no trouble with the quadratically divergent term in $\Pi^{\mu\nu}$. The problem of a proper definition of the current operator is also considered. It is shown that the definitions proposed by Heisenberg and Euler, Schwinger and Brandt can be generalized to the case of an arbitrary compensating current, described previously. The definition of the current gives a certain prescription of the regularization, which ensures transversity of the vacuum polarization tensor at every step of calculations. We also discuss the difference between this method of regularization and other methods. A perturbation expansion of the current operator is calculated up to the third order and it is shown that it is independent of the compensating current.

1. Introduction

In the previous paper [2] a general outline of the gauge invariant formulation of quantum electrodynamics has been given. We have shown that local charge conservation can be explicitly expressed by means of a certain kind of a *c*-number current a^{λ}, the so-called compensating current, which can be thought of as a model of the sources and the detectors of charged particles. Due to the presence of such a current all the propagators and field operators become gauge-invariant and renormalizable in the state space with positive definite norm squared.

Since the compensating current dependent quantum electrodynamics is a fully gauge invariant theory, it would be interesting to investigate properties of those objects, which are gauge independent also in the usual formulation. Some typical examples of such objects are the vacuum polarization tensor and the current operator. It is known [6] that the unrenormalized vacuum polarization tensor, when calculated in a most straightforward way, contains a quadratically divergent term which violates the transversity condition. This term occurs as one of the consequences of the violation of charge conservation law

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and it can be removed by means of, say, the Pauli-Villars regularization. Then, the regularized polarization tensor can be renormalized with the help of the charge renormalization constant.

In this paper we shall investigate properties of the current operator and the vacuum polarization tensor in quantum electrodynamics with compensating current. In Section 2 a general expression for the photon propagator is given together with the differential equation fulfilled by this object. Similarly as in the usual theory, the right-hand side of this equation contains a propagator with two arguments equal. A correct definition of such a quantity can be given in terms of the current operator. Its definition is a natural generalization of the expressions proposed by Heisenberg and Euler [1], Schwinger [3], and Brandt [4]. Namely, we replace the linear integral of the potential by an integral containing the compensating current a^λ, just obtaining the expression given already in [2]:

$$j^\mu(x) = \lim_{\xi\to 0} e[\bar{\psi}(x)e^{-ie\int d^4z[a^\lambda(z-x-\xi)-a^\lambda(z-x+\xi)]A_\lambda(z)}\psi(x-\xi)]. \tag{1.1}$$

In this section we also introduce the vacuum polarization tensor $\Pi^{\mu\lambda}$ which is defined in the same way as in usual quantum electrodynamics:

$$\mathscr{G}_{\mu\nu} = \Delta_{F\mu\nu}+\Delta_{F\mu\lambda}\Pi^{\lambda\sigma}\mathscr{G}_{\sigma\nu}, \tag{1.2}$$

where $\mathscr{G}_{\mu\lambda}$ is the photon propagator and $\Delta_{F\mu\nu}$ is the free photon propagator. We show that the correct definition of $\Pi^{\mu\nu}$ can be given in terms of the limiting procedure strictly connected with j^μ. In Section 3 we consider the problem of the quadratic divergence in the vacuum polarization tensor. It turns out that the quadratically divergent term either fulfills the transversity condition or does not appear in $\Pi^{\mu\nu}$. In particular, it can be shown that $\Pi^{\mu\nu}$ is free of the quadratic divergence for the current $\tilde{a}^\lambda$ equal to [7]:

$$\tilde{a}^\lambda(k) = i\frac{n^\lambda}{nk-i\varepsilon}, \quad n^2 = 1 \tag{1.3}$$

and for the compensating current describing the Coulomb gauge [2]. For other forms of the compensating current the transverse quadratically divergent term is present in the vacuum polarization tensor. In Section 4 we calculate the renormalization function which cancel the singularities of (1.1). Finally, in Section 5 we calculate the perturbation expansion of the current operator up to the third order.

2. *Vacuum polarization tensor and current operator*

The starting point of our considerations is the formula for the perturbation expansion of a general propagator obtained in [2]:

$$G_{\mu_1\ldots\mu_k}[x_1, \ldots, x_n, y_n, \ldots, y_1, z_1, \ldots, z_k|a\mathscr{A}] = \exp(-ie\textstyle\int a\mathscr{A})\times$$
$$\times\frac{\delta^k}{\delta J^{\mu_1}(z_1)\ldots\delta J^{\mu_k}(z_k)}\left\{(V[\mathscr{A}])^{-1}\exp\left(-\frac{i}{2}\int J\Delta_F J\right)\exp\left(-\int JD_F^{(a)}\frac{\delta}{\delta\mathscr{A}}\right)\times\right.$$
$$\left.\times\exp\left(-\frac{i}{2}\int\frac{\delta}{\delta\mathscr{A}}\mathscr{D}^F\frac{\delta}{\delta\mathscr{A}}\right)C[\mathscr{A}]K_F[x_1, \ldots, x_n, y_n, \ldots, y_1|a\mathscr{A}]\right\}\Bigg|_{J=0}. \tag{2.1}$$

The notation is explained in detail in [2]. We will only remind here that $D^{(a)}_{F\mu\nu}$ and $\mathscr{D}^F_{\mu\nu}$ are the free photon propagators corresponding to the external and the internal lines respectively. The photon propagator is defined as:

$$\mathscr{G}_{\mu\nu}[z_1, z_2|a\mathscr{A}] = iG^C_{\mu\nu}[z_1, z_2|a\mathscr{A}], \tag{2.2}$$

where the index C means that only the connected part of $G_{\mu\nu}$ must be taken into account. From (2.1) we obtain:

$$G_{\mu\nu}[z_1, z_2|a\mathscr{A}] = \exp\left(-ie\int a\mathscr{A}\right) i^2 \frac{\delta^2}{\delta J^{\mu_1}(z_1)\delta J^{\mu_2}(z_2)} \times$$
$$\times \left\{(V[\mathscr{A}])^{-1} \exp\left(-\frac{i}{2}\int J\Delta_F J\right)\exp\left(-\int J\Delta_F \frac{\delta}{\delta\mathscr{A}}\right) \times\right.$$
$$\left.\left.\times \exp\left(-\frac{i}{2}\int \frac{\delta}{\delta\mathscr{A}}\Delta_F \frac{\delta}{\delta\mathscr{A}}\right) C[\mathscr{A}]\right\}\right|_{J=0}. \tag{2.3}$$

Replacement of $D^{(a)}_F$ and $\mathscr{D}^F$ by the Proca propagator $\Delta_{F\mu\nu}$ is justified owing to the gauge invariance of $C[\mathscr{A}]$. Therefore, the compensating current dependent gradient term will not appear in $D^{(a)}_F$ and $\mathscr{D}^F$ (*cf.* formulae 2.14 in [2]). Performing the indicated functional differentiation over J and using the following formula [8]:

$$\frac{\delta}{\delta\mathscr{A}_\mu(x)} C[\mathscr{A}] = e\,\mathrm{Tr}\,(\gamma^\mu K_F[x,x|\mathscr{A}])C[\mathscr{A}] \tag{2.4}$$

we obtain:

$$\mathscr{G}_{\mu\nu}[z_1, z_2|a\mathscr{A}] = \Delta_{F\mu\nu}(z_1-z_2)+e\int d^4w\Delta_{F\mu\lambda}(z_1-w)\,\mathrm{Tr}\,(\gamma^\lambda G_\nu[w, w, z_2|a\mathscr{A}])^C. \tag{2.5}$$

This equation can be easily converted into the usual differential equation obeyed by the photon propagator:

$$[-g^{\lambda\mu}(\Box_{z_1}+\mu^2)+\partial^\lambda_{z_1}\partial^\mu_{z_1}]\mathscr{G}_{\mu\nu}[z_1, z_2|a\mathscr{A}] =$$
$$= \delta^\lambda_\nu\delta^{(4)}(z_1-z_2)+e\,\mathrm{Tr}\,(\gamma^\lambda G_\nu[z_1, z_1, z_2|a\mathscr{A}]). \tag{2.6}$$

Since we shall investigate further the photon propagator in the absence of an external electromagnetic field, the propagators in (2.6) must be written in a form independent on $\mathscr{A}_\mu$. In this case $G_{\mu\nu}[z_1, z_2|a]$ contains no disconnected part. The right hand side of (2.6) is expressed by the formal expression of the following form:

$$\mathrm{Tr}\,(\gamma^\lambda G'_\nu[z_1, z_1, z_2|a\mathscr{A}]) = i\,\mathrm{Tr}\,(\gamma^\lambda\langle 0|T(\psi(z_1)\bar\psi(z_1)A_\nu(z_2))\,|0\rangle =$$
$$= -i\langle 0|T(\bar\psi(z_1)\gamma^\lambda\psi(z_1)A_\nu(z_2))\,|0\rangle. \tag{2.7}$$

Since this propagator, as it is written in (2.7), has no mathematical meaning (it contains the product of two operator-valued distributions in the same space-time point) we must be careful about its proper definition. It is obvious that the right-hand side of (2.6) is closely connected to the current operator. The problem of the proper definition of this observable has been widely considered in literature [1, 3, 4, 9] and therefore we adopt these definitions. Namely, we will use a generalization of the expressions given by Schwinger [3] and Brandt [4], writing:

$$j^{\mu}(x) = \lim_{\xi \to 0} j^{\mu}(x; \xi), \tag{2.8a}$$

$$\begin{aligned} j^{\mu}(x; \xi) = & -\frac{e}{2} \operatorname{Tr}\left[\gamma^{\mu} \mathscr{F}(x, x+\xi)+\gamma^{\mu} \mathscr{F}(x, x-\xi)\right]-C_1^{\mu}(\xi)- \\ & -C_2^{\mu\nu}(\xi) B_{\nu}(x)-C_3^{\mu\nu\lambda}(\xi) B_{\nu,\lambda}(x)- \\ & -C_4^{\mu\nu\lambda\sigma}(\xi) B_{\nu,\lambda\sigma}(x)-C_6(\xi) j^{\mu}(x), \end{aligned} \tag{2.8b}$$

$$\mathscr{F}(x, y) = T(\psi(x) \exp(-ie \textstyle\int d^4z(a^{\lambda}(z-x)-a^{\lambda}(z-y))A_{\lambda}(z))\bar{\psi}(y)). \tag{2.8c}$$

It can be easily seen that main difference in our expression as compared with expressions used up to now [3, 4] consists in using $\exp(-ie\int a\mathscr{A})$ instead of a linear integral of the potential. Singular renormalization functions $C_i^{\mu\ldots\nu}(\xi)$ cancel the singularities which occur in the first term in (2.8b) after putting $\xi = 0$. Finally, the B_ν-field is given as the compensating current dependent gauge transformation of the Proca field A_μ:

$$B_{\mu}(x) = A_{\mu}(x)-\int d^4z \frac{\partial}{\partial z^{\mu}} a^{\nu}(z-x) A_{\nu}(z). \tag{2.9}$$

Therefore, if only the functions C_i are chosen in a gauge-invariant manner, gauge independence of the current operator will follow from gauge independence of every term in (2.8) separately. Although in the usual theory the singularities of the current operator depend on the choice of gauge, we will show that due to presence of the compensating current our renormalization functions are gauge invariant. On the other hand, since $j^{\mu}(x)$ is an observable it should not depend on the compensating current, and we will show that this is actually the case, all the a^{λ}-dependence being contained in the singularities which are removed in the process of renormalization. It is clear from (2.8) that the right-hand side of the photon propagator equation (2.6) should also be defined in terms of the limiting procedure. We put therefore:

$$\begin{aligned} \operatorname{Tr}(\gamma^{\lambda} G_{\nu}[z_1, z_1, z_2|a]) = & \lim_{\xi \to 0} \{\tfrac{1}{2} \operatorname{Tr}(\gamma^{\lambda} G_{\nu}[z_1, z_1+\xi, z_2|a])+ \\ & +\gamma^{\lambda} G_{\nu}[z_1, z_1-\xi, z_2|a])+\ldots\}. \end{aligned} \tag{2.10}$$

The dots denote here the renormalization functions which must be taken into account in (2.10) and which are, of course, closely connected to C_i of (2.8).

The vacuum polarization tensor $\Pi^{\mu\lambda}$ is defined in our formulation of quantum electrodynamics in the same way as in the usual version of the theory:

$$\begin{aligned} \mathscr{G}_{\mu\nu}[z_1, z_2|a] = & \Delta_{F\mu\nu}(z_1-z_2)+\int d^4w \int d^4v \Delta_{F\mu\lambda}(z_1-w) \times \\ & \times \Pi^{\lambda\sigma}[w, v|a] \mathscr{G}_{\sigma\nu}(v, z_2|a]. \end{aligned} \tag{2.11}$$

To separate the $\Pi^{\mu\lambda}$ tensor from (2.5) we must express the propagator $G_\nu[w, v, z|a]$ in terms of the vertex function. It can be written in the same form as in usual quantum electrodynamics apart from the propagator corresponding to the effective external photon line. It follows from the formula (2.13) in [2] that the free propagator connected with an external photon line is given by:

$$D^{(a)}_{F\mu\nu}(z-w) = \int d^4z' \lambda^\sigma_\nu(z'-w) \Delta_{F\mu\sigma}(z-z'), \tag{2.12a}$$

$$\lambda_{\mu\nu}(z'-w) = g_{\mu\nu}\delta^{(4)}(z'-w) - \frac{\partial}{\partial z'^\mu} a_\nu(z'-w). \tag{2.12b}$$

We can therefore write:

$$G_\nu[w, v, z_2|a] = ie \int d^4x' d^4y' d^4v' \mathscr{G}^{(a)}_{\nu\varrho}[z_2, v'|a] G[w, x'|a] \times$$
$$\times \Gamma^\varrho[x', y', v'|a] G[y', v|a], \tag{2.13}$$

where:

$$\mathscr{G}^{(a)}_{\nu\varrho}[z_2, v'|a] = \int d^4z'' \lambda_{\varrho\kappa}(z''-v') \mathscr{G}^\kappa_\nu[z_2, z''|a]. \tag{2.14}$$

Perturbation expansion of the compensating current dependent vertex function $\Gamma^\varrho[a]$ can be obtained with the use of the same Feynman diagrams as in conventional electrodynamics. The only difference consists in the fact that expressions corresponding to the internal photon lines are given by (formula 2.14 of [2]):

$$\mathscr{D}^F_{\lambda\varrho}(w-w') = \int d^4z d^4z' \lambda_{\lambda\mu}(z-w) \lambda_{\varrho\nu}(z'-w') \Delta^{\mu\nu}_F(z-z'). \tag{2.15}$$

A vertex function defined in such a way obeys the generalized Ward identity in the usual form:

$$\frac{1}{i} \frac{\partial}{\partial z^\lambda} \Gamma^\lambda[x, y, z|a] = [\delta^{(4)}(x-z) - \delta^{(4)}(y-z)] G^{-1}[x, y|a]. \tag{2.16}$$

However, it is necessary to stress that equation (2.13) does not define the vertex function unambiguously, the ambiguity being connected with the projective nature of the $\lambda_{\mu\nu}$-operator. Using the divergence condition for the compensating current

$$k^\lambda \tilde{a}^\lambda(k) = i$$

it is easy to verify that $\lambda_{\mu\nu}$ is a projector:

$$\lambda^2 = \lambda$$

and therefore its inversion does not exist. Thus, the vertex function $\Gamma^\varrho[a]$ cannot be defined by simply cutting external photon lines in corresponding Feynman graphs, since this operation is equivalent to multiplication by the inverse of $\mathscr{G}^{(a)}_{\mu\nu}$. In spite of that it can be shown that it is possible to define the a^λ-dependent vertex function in an unambiguous way. This problem will be considered in a separate note. All we need here is the Ward identity (2.16).

We are now in a position to obtain an integral expression for the vacuum polarization tensor analogous to the one given in the paper of Brandt [4]. It follows from (2.10), (2.11), (2.13) and (2.14) that cutting the photon propagator $\mathcal{G}_{\mu\nu}$ in (2.5) we obtain:

$$\Pi^{\lambda\sigma}[w, v|a] = \lim_{\xi=0}\left\{\frac{ie^2}{2}\int d^4x'd^4y'd^4v' \operatorname{Tr}(\gamma^\lambda G[w, x'|a]\Gamma^\varrho[x', y', v'|a]\times\right.$$
$$\times G[y', w+\xi|a]+\gamma^\lambda G[w, x'|a]\Gamma^\varrho[x', y', v'|a]\times$$
$$\left.\times G[y', w-\xi|a])\lambda_\varrho^{\ \sigma}(v-v')+\ldots\right\}. \tag{2.17}$$

Since we have assumed that $\mathcal{A}_\mu = 0$, the vacuum polarization tensor is, in fact, a function of one argument equal to $w-v$. Treating now $x'-y'$ and $x'-v'$ in Γ^ϱ as independent variables we are led to the following expression for the $\Pi^{\lambda\sigma}$-tensor in momentum space:

$$\tilde{\Pi}^{\lambda\sigma}[k|a] = \lim_{\xi\to 0}\left\{\frac{ie^2}{2}\int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\times\right.$$
$$\left.\times\operatorname{Tr}(\gamma^\lambda\tilde{G}[q+k|a]\tilde{\Gamma}^\varrho[q, k|a]\tilde{G}[q|a])\tilde{\lambda}_\varrho^{\ \sigma}(-k)+\ldots\right\}. \tag{2.18}$$

Formulae (2.17) and (2.18) are basic for further considerations.

3. *Singularities of the unrenormalized polarization tensor*

In this Section we consider in detail the problem of the singularities which appear in the unrenormalized vacuum polarization tensor. It is well known that this tensor in the theory without a compensating current, when calculated in a most straightforward way, does not fulfill the transversity condition and can be written in the following form:

$$\Pi_{\mu\nu} = Ag_{\mu\nu}+(g_{\mu\nu}k^2-k_\mu k_\nu)B. \tag{3.1}$$

The gauge-dependent term proportional to A is quadratically divergent and we can see that the transversity condition $k^\mu\Pi_{\mu\nu} = 0$ does not make sense. This difficulty can be overcome by imposing the transversity condition only on the renormalized (*i.e.* physical) polarization tensor.

In the charge conserving theory with a compensating current we hope that the transversity condition is fulfilled from the very beginning, no matter by which method the $\Pi^{\mu\nu}$-tensor is calculated. To show this explicitly we must write (2.17) in the following form:

$$\Pi^{\lambda\sigma}[w-v|a] = \frac{ie^2}{2}\lim_{\xi\to 0}\int d^4x'd^4y' \operatorname{Tr}(\gamma^\lambda G[w-x'|a]\Gamma^\sigma[x'-y', x'-v|a]\times$$
$$\times G[y'-w-\xi|a]+\gamma^\lambda G[w-x'|a]\Gamma^\sigma[x'-y', x'-v|a]G[y'-w+\xi|a])+$$
$$+\frac{e^2}{2}\lim_{\xi\to 0}[a^\sigma(v-w+\xi)-a^\sigma(v-w-\xi)]\operatorname{Tr}(\gamma^\lambda G[\xi|a])+\ldots, \tag{3.2}$$

where we have used the following formula:

$$\mathrm{Tr}\,(\gamma^\lambda G[\xi|a]) = -\mathrm{Tr}\,(\gamma^\lambda G[-\xi|a]). \tag{3.3}$$

Using the generalized Ward identity and the divergence condition for the compensating current a^λ one can easily show that:

$$\frac{\partial}{\partial w^\sigma}\Pi^{\lambda\sigma}[w-v|a] = 0 \tag{3.4}$$

and therefore the transversity condition is fulfilled from the very beginning, before renormalization. The difference between the vacuum polarization tensor in our theory and in the usual one can be easily seen from (3.2). The a^λ-dependent tensor is constructed of two terms, first of which has the same form as in the conventional theory and the second one being linear in a^λ. It is this last term due to which (3.4) is fulfilled and it can be thought of as a remainder of the gauge dependence of the unrenormalized tensor in the absence of the compensating current. Since $\Pi^{\lambda\sigma}$ is transverse from the very beginning, it is interesting to verify whether it contains quadratic divergence, or not (by quadratic divergence we mean a singularity of a ξ^{-2}-type). It is known [7] that $\Pi^{\mu\lambda}$ is free of such a singularity if the current a^λ is created by a point particle *i.e.* when:

$$a^\sigma(v-w+\xi)-a^\sigma(v-w-\xi) = \int_{-\xi}^{+\xi} d\eta^\sigma\delta^{(4)}(\eta+v-w). \tag{3.5}$$

The Fourier transform of $\Pi^{\mu\nu}$ can then be written as [7]:

$$\tilde{\Pi}^{\lambda\sigma}[k|a] = \lim_{\xi\to 0}\frac{ie^2}{2}\int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\Big\{\mathrm{Tr}\,(\gamma^\lambda\tilde{G}[q+k|a]\tilde{\Gamma}^\sigma[q,k|a]\tilde{G}[q|a])-$$
$$-\frac{\partial}{\partial q_\sigma}\mathrm{Tr}\,(\gamma^\lambda\tilde{G}[q|a])-\frac{1}{6}\frac{\partial}{\partial q_\sigma}\left(k\cdot\frac{\partial}{\partial q}\right)^2\mathrm{Tr}\,(\gamma^\lambda\tilde{G}[q|a])\Big\}+\dots \tag{3.6}$$

The ξ^{-2}-singularity is cancelled by the second term in the large bracket in (3.6). To investigate the quadratic divergence for an arbitrary compensating current we will add to (3.2) and then subtract from it the linear current expression (3.5). We will therefore deal with the following tensor:

$$Q^{\lambda\sigma}(\xi) = \frac{e^2}{2}\{[a^\sigma(v-w+\xi)-a^\sigma(v-w-\xi)]-$$
$$-[a^\sigma_{(l)}(v-w+\xi)-a^\sigma_{(l)}(v-w-\xi)]\}\,\mathrm{Tr}\,(\gamma^\lambda G[\xi|a]), \tag{3.7}$$

where $a^\sigma_{(l)}$ is the point particle current:

$$a^\sigma_{(l)}(z) = \int_{-\infty}^{z} d\eta^\sigma\delta^{(4)}(\eta). \tag{3.8}$$

If $Q^{\lambda\sigma}(\xi)\to\xi^{-2}$ when $\xi\to 0$ then also the vacuum polarization tensor possesses a singularity of this type, and if $Q^{\lambda\sigma}$ is less singular (*e.g.* like $\ln\xi$) at $\xi\to 0$ then the $\Pi^{\lambda\mu}$ tensor is free of the quadratic divergence. Nevertheless, the quadratically divergent term, if present in $\Pi^{\lambda\sigma}$, fulfills the transversity condition. The tensor $Q^{\lambda\sigma}$ can be written in the following form:

$$Q^{\lambda\sigma}(\xi) = ie^2 \int \frac{d^4k}{(2\pi)^4} e^{-ik(w-v)}[a^\sigma(-k)-\tilde{a}^\sigma_{(l)}(-k)] \sin k\xi \,\mathrm{Tr}\,(\gamma^\lambda G[\xi|a]) =$$

$$= \frac{F^\sigma(w-v-\xi)-F^\sigma(w-v+\xi)}{2i} \mathrm{Tr}\,(\gamma^\lambda G[\xi|a]), \tag{3.9}$$

where

$$F^\sigma(x) = ie^2 \int \frac{d^4k}{(2\pi)^4} e^{-ikx}[\tilde{a}^\sigma(-k)-\tilde{a}^\sigma_{(l)}(-k)]. \tag{3.10}$$

Since $\mathrm{Tr}\,(\gamma^\lambda G[\xi|a]) \approx \xi^{-3}$ for $\xi \to 0$, the singularity of (3.9) is determined by the first factor in this formula. We shall now confine ourselves to the following class of compensating currents:

$$\tilde{a}^\lambda(k) = i\,\frac{n^\lambda(nk)-\varrho k^\lambda}{(nk)^2-\varrho k^2}, \qquad n^2 = 1, \qquad 0 < \varrho < 1. \tag{3.11}$$

Using:

$$a^\lambda_{(l)}(k) = i\,\frac{n^\lambda}{nk-i\varepsilon} \tag{3.12}$$

we obtain:

$$F^\sigma(x) = e^2\varrho \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{k^\sigma(nk)-n\sigma k^2}{(nk-i\varepsilon)\,[(nk)^2-\varrho k^2]}. \tag{3.13}$$

The four-momentum integration can be easily performed by contour integration in the k_0-plane. A prescription for omitting the singularities of the integrand is not needed, since for $0 < \varrho < 1$ poles appear only for purely imaginary k_0. We obtain:

$$F^\sigma(x) = i\,\frac{\sqrt{\varrho(1-\varrho)^3}}{2\pi^2}\,\frac{x^\sigma}{[(nx)^2-(1-\varrho)x^2]^2}. \tag{3.14}$$

It follows therefore that in general $Q^{\lambda\sigma}(\xi)$ and the vacuum polarization tensor behave like ξ^{-2} for $\xi \to 0$. However, for $\varrho = 0$ (point-particle current) and $\varrho = 1$ (Coulomb gauge current) $F^\sigma(x)$ is equal to zero and $\Pi^{\lambda\sigma}$ contains a singularity of logarithmic type only. But, in contrast to the usual theory, the vacuum polarization tensor always fulfills the transversity condition, no matter whether renormalized or not.

To end this section we add a few words about the problem of regularization in quantum electrodynamics with a compensating current. One can see from (2.17) and (2.18) that the limiting procedure needed for the definition of the current operator gives also a prescription for the regularization of the divergent expression for the vacuum polarization tensor. In contrast to the usual, say Pauli-Villars regularization method, this prescription has not been introduced to the theory "*ad hoc*", but it is a natural consequence of properly defined field equations. It is obvious that this method will work also for other important quantities in quantum electrodynamics like the self energy part or the vertex function.

4. *Renormalization of the vacuum polarization tensor and current operator*

To find the renormalization functions $C_i(\xi)$ introduced in (2.8) we will use the following identity:

$$e\,\mathrm{Tr}\,(\gamma^\lambda G_\nu[z_1, z_1, z_2|a]) = -i\langle 0|T(j^\lambda(z_1)A_\nu(z_2))\,|0\rangle. \tag{4.1}$$

Thus, the limiting procedure in (2.10) is closely connected to the limiting procedure encountered already in the current definition (2.8). Substituting this definition to (2.5) and then cutting the external photon propagator $\mathcal{G}_{\mu\nu}$ (formula 2.11), we get:

$$\begin{aligned}\Pi^{\lambda\sigma}[w-v|a] &= \lim_{\xi\to 0}\Big\{\frac{ie^2}{2}\int d^4x'd^4y'd^4v'\lambda_\varrho^\sigma(v-v')\times\\ &\times[\mathrm{Tr}\,(\gamma^\lambda G[w-x'|a]\Gamma^\varrho[x',y',v'|a]G[y'-w-\xi|a])+(\xi\to-\xi)]+\\ &+\Big[C_2^{\lambda\varrho}(\xi)-C_3^{\lambda\varrho\alpha}(\xi)\frac{\partial}{\partial v^\alpha}+C_4^{\lambda\varrho\alpha\beta}(\xi)\frac{\partial}{\partial v^\alpha}\frac{\partial}{\partial v^\beta}\Big]\lambda_\varrho^\sigma(v-w)-C_6(\xi)\Pi^{\lambda\sigma}[w-v|a]\Big\}.\end{aligned} \tag{4.2}$$

This expression can be written in a much more transparent form in momentum space:

$$\begin{aligned}\tilde\Pi^{\lambda\sigma}[k|a] &= \lim_{\xi\to 0}\Big\{\Big[\frac{ie^2}{2}\int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\,\mathrm{Tr}\,(\gamma^\lambda\tilde G[q+k|a]\times\\ &\times\tilde\Gamma^\varrho[q,k|a]\tilde G[q|a])+C_2^{\lambda\varrho}(\xi)-iC_3^{\lambda\varrho\alpha}(\xi)k_\alpha-\\ &-C_4^{\lambda\varrho\alpha\beta}(\xi)k_\alpha k_\beta\Big]\tilde\lambda_\varrho^\sigma(-k)-C_6(\xi)\tilde\Pi^{\lambda\sigma}[k|a]\Big\}.\end{aligned} \tag{4.3}$$

Similarly as in the usual formulation of quantum electrodynamics [4] the renormalization functions are given by the following requirements:

$$\begin{aligned}&\tilde\Pi^{\lambda\sigma}[0|a]=0,\qquad \frac{\partial}{\partial k_\mu}\tilde\Pi^{\lambda\sigma}[k|a]\Big|_{k=0}=0,\\ &\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k_\nu}\tilde\Pi^{\lambda\sigma}[k|a]\Big|_{k=0}=0.\end{aligned} \tag{4.4}$$

Thus:

$$\begin{aligned}C_2^{\lambda\varrho}(\xi) &= -\frac{ie^2}{2}\int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\,\mathrm{Tr}\,(\gamma^\lambda\tilde G[q|a]\tilde\Gamma^\varrho[q,0|a]\tilde G[q|a]) =\\ &= e^2\xi^\varrho\,\mathrm{Tr}\,(\gamma^\lambda G[\xi|a]),\end{aligned} \tag{4.5a}$$

$$C_3^{\lambda\varrho\mu}(\xi) = \frac{e^2}{2}\int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\frac{\partial}{\partial k_\mu}\mathrm{Tr}\,(\gamma^\lambda\tilde G[q+k|a]\tilde\Gamma^\varrho[q,k|a]\tilde G[q|a])\Big|_{k=0}, \tag{4.5b}$$

$$C_4^{\lambda\varrho\mu\nu}(\xi) = \frac{ie^2}{2}\int \frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k_\nu}\mathrm{Tr}\,(\gamma^\lambda\tilde{G}[q+k|a]\tilde{\Gamma}^\varrho[q,k|a]\tilde{G}[q|a])\Big|_{k=0}. \tag{4.5c}$$

The explicit form of the $C_2^{\lambda\varrho}(\xi)$ function follows from the ordinary Ward identity:

$$\tilde{G}[q|a]\tilde{\Gamma}^\mu[q,0|a]\tilde{G}[q|a] = \frac{\partial\tilde{G}[q|a]}{\partial q_\mu}. \tag{4.6}$$

The function $C_1^\mu(\xi)$ has been introduced to the definition of the operator just to guarantee the vanishing of the current vacuum expectation value. Therefore:

$$C_1^\mu(\xi) = -ie\,\mathrm{Tr}\,(\gamma^\mu G[\xi|a]). \tag{4.7}$$

The only function that has not yet been determined is $C_6(\xi)$. However, there is no way of calculating this quantity on the basis of requirements imposed on the vacuum polarization tensor only. As it has been shown in [4] this function can be determined with the help of the renormalization condition imposed upon the vertex function and the same method can, of course, be applied in the theory with compensating current. This problem will be investigated in a separate note.

To simplify expressions (4.2) and (4.3) for the renormalized vacuum polarization tensor we use formula (3.2) for its unrenormalized part and then separate the $Q^{\lambda\sigma}$-tensor (3.7). Writing the difference between compensating current a^λ and the point-particle current $a^\lambda_{(l)}$ as:

$$a^\sigma(v-w+\xi)-a^\sigma(v-w-\xi)-[a^\sigma_{(l)}(v-w+\xi)-a^\sigma_{(l)}(v-w-\xi)] =$$

$$= \int_{-\xi}^{+\xi} d\eta^\varrho\left[\frac{\partial}{\partial\eta^\varrho}a^\sigma(\eta+v-w)-g^\sigma_\varrho\delta^{(4)}(\eta+v-w)\right] = -\int_{-\xi}^{+\xi} d\eta^\varrho\tilde{\lambda}_\varrho^{\ \sigma}(\eta+v-w) =$$

$$= -\int\frac{d^4k}{(2\pi)^4}e^{-ik(w-v)}\xi^\varrho\frac{\sin k\xi}{k\xi}\tilde{\lambda}^\sigma_\varrho(-k) \simeq -\int\frac{d^4k}{(2\pi)^4}e^{-ik(w-v)}\xi^\varrho\left[1-\frac{(k\xi)^2}{6}\right]\tilde{\lambda}_\varrho^{\ \sigma}(-k), \tag{4.8}$$

we obtain for the $\Pi^{\lambda\sigma}$-tensor:

$$\tilde{\Pi}^{\lambda\sigma}[k|a] = \frac{ie^2}{2}\int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\Bigg[\mathrm{Tr}\,(\gamma^\lambda\tilde{G}[q+k|a]\tilde{\Gamma}^\sigma[q,k|a]\tilde{G}[q|a])-$$

$$-\frac{\partial}{\partial q_\sigma}\mathrm{Tr}\,(\gamma^\lambda\tilde{G}[q|a])-\frac{1}{6}\frac{\partial}{\partial q_\sigma}\left(k\cdot\frac{\partial}{\partial q}\right)^2\mathrm{Tr}\,(\gamma^\lambda\tilde{G}[q|a])\Bigg]+$$

$$+\Bigg[-e^2\left(1-\frac{(k\xi)^2}{6}\right)\xi^\varrho\,\mathrm{Tr}\,(\gamma^\lambda G[\xi|a])+C_2^{\lambda\varrho}(\xi)-iC_3^{\lambda\varrho\alpha}(\xi)k_\alpha-$$

$$-C_4^{\lambda\varrho\alpha\beta}(\xi)k_\alpha k_\beta\Bigg]\tilde{\lambda}_\varrho^{\ \sigma}(-k)-C_6(\xi)\tilde{\Pi}^{\lambda\sigma}[k|a]. \tag{4.9}$$

The first term in this expression can be easily recognized as (3.6), which is equal to the vacuum polarization tensor with the compensating current equal to the point-particle current $a^{\lambda}_{(l)}$. Using now the explicit form of the C_2-function (4.5a) we obtain:

$$\tilde{\Pi}^{\lambda\sigma}[k|a] = \lim_{\xi\to 0}\left\{\frac{ie^2}{2}\int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\left[\mathrm{Tr}\,(\gamma^{\lambda}\tilde{G}[q+k|a]\times\right.\right.$$

$$\left.\times\tilde{\Gamma}^{\sigma}[q,k|a]\tilde{G}[q|a])-\frac{\partial}{\partial q_{\sigma}}\mathrm{Tr}\,(\gamma^{\lambda}\tilde{G}[q|a]-\frac{1}{6}\frac{\partial}{\partial q_{\sigma}}\left(k\cdot\frac{\partial}{\partial q}\right)^2\mathrm{Tr}\,(\gamma^{\lambda}\tilde{G}[q|a])\right]+$$

$$+[-iC_3^{\lambda\varrho\alpha}(\xi)k_{\alpha}-(C_4^{\lambda\varrho\alpha\beta}(\xi)-\tfrac{1}{6}e^2\xi^{\alpha}\xi^{\beta}\xi^{\varrho}\,\mathrm{Tr}\,(\gamma^{\lambda}G[\xi|a]))k_{\alpha}k_{\beta}]\times$$

$$\left.\times\tilde{\lambda}_{\varrho}{}^{\sigma}(-k)-C_6(\xi)\tilde{\Pi}^{\lambda\sigma}[k|a]\right\}. \tag{4.10}$$

It has been shown by Brandt [4] that the $C_4^{\lambda\varrho\alpha\beta}$-function can be written as a sum of two terms, one of which is symmetric in ϱ, α and ϱ, β and the second one is antisymmetric with respect to the permutation of indices in each of those pairs. The symmetric part is equal to $(e^2/6)\xi^{\varrho}\xi^{\alpha}\xi^{\beta}\,\mathrm{Tr}\,(\gamma^{\lambda}G[\xi|a])$ and does not fulfill the transversity condition when multiplied by $k_{\alpha}k_{\beta}$. It follows from (4.10) that our expression for the vacuum polarization tensor contains only the skew symmetric part of this function, which is connected with the charge renormalization. Similarly as C_4, the $C_3^{\lambda\varrho\alpha}$ function can be split into parts that are symmetric in ϱ, α and antisymmetric in these indices. The symmetric part has the following form:

$$C_3^{\lambda(\varrho\alpha)}(\xi) \sim \int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\frac{\partial}{\partial q_{\varrho}}\frac{\partial}{\partial q_{\alpha}}\mathrm{Tr}\,(\gamma^{\lambda}\tilde{G}[q|a])$$

and is equal to zero due to the antisymmetric character of the integrand as a function of q. Further:

$$C_3^{\lambda[\varrho\alpha]}(\xi)k_{\alpha}\tilde{\lambda}_{\varrho}{}^{\sigma}(-k) = C_3^{\lambda[\sigma\alpha]}k_{\alpha}.$$

The transversity condition $k^{\sigma}\tilde{\Pi}_{\lambda\sigma}[k|a] = 0$ fulfilled by the renormalized tensor can now be easily verified.

The vacuum polarization tensor can be finally written in the following simple and transparent form:

$$\tilde{\Pi}^{\lambda\sigma}[k|a] = \lim_{\xi\to 0}\left\{\frac{ie^2}{2}\int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\times\right.$$

$$\times\mathrm{Tr}\,(\gamma^{\lambda}\tilde{G}[q+k|a]\tilde{\Gamma}^{\sigma}[q,k|a]\tilde{G}[q|a])+C_2^{\lambda\sigma}(\xi)-iC_3^{\lambda\sigma\alpha}(\xi)k_{\alpha}-$$

$$\left.-C_4^{\lambda\sigma\alpha\beta}(\xi)k_{\alpha}k_{\beta}-C_6(\xi)\tilde{\Pi}^{\lambda\sigma}[k|a]\right\}, \tag{4.11}$$

where $(\partial/\partial q_{\sigma})\,\mathrm{Tr}\,(\gamma^{\lambda}\tilde{G}[q|a])$ has been denoted by $C_2^{\lambda\sigma}$ and the third derivative of $\mathrm{Tr}\,(\gamma^{\lambda}\tilde{G}[q|a])$ has been included into the symmetric part of C_4. Expression (4.11) is of exactly the same form as in quantum electrodynamics without a compensating current. This result is not surprising, for quantum electrodynamics with a compensating current, being explicitly gauge invariant, cannot change expressions which are gauge independent in the usual theory.

5. Perturbation expansion of the current operator

This section is devoted to the verification of the current definition (2.8) in the perturbation theory. In particular, we will show that the renormalized current operator can be obtained with the help of the renormalization functions given by (4.5) and that the only compensating current dependence is contained in the singularities of unrenormalized j^μ-operator.

To find expression for the current operator one must perform the limit $y \to x$ in the following quantity:

$$\mathscr{F}(x, y) = T[\psi(x) \exp(-ie \int d^4z[a^\lambda(z-x)-a^\lambda(z-y)]A_\lambda(z))\bar{\psi}(y)], \tag{5.1}$$

which can be given in terms of incoming field operators as:

$$\mathscr{F}(x, y) = S^\dagger T[\psi^{\text{in}}(x)\bar{\psi}^{\text{in}}(y) \exp(i \int d^4z[-ea^\lambda(z; x, y)-j^\lambda_{\text{in}}(z)]A^{\text{in}}_\lambda(z))], \tag{5.2}$$

where:

$$a^\lambda(z; x, y) = a^\lambda(z-x)-a^\lambda(z-y) \tag{5.3}$$

and j^λ_{in} is the incoming current operator:

$$j^\lambda_{\text{in}}(z) = e:\bar{\psi}^{\text{in}}(z)\gamma^\lambda\psi^{\text{in}}(z):. \tag{5.4}$$

One can show with the help of straightforward calculation that the singular terms of the type $\mu^{-2}\partial_\mu\partial_\nu\Delta_F$ cancel in every order of perturbation theory. This cancellation occurs because of the presence of the compensating current. The same statement follows also from (2.1) since, due to the asymptotic condition, all matrix elements of $\mathscr{F}(x, y)$ can be expressed by propagators.

Perturbative expressions for $\mathscr{F}$ up to the second order are given by:

$$\mathscr{F}^{(0)}(x, y) = T(\psi^{\text{in}}(x)\bar{\psi}^{\text{in}}(y)) = :\psi^{\text{in}}(x)\bar{\psi}^{\text{in}}(y): -iS_F(x-y), \tag{5.5a}$$

$$\mathscr{F}^{(1)}(x, y) = -i \int d^4z T(\psi^{\text{in}}(x)\bar{\psi}^{\text{in}}(y)j^\lambda_{\text{in}}(z)A^{\text{in}}_\lambda(z)) -$$

$$-ie \int d^4z[a^\lambda(z-x)-a^\lambda(z-y)]T(\psi^{\text{in}}(x)\bar{\psi}^{\text{in}}(y)A^{\text{in}}_\lambda(z)), \tag{5.5b}$$

$$\mathscr{F}^{(2)}(x, y) = \frac{i^2}{2} \int d^4z d^4z' T[\psi^{\text{in}}(x)\bar{\psi}^{\text{in}}(y) (-ea^\lambda(z; x, y)-j^\lambda_{\text{in}}(z)) \times$$

$$\times (-ea^{\lambda'}(z'; x, y)-j^{\lambda'}_{\text{in}}(z'))A^{\text{in}}_\lambda(z)A^{\text{in}}_{\lambda'}(z')]+(S^\dagger)^{(2)}\mathscr{F}^{(0)}(x, y). \tag{5.5c}$$

The first approximation to the current operator leads simply to the incoming current:

$$j_\mu^{(1)}(x;\xi) = -\frac{e}{2}\operatorname{Tr}(\gamma_\mu \mathscr{F}^{(0)}(x, x+\xi)+\gamma_\mu \mathscr{F}^{(0)}(x, x-\xi)) =$$

$$= \tfrac{1}{2}\left[e: \bar{\psi}^{\text{in}}(x)\gamma_\mu \psi^{\text{in}}(x+\xi): + e: \bar{\psi}^{\text{in}}(x)\gamma_\mu \psi^{\text{in}}(x-\xi): + \right.$$

$$\left. + ie \operatorname{Tr}(\gamma_\mu S_F(-\xi)+\gamma_\mu S_F(\xi))\right] \xrightarrow[\xi\to 0]{} e: \bar{\psi}^{\text{in}}(x)\gamma_\mu \psi^{\text{in}}(x):. \tag{5.6}$$

Singular terms do not appear in this order of perturbation. Were we performing the limit in a slightly different form, for instance as:

$$j^\mu(x;\xi) = -e \operatorname{Tr}(\gamma^\mu \mathscr{F}(x+\xi, x-\xi)),$$

the unrenormalized first order current would possess the singularity.

Contribution to the second order expression for j^μ will be given by $\mathscr{F}^{(1)}(x, y)$. Of course, not every term in (5.5b), when expanded according to the Wick theorem, will give rise to a singularity after putting $y = x$. One can easily show that singularities will appear only in the following one photon-to vacuum matrix element of $\mathscr{F}^{(1)}$:

$$ie \int d^4z A_\nu^{\text{in}}(z) S_F(x-z)\gamma^\nu S_F(z-y) - e \int d^4z [a^\nu(z-x) - a^\nu(z-y)] A_\nu^{\text{in}}(z) S_F(x-y). \tag{5.7}$$

It contributes the following quantity to the current operator:

$$S_\mu^{(2)}(x;\xi) = -\frac{ie^2}{2}\int d^4z A_\nu^{\text{in}}(z)\left\{\operatorname{Tr}\left[\gamma_\mu S_F(x-z)\gamma^\nu S_F(z-x-\xi) + \right.\right.$$

$$\left.\left. + \gamma_\mu S_F(x-z)\gamma^\nu S_F(z-x+\xi)\right]\right\} + \frac{e^2}{2}\int d^4z A_\nu^{\text{in}}(z)\left\{\left[a^\nu(z-x) - a^\nu(z-x-\xi)\right] \times \right.$$

$$\left. \times \operatorname{Tr}(\gamma_\mu S_F(-\xi)) + \left[a^\nu(z-x) - a^\nu(z-x+\xi)\right] \operatorname{Tr}(\gamma_\mu S_F(\xi))\right\}. \tag{5.8}$$

Expression (5.8) can be represented graphically by the photon self-energy diagram (Fig. 1) [9].

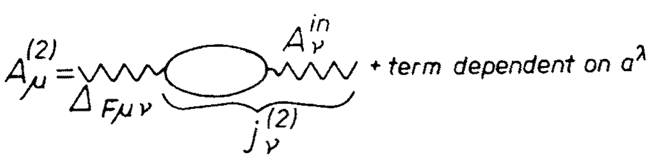

Fig. 1

Using Fourier transforms one can write (5.8) as:

$$S_\mu^{(2)}(x;\xi) = -\int \frac{d^4k}{(2\pi)^4} e^{-ikx} A_\nu^{\text{in}}(k)\left\{\Pi_\mu^{(2)\nu}(k;\xi) - \right.$$

$$\left. - \frac{e^2}{2}\tilde{a}^\nu(-k)(e^{-ik\xi} - e^{ik\xi}) \operatorname{Tr}(\gamma_\mu S_F(\xi))\right\}, \tag{5.9}$$

where $\Pi_{\mu\nu}$ is equal to the lowest order approximation for the singular expression defining vacuum polarization tensor (4.11):

$$\Pi^{(2)}_{\mu\nu}(k;\xi) = \frac{ie^2}{2}\int \frac{d^4q}{(2\pi)^4}\,(e^{iq\xi}+e^{-iq\xi})\,\mathrm{Tr}\,[\gamma_\mu \tilde{S}_F(q+k)\gamma_\nu \tilde{S}_F(q)]. \tag{5.10}$$

To separate singularities we expand the integrand in a Taylor series around $k = 0$:

$$\Pi^{(2)}_{\mu\nu}(k;\xi) = \frac{ie^2}{2}\int \frac{d^4q}{(2\pi)^4}\,(e^{iq\xi}+e^{-iq\xi})\times$$

$$\times\left\{\mathrm{Tr}\,[\gamma_\mu \tilde{S}_F(q)\gamma_\nu \tilde{S}_F(q)] + \frac{\partial}{\partial k_\alpha}\mathrm{Tr}\,[\gamma_\mu \tilde{S}_F(q+k)\gamma_\nu \tilde{S}_F(q)]\Big|_{k=0} k_\alpha + \right.$$

$$\left. + \frac{1}{2}\frac{\partial}{\partial k_\alpha}\frac{\partial}{\partial k_\beta}\mathrm{Tr}\,[\gamma_\mu \tilde{S}_F(q+k)\gamma_\nu \tilde{S}_F(q)]\Big|_{k=0} k_\alpha k_\beta + \ldots\right\}. \tag{5.11}$$

The first term of this expansion contributes a singularity of the type ξ^2 for $\xi \to 0$, the second one $-\xi^{-1}$ and the third one increases as $\ln \xi$. Further terms, which are denoted in (5.11) by dots, are regular for $\xi = 0$. It follows from the lowest order ordinary Ward identity that the quadratically divergent term is equal to:

$$\mathrm{Tr}\,[\gamma_\mu \tilde{S}_F(q)\gamma_\nu \tilde{S}_F(q)] = \frac{\partial}{\partial q_\nu}\mathrm{Tr}\,[\gamma_\mu \tilde{S}_F(q)],$$

whereas the second one is:

$$\mathrm{Tr}\,[\gamma_\mu \tilde{S}_F(q)\gamma^\alpha \tilde{S}_F(q)\gamma_\nu \tilde{S}_F(q)].$$

Since its dominant large q part is antisymmetric in q, it contributes nothing to the singularities of $\Pi^{(2)}_{\mu\nu}$. The logarithmically divergent part of $\Pi^{(2)}_{\mu\nu}$ has a much more complicated structure than the last two just considered. It has been shown in Appendix that it can be written in the following form:

$$\frac{1}{6}\frac{\partial^3}{\partial q_\alpha \partial q_\beta \partial q_\nu}\mathrm{Tr}\,[\gamma_\mu \tilde{S}_F(q)] + \left\{\frac{4}{3}\,q^2[q_\mu(g^{\alpha[\nu}q^{\beta]} + g^{\beta[\nu}q^{\alpha]} + \right.$$

$$\left. + q^\alpha g^{[\beta}_\mu q^{\nu]} + q^\beta g^{[\alpha}_\mu q^{\nu]})] + \frac{2}{3}\,q^4(g^{[\nu}_\mu g^{\beta]\alpha} + g^{[\nu}_\mu g^{\alpha]\beta})\right\}\frac{1}{(q^2-m^2)^4}. \tag{5.12}$$

It is obvious that the logarithmically divergent term in (5.11) corresponds to $C_4^{\mu\nu\alpha\beta}$ and its splitting into the symmetric and the antisymmetric part is now evident. Comparing (5.11) with (4.5) we obtain:

$$S^{(2)}_\mu(x;\xi) = C^{(2)\nu}_{2\mu}(\xi)A^{\mathrm{in}}_\nu(x) + C^{(2)\nu\alpha\beta}_{4\mu}(\xi)A^{\mathrm{in}}_{\nu,\alpha\beta}(x) +$$

$$+ \frac{e^2}{2}\int \frac{d^4k}{(2\pi)^4}\,e^{-ikx}A^{\mathrm{in}}_\nu(k)\tilde{a}^\nu(-k)\,(e^{-ik\xi}-e^{ik\xi})\,\mathrm{Tr}\,[\gamma_\mu S_F(\xi)]. \tag{5.13}$$

Since we are interested only in the small ξ behaviour of (5.13), we will expand a^λ-dependent term around $\xi = 0$:

$$e^{-ik\xi} - e^{ik\xi} = -2ik\xi + \frac{i}{3}(k\xi)^3 + \dots . \tag{5.14}$$

Thus, the a^λ-dependent part of $S_\mu^{(2)}$ can now be written as:

$$e^2 \int \frac{d^4k}{(2\pi)^4}(-ik_\nu)\tilde{a}_\lambda(-k)A^{\text{in}\,\lambda}(k)e^{-ikx}[\xi^\nu \operatorname{Tr}(\gamma_\mu S_F(\xi)) -$$
$$- \tfrac{1}{6} k_\alpha k_\beta \xi^\nu \xi^\alpha \xi^\beta \operatorname{Tr}(\gamma_\mu S_F(\xi))] = C_{2\mu}^{(2)\nu}(\xi)\,[B_\nu^{\text{in}}(x) - A_\nu^{\text{in}}(x)] +$$
$$+ \frac{e^2}{6} \xi^\nu \xi^\alpha \xi^\beta \operatorname{Tr}[\gamma_\mu S_F(\xi)]\,[B_{\nu,\alpha\beta}^{\text{in}}(x) - A_{\nu,\alpha\beta}^{\text{in}}(x)], \tag{5.15}$$

where B_ν is defined by (2.9). Since the antisymmetric part of $C_4^{\mu\nu\alpha\beta}$ gives zero when contracted with $B_{\nu,\alpha\beta} - A_{\nu,\alpha\beta}$, we obtain for the singular part of unrenormalized $j_\mu^{(2)}$-operator:

$$S_\mu^{(2)}(x;\xi) = C_{2\mu}^{(2)\nu}(\xi)B_\nu^{\text{in}}(x) + C_{4\mu}^{(2)\nu\alpha\beta}(\xi)B_{\nu,\alpha\beta}^{\text{in}}(x). \tag{5.16}$$

It is now evident that the renormalization performed according to (2.8) and (4.5) leads to finite expressions for the second order current operator. Moreover, we can easily see that the only compensating current dependence is contained in singularities of $j_\mu^{(2)}(x;\xi)$ and it is removed in the process of renormalization. Thus, the final expression for the second order current operator does not contain a compensating current, in accordance with the gauge independence of j^μ in the usual theory.

The next term in the perturbation series for j^μ is given by $\mathscr{F}^{(2)}(x, y)$. Singularities are now connected with the vacuum polarization tensor and with the vertex function [9]. In this order will also appear the singularity of the vacuum expectation value of $j^\mu(x, \xi)$:

$$\frac{ie}{2}\{\operatorname{Tr}(\gamma_\mu G^{(2)}[\xi|a]) + \operatorname{Tr}(\gamma_\mu G^{(2)}[-\xi|a])\} = C_{1\mu}^{(2)}(\xi). \tag{5.17}$$

Since $\langle 0|j_\mu|0\rangle = 0$, this term will be removed by renormalization. Vacuum polarization singularities arise from the following two fermion to vacuum matrix elements of j^μ:

$$-ie^2 \int d^4z d^4z' D_{F\lambda\nu}(z-z')S_F(x-z)\gamma^\lambda S_F(z-y) : \bar{\psi}^{\text{in}}(z')\gamma^\nu\psi^{\text{in}}(z') : +$$
$$+ e^2 \int d^4z d^4z' D_{F\lambda\nu}(z-z')\,[a^\lambda(z-x) - a^\lambda(z-y)]S_F(x-y) : \bar{\psi}^{\text{in}}(z')\gamma^\nu\psi^{\text{in}}(z') : =$$
$$= ie \int d^4z A_\nu^{(1)}(z)S_F(x-z)\gamma^\nu S_F(z-y) -$$
$$- e \int d^4z A_\nu^{(1)}(z)\,[a^\nu(z-x) - a^\nu(z-y)]S_F(x-y), \tag{5.18}$$

where:

$$\Box A_\nu^{(1)}(z) = j_\nu^{(1)}(z). \tag{5.19}$$

Vacuum polarization contribution to $j_\mu^{(3)}$ is given by:

$$S_{1\mu}^{(3)}(x;\xi) = -\frac{ie^2}{2}\int d^4z A_\nu^{(1)}(z)\,\{\mathrm{Tr}\,[\gamma_\mu S_F(x-z)\gamma^\nu S_F(z-x-\xi)]+(\xi\to-\xi)\}+$$

$$+\frac{e^2}{2}\int d^4u A_\nu^{(1)}(z)\,\{[a^\nu(z-x)-a^\nu(z-x-\xi)]\,\mathrm{Tr}\,(\gamma_\mu S_F(\xi))+(\xi\to-\xi)\} \qquad (5.20)$$

and can be represented by the photon self energy diagram (Fig. 2).

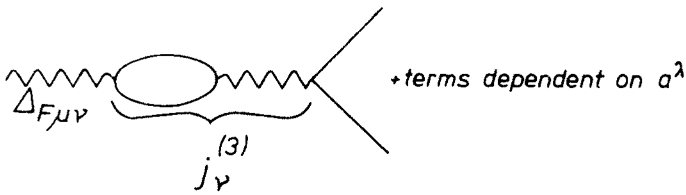

Fig. 2

Expression (5.20) is of the same form as (5.8), apart from $A_\lambda^{(1)}$ which plays now the role of A_λ^{in}. Thus:

$$S_{1\mu}^{(3)}(x;\xi) = C_{2\mu}^{(2)\nu}(\xi)B_\nu^{(1)}(x)+C_{4\mu}^{(2)\nu\alpha\beta}(\xi)B_{\nu,\alpha\beta}^{(1)}(x)+\ldots, \qquad (5.21)$$

where dots denote this part of $S_{1\mu}^{(3)}$ which is regular at $\xi = 0$. Since terms dependent on the compensating current contribute only to the singularity, they will not appear in the renormalized vacuum polarization part of $j_\mu^{(3)}$-operator.

The vertex function part of the current operator can be represented graphically by the diagram of the vertex type (Fig. 3)

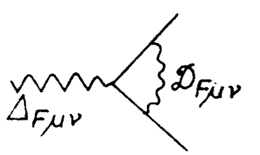

Fig. 3

It results as a contribution from the remaining part of the two fermion to vacuum matrix elements of $\mathscr{F}^{(2)}(x,y)$:

$$-ie^2\int d^4zd^4z'D_{F\lambda\nu}(z-z')S_F(x-z)\gamma^\lambda :\psi^{\mathrm{in}}(z)\bar\psi^{\mathrm{in}}(z'):\gamma^\nu S_F(z'-y)-$$

$$-e^2\int d^4zd^4z'D_{F\lambda\nu}(z-z')\,\{S_F(x-z)\gamma^\lambda:\psi^{\mathrm{in}}(z)\bar\psi^{\mathrm{in}}(y):a^\nu(z'-y)-$$

$$-:\psi^{\mathrm{in}}(x)\bar\psi^{\mathrm{in}}(z):\gamma^\lambda S_F(z-y)a^\nu(z'-x)\}-$$

$$-ie^2\int d^4zd^4z'D_{F\lambda\nu}(z-z')a^\lambda(z-x)a^\nu(z'-y):\psi^{\mathrm{in}}(x)\bar\psi^{\mathrm{in}}(y):. \qquad (5.22)$$

The part of $j_\mu^{(3)}(x, \xi)$ which is connected with the vertex singularity can be therefore written as:

$$S_{2\mu}^{(3)}(x;\xi) = -\frac{e^3}{2}\int \frac{d^4q}{(2\pi)^4}\frac{d^4q'}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} e^{-i(q-q')x}(e^{ik\xi}+e^{-ik\xi})\tilde{D}_{F\lambda\nu}(k)\times$$

$$\times\{i:\overline{\psi}^{\text{in}}(q')\gamma^\lambda\tilde{S}_F(q'+k)\gamma_\mu\tilde{S}_F(q+k)\gamma^\nu\psi^{\text{in}}(q):+$$

$$+:\overline{\psi}^{\text{in}}(q')\gamma_\mu\tilde{S}_F(q+k)\gamma^\lambda\psi^{\text{in}}(q):\tilde{a}^\nu(k)-:\overline{\psi}^{\text{in}}(q')\gamma^\lambda\tilde{S}_F(q'+k)\gamma_\mu\psi^{\text{in}}(q):\tilde{a}^\nu(-k)+$$

$$+i\tilde{a}^\lambda(k)\tilde{a}^\nu(-k):\overline{\psi}^{\text{in}}(q')\gamma_\mu\psi^{\text{in}}(q):\}. \tag{5.23}$$

Using now the well-known formulae

$$-\gamma k\tilde{S}_F(q'+k)|_{\gamma_{q'}=m} = 1 = -\tilde{S}_F(q+k)\gamma k|_{\gamma_q=m}, \tag{5.24}$$

we obtain:

$$S_{2\mu}^{(3)}(x;\xi) = e\int\frac{d^4p}{(2\pi)^4}e^{ipx}\int\frac{d^4q}{(2\pi)^4}:\overline{\psi}^{\text{in}}(q+p)\Lambda_\mu^{(2)}(q,p;\xi)\psi^{\text{in}}(q):, \tag{5.25}$$

where:

$$\Lambda_\mu^{(2)}(q,p;\xi) = -\frac{ie^2}{2}\int\frac{d^4k}{(2\pi)^4}(e^{ik\xi}+e^{-ik\xi})\times$$

$$\times\gamma^\lambda\tilde{S}_F(p+q+k)\gamma_\mu\tilde{S}_F(q+k)\gamma^\nu\tilde{\mathscr{D}}^F_{\lambda\nu}(k). \tag{5.26}$$

This formula can be easily recognized as the lowest order expression for the regularized, but not renormalized, vertex function. Its singular part can be obtained by putting $p = 0$ and is given by $C_6^{(2)}(\xi)$, where:

$$C_6^{(2)}(\xi)\gamma_\mu = -\frac{ie^2}{2}\int\frac{d^4k}{(2\pi)^4}(e^{ik\xi}+e^{-ik\xi})\gamma^\lambda\tilde{S}_F(q+k)\gamma_\mu\tilde{S}_F(q+k)\gamma^\nu\tilde{\mathscr{D}}_{F\lambda\nu}(k)|_{\gamma_q=m}. \tag{5.27}$$

A more detailed discussion of the renormalization procedure for the compensating current dependent vertex function will be given in further publication. Here we quote only the final formula for $C_6^{(2)}$; one can easily see that by putting formally $\xi = 0$ in (5.27) we obtain a divergent expression for the usual $Z_2^{(2)}$ renormalization constant.

Opposite to the renormalized vacuum polarization part, the vertex function, as gaude dependent in the usual formulation of quantum electrodynamics, will in general depend on the compensating current a^λ. However, this is not the case when $p+q$ and q are on their mass-shells since it follows then from (5.24) that a^λ-dependent terms in (5.26) contribute only to the singularities.

We have shown that the singular part of the third order unrenormalized current operator can be written as:

$$S_\mu^{(3)}(x;\xi) = S_{1\mu}^{(3)}(x;\xi)+S_{2\mu}^{(3)}(x;\xi)+\langle 0|j_\mu^{(3)}(x;\xi)|0\rangle = C_{1\mu}^{(3)}(\xi)+$$

$$+C_{2\mu\nu}^{(2)}(\xi)B^{(1)\nu}(x)+C_{4\mu}^{(2)\nu\alpha\beta}(\xi)B^{(1)\nu}{}_{,\alpha\beta}(x)+C_6^{(2)}(\xi)j_\nu^{(1)}(x), \tag{5.28}$$

where all the a^λ-dependence is contained in this expression. It follows from (4.5) that (2.8) gives a finite result for the current operator also in the third order of perturbation.

6. *Conclusions*

We have shown in this paper that the vacuum polarization tensor and current operator in quantum electrodynamics with compensating current can be defined in a way analogous to that in conventional theory. Furthermore, it follows from our considerations that earlier definitions can be obtained from ours as a particular case, namely this one in which $a^\lambda = a^\lambda_{(l)}$. Since quantum electrodynamics with compensating current expresses explicitly charge conservation, we have no trouble with the quadratically divergent term in the $\Pi^{\mu\nu}$-tensor; this term either does not appear at all or fulfills the transversity condition. This property of the vacuum polarization tensor results from the presence of the compensating current as well as from the natural prescription of regularization, connected with the definition of the current operator. Our definition of j^μ(2.8) has been verified up to the third order of perturbation theory with the renormalization subtractions resulting from the photon mass, charge and vertex renormalization, similarly as in the usual theory.

It follows from calculations performed in Sec. 5 that the current operator is independent on a^λ, at least up to the third order. This result is consistent with the gauge invariance of this observable. To prove the independence of j^μ on the compensating current in every order one should show that equation of the following type is fulfilled:

$$\frac{\delta}{\delta a^\mu} j^\nu[a] = 0. \tag{6.1}$$

In the formulation without a compensating current, *e. g.* that of Zimmerman [10] with ghost states, gauge invariance is equivalent to independence on the mass of the ghost state m_0 [11]:

$$\frac{\partial}{\partial m_0} j^\nu = 0. \tag{6.2}$$

Since the a^λ-current plays a role in a sense similar to that of ghost states in Zimmerman's formulation, it would be interesting to verify whether formulae (6.1) and (6.2) are equivalent.

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APPENDIX

We shall calculate the following quantity:

$$\tilde{C}_4^{(2)\mu\nu\sigma\kappa}(q) = \frac{ie^2}{2}\frac{\partial}{\partial k_\sigma}\frac{\partial}{\partial k_\kappa}\,\mathrm{Tr}\,[\gamma^\mu\tilde{S}_F(q+k)\gamma^\nu\tilde{S}_F(q)]|_{k=0} =$$

$$= \frac{ie^2}{2}\,\mathrm{Tr}\left[\gamma^\mu\left(\frac{\partial}{\partial q_\sigma}\frac{\partial}{\partial q_\kappa}\tilde{S}_F(q)\right)\gamma^\nu\tilde{S}_F(q)\right]. \tag{A1}$$

It follows from the ordinary Ward identity that:

$$\tilde{C}_4^{(2)\mu\nu\sigma\kappa}(q) = \frac{ie^2}{2}\operatorname{Tr}[\gamma^\mu \tilde{S}_F(q)\gamma^\sigma \tilde{S}_F(q)\gamma^\kappa \tilde{S}_F(q)\gamma^\nu \tilde{S}_F(q)] +$$

$$+\frac{ie^2}{2}\operatorname{Tr}[\gamma^\mu \tilde{S}_F(q)\gamma^\kappa \tilde{S}_F(q)\gamma^\sigma \tilde{S}_F(q)\gamma^\nu \tilde{S}_F(q)]. \tag{A2}$$

The first term in (A2) can be written as:

$$\tfrac{1}{3}\operatorname{Tr}(\gamma^\mu \tilde{S}_F\gamma^\sigma \tilde{S}_F\gamma^\kappa \tilde{S}_F\gamma^\nu \tilde{S}_F) + \tfrac{2}{3}\operatorname{Tr}(\gamma^\mu \tilde{S}_F\gamma^\sigma \tilde{S}_F\gamma^\kappa \tilde{S}_F\gamma^\nu \tilde{S}_F). \tag{A3}$$

The first term in (A3) can be further transformed:

$$\tfrac{1}{3}\operatorname{Tr}(\gamma^\mu \partial^\sigma S\gamma^\kappa \partial^\nu S) = \tfrac{1}{3}\operatorname{Tr}(\gamma^\mu \partial^\sigma(S\gamma^\kappa \partial^\nu S)) - \tfrac{1}{3}\operatorname{Tr}(\gamma^\mu S\gamma^\kappa \partial^\sigma \partial^\nu S) =$$

$$= \tfrac{1}{3}\operatorname{Tr}[\gamma^\mu \partial^\sigma \partial^\nu(S\gamma^\kappa S)] - \tfrac{1}{3}\operatorname{Tr}[\gamma^\mu \partial^\sigma(\partial^\nu S\gamma^\kappa S)] - \tfrac{1}{3}\operatorname{Tr}(\gamma^\mu S\gamma^\kappa \partial^\sigma \partial^\nu S) =$$

$$= \frac{1}{3}\frac{\partial}{\partial q_\sigma}\frac{\partial}{\partial q_\nu}\frac{\partial}{\partial q_\kappa}\operatorname{Tr}[\gamma^\mu \tilde{S}_F(q)] - \frac{1}{3}\operatorname{Tr}[\gamma^\mu \partial^\sigma(\partial^\nu S\gamma^\kappa S)] - \frac{1}{3}\operatorname{Tr}[\gamma^\mu S\gamma^\kappa \partial^\sigma \partial^\nu S], \tag{A4}$$

where

$$S \equiv \tilde{S}_F(q), \qquad \partial_\nu \equiv \frac{\partial}{\partial q^\nu}.$$

Substituting (A4) to (A3), and then (A3) to (A2) we obtain:

$$\tilde{C}_4^{(2)\mu\nu\sigma\kappa}(q) = \frac{ie^2}{6}\frac{\partial}{\partial q_\nu}\frac{\partial}{\partial q_\sigma}\frac{\partial}{\partial q_\kappa}\operatorname{Tr}[\gamma^\mu \tilde{S}_F(q)] + \tilde{C}_{4a}^{(2)\mu\nu\sigma\kappa}(q), \tag{A5}$$

where the antisymmetric part is given by:

$$\tilde{C}_{4a}^{(2)\mu\nu\sigma\kappa}(q) = -\tfrac{1}{3}[\operatorname{Tr}(\gamma^\mu S\gamma^\sigma S\gamma^\nu S\gamma^\kappa S) - (\nu\leftrightarrow\kappa)] -$$

$$-\tfrac{1}{3}[\operatorname{Tr}(\gamma^\mu S\gamma^\nu S\gamma^\sigma S\gamma^\kappa S) - (\nu\leftrightarrow\kappa)] - \tfrac{1}{3}[\operatorname{Tr}(\gamma^\mu S\gamma^\nu S\gamma^\kappa S\gamma^\sigma S) - (\sigma\leftrightarrow\nu)] -$$

$$-\tfrac{1}{3}[\operatorname{Tr}(\gamma^\mu S\gamma^\kappa S\gamma^\nu S\gamma^\sigma S) - (\sigma\leftrightarrow\nu)]. \tag{A6}$$

Since for the renormalization only large q behaviour of $\tilde{C}_4$ is important, it is enough to evaluate quantities of the following type:

$$\operatorname{Tr}(\gamma^\mu \gamma_q \gamma^\nu \gamma_q \gamma^\sigma \gamma_q \gamma^\kappa \gamma_q).$$

Straightforward calculation leads to formula (5.12).

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