THE MULTIPLICATION LAW IN QUANTUM PHYSICS

PART I. THE NON-RELATIVISTIC QUANTUM MECHANICS

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A generalization of the Heisenberg quantum theoretical multiplication law is proposed. What kind of modification this introduces to the usual results of quantum mechanics is shown. The large domain of applicability of the generalized product is indicated.

1. Introduction

In relativistic quantum field theory with contact interaction it is necessary to use the products of field operators at the same spacetime points. It is, however, well known that, owing to the distributional character of the fields, such products are ill-defined quantities. Although considerable efforts have been made recently to obtain some satisfactory definition of the product in the framework of the perturbation approach [1], this does not solve the fundamental problem for the theory to be well-defined and self-consistent from the beginning.

It is known that the multiplication law for the representatives of physical quantities was a cornerstone for the non-relativistic quantum mechanics. The lack of such a law in field theory makes it difficult to speak of quantum field theory, since at the beginning these two theories are different from the logical point of view. It is necessary, therefore, to investigate why these two theories are so different and the discovery of the origin of this difference should give some hint of the solution of the multiplication problem.

In the present paper, which is the first of a series of three, we discuss the general multiplication law in non-relativistic quantum mechanics. Our approach is a direct generalization of the original Heisenberg approach to quantum mechanics [2]. It is shown that by using the same kind of arguments as Heisenberg did it is possible to construct a much more general quantum theoretical scheme, which, as a particular case, contains the conventional quantum mechanics and at the same time works for a much larger class of physical systems. In the second paper of the series we shall show that quantum

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field theory is an element of such a larger class, which is beyond the conventional quantum mechanics. This explains the above-mentioned difference between these two theories.

The last paper of our series is devoted to the perturbation theory applied to the interacting fields. Detailed calculations performed for the usually adopted $\lambda\varphi^3$ and $\lambda\varphi^4$ theories up to the second order in the perturbation are then compared with the conventional renormalized perturbation series.

2. *The generalization of the Heisenberg quantum mechanical scheme*

The crucial Heisenberg discovery was that in the quantum case all kinematical quantities should be represented by a double indexed set of the form

$$x(t) \to \{x_{mn} e^{i\omega_{mn} t}\}. \tag{2.1}$$

where n, $m \in Z$ and Z is the set of all natural numbers. For these representatives of kinematical quantities Heisenberg next discovered the fundamental multiplication law given by

$$(XY)_{mn} = \sum_{k \in Z} X_{mk} Y_{kn}. \tag{2.2}$$

In proposing this multiplication law, however, Heisenberg made an important reservation that this type of combination is "an almost necessary consequence of the (Rydberg-Ritz) frequency combination rules" given by the formula

$$\omega_{mk} + \omega_{kn} = \omega_{mn} \tag{2.3}$$

for all m, n and k. It is therefore interesting to look for more general multiplication laws which agree with the Rydberg-Ritz combination rules as well.

To find the solution to this problem we shall follow the method of Dirac [3] applied in the derivation of the general expression for the derivatives in Heisenberg quantum mechanics. First of all we write the most general expression for the representatives of the product of two quantities X and Y in the form

$$(X \circ Y)_{mn} = \sum_{k,l,s,u \in Z} a_{mnlsu} X_{kl} Y_{su}. \tag{2.4}$$

Requiring now that the mn-th representative of the product after using (2.3) should be accompanied only by the right ω_{mn} frequency when we insert into (2.4) the representatives of X and Y together with the time factors, we may reduce the general case (2.4) to the following one:

$$(X \circ Y)_{mn} = \sum_{k \in Z} (A_{mnk} X_{mk} Y_{kn} + B_{mnk} X_{kk} Y_{mn} + C_{mnk} Y_{kk} X_{mn} + D_{mnk} Y_{mk} X_{kn}), \tag{2.5}$$

where the A, B, C and D's are the corresponding non-zero values of the previous coefficients in (2.4). The multiplication law (2.5) is therefore the only necessary consequence of the Rydberg-Ritz combination principle (2.3) and the restricted character of the Heisenberg definition (2.2) is obvious.

Our further steps consist now in imposing some reasonable restrictions on the general multiplication law (2.5).

First, since all physical constants are represented in the theory by a special set in the class (2.1) given by a matrix proportional to the unit matrix and the multiplication by a constant should not change the character of the set representing a given kinematical quantity, we shall require that the unit matrix be a unit for the algebra defined by the multiplication law (2.5). This gives us rather a weak restriction in the form

$$B_{mnk} = C_{mnk} = 0 \tag{2.6}$$

and

$$A_{mnm} + D_{mnn} = 1, \quad A_{mnn} + D_{mnm} = 1, \tag{2.7}$$

for all m and n.

Second, we shall require that the square of any hermitian set (2.1) should be hermitian, which gives

$$A_{mnk} + D_{mnk} = A_{nmk} + D_{nmk}. \tag{2.8}$$

Our last restriction has a strong character and is based on the following physical fact. Born and Jordan [4] have shown that the quantum mechanical commutation relation

$$[p, x] = -i\hbar \tag{2.9}$$

is a consequence of the experimentally verified Thomas-Kuhn sum rule

$$\sum_k \omega_{mk} |X_{mk}|^2 = \frac{\hbar}{2M}. \tag{2.10}$$

Furthermore, all commutation relations in the conventional quantum mechanics are closely related to the invariance properties of the physical system under different symmetry operations. Since we do not want to change this part of the theory we shall require that the commutation relations of two arbitrary quantities calculated according to the law (2.5) are exactly the same as the corresponding ones calculated according to the law (2.2). This gives us the condition that

$$A_{mnk} = 1 + D_{mnk}. \tag{2.11}$$

Summarizing, we obtain the general multiplication law in the form

$$(X \circ Y)_{mn} = (XY)_{mn} + \sum_{k \in Z} D_{mnk}(X_{mk} Y_{kn} + Y_{mk} X_{kn}), \tag{2.12}$$

where D_{mnk} are arbitrary coefficients satisfying the relations following from (2.7) and (2.8) and by XY we denote the usual matrix multiplication.

The multiplication law (2.12) is distributive with respect to addition but in general is neither commutative nor associative. The associativity requirement has only one solution

$$D_{mnk} = 0, \tag{2.13}$$

but since this requirement is not implied by any physical fact we do not impose it. This, however, gives some difficulties in defining the repeated products of more than two factors. In order to see how to proceed in this case, we shall illustrate our method on the

example of the product of three factors. First we start with the most general expression for such a product given by

$$(X \circ Y \circ Z)_{mn} = \sum_{k,l,s,u,r,w \in Z} a_{mnklsurw} X_{kl} Y_{su} Z_{rw}. \tag{2.14}$$

Proceeding now in a similar way as above we may reduce this expression to the following one

$$\begin{aligned}(X \circ Y \circ Z)_{mn} &= (XYZ)_{mn} + \\ &+ \sum_{k,l \in Z} D_{mnkl}(X_{mk} Y_{kl} Z_{ln} + X_{mk} Z_{kl} Y_{ln} + \\ &+ Y_{mk} X_{kl} Z_{ln} + Y_{mk} Z_{kl} X_{ln} + \\ &+ Z_{mk} X_{kl} Y_{ln} + Z_{mk} Y_{kl} X_{ln}).\end{aligned} \tag{2.15}$$

The condition that in the case when one of the factors is a unit matrix this product should reduce to the already defined product (2.12) gives a connection between the coefficients in (2.15) and (2.12) in the form

$$D_{mnkn} + D_{mnkk} + D_{mnmk} = \tfrac{1}{2} D_{mnk}. \tag{2.16}$$

We get a further connection between coefficients by requiring that all powers of a given matrix X calculated by means of our definitions of products should commute with X. This gives

$$:X^n: = \sum_{j=0}^{n} \alpha_j(X) X^j, \tag{2.17}$$

where $:X^n:$ denotes the n-th power calculated by our definition and $\alpha_j(X)$ are homogeneous numerical functions of order n-j of the matrix elements of the matrix X with the normalization $a_n(X) = 1$. Then we get

$$:X^2: \circ X = X \circ :X^2: \tag{2.18}$$

and we may impose the condition

$$:X^3: = :X^2: \circ X. \tag{2.19}$$

We may proceed similarly in the case of higher powers. Instead of elaborating such cases we now switch to some special examples which perfectly illustrate our method.

3. The harmonic and anharmonic oscillators

As a first example we consider here the harmonic oscillator dynamics with mass M and frequency ω. It is well known that the solution of the equation of motion is given by

$$X_{mn} = x_m \delta_{m+1,n} + x_n \delta_{m,n+1},$$

$$x_m^2 = \frac{\hbar}{2M\omega}(m+1), \qquad E_n = E_0 + n\hbar\omega. \tag{3.1}$$

In the conventional formulation

$$E_0 = \tfrac{1}{2}\hbar\omega, \tag{3.2}$$

which, as we shall see in a moment, is a consequence of the usual definition of the product (2.2) rather than of the canonical commutation relation (2.9) [5]. In fact, let us insert the solution (3.1) to the Hamiltonian

$$H = \frac{1}{2M} :P^2: + \frac{\omega^2 M}{2} :X^2:. \tag{3.3}$$

We then get

$$\frac{\alpha_0(P)}{2M} + \frac{\alpha_0(X)\omega^2 M}{2} = E_0 - \frac{1}{2}\hbar\omega. \tag{3.4}$$

Now from (2.12) we get

$$\begin{aligned} \alpha_0(X) &= 2(D_{m,m,m+1}x_m^2 + D_{m,m,m-1}x_{m-1}^2), \\ \alpha_0(P) &= 2M^2\omega^2(D_{m,m,m+1}x_m^2 + D_{m,m,m-1}x_{m-1}^2), \end{aligned} \tag{3.5}$$

and

$$D_{m,m+2,m+1} = D_{m+2,m,m+1} = 0.$$

Therefore

$$D_{m,m,m+1}x_m^2 + D_{m,m,m-1}x_{m-1}^2 = \frac{1}{2M\omega^2}\left(E_0 - \frac{1}{2}\hbar\omega\right), \tag{3.6}$$

and we get

$$\alpha_0(X) = \langle X^2\rangle_0\left(\frac{2E_0}{\hbar\omega} - 1\right), \quad \alpha_0(P) = \langle P^2\rangle_0\left(\frac{2E_0}{\hbar\omega} - 1\right). \tag{3.7}$$

Thus for $:X^2:$ and $:P^2:$ we finally get

$$:X^2: = X^2 + \langle X^2\rangle_0\left(\frac{2E_0}{\hbar\omega} - 1\right), \quad :P^2: = P^2 + \langle P^2\rangle_0\left(\frac{2E_0}{\hbar\omega} - 1\right). \tag{3.8}$$

Now we see that by choosing (3.2) we get the conventional multiplication law and calculating with this choice the energy levels we again get (3.2).

Generalizing the above treatment for N harmonic oscillators, we get essentially the same result with a factor

$$\frac{2E_0}{N\hbar\omega} - 1, \tag{3.9}$$

where E_0 is now the total energy of the ground state. If we pass to the limit when N is infinite with finite E_0 we see that the multiplication law has the form

$$:X^2: = X^2 - \langle X^2\rangle_0. \tag{3.10}$$

which strongly resembles the Wick ordered product used in field theory. We get exactly the same result, however, if we put $E_0 = 0$ in (3.9). In fact, when calculating (3.3) with such a choice of E_0 we get the result that the only possible finite value of the ground state energy for an infinite number of oscillators is the zero value.

The obtained values of the coefficients D_{mnk} do not however allow the calculation of higher order powers of X, since for these powers different coefficients from those in (3.5) are needed. In order to get some information on them we must consider dynamical problems into which such higher powers enter. Examples of such systems are the anharmonic oscillators. In order to know how to calculate the third power of the harmonic oscillator matrix X we have to consider the anharmonic oscillator

$$H = \frac{1}{2M} :P^2: + \frac{\omega^2 M}{2} :X^2: + \frac{\lambda}{3} :X^3: \tag{3.11}$$

in the first order of perturbation theory. The zero order approximation to (3.11) is the harmonic oscillator and this gives us the shape of $:X^2:$ in the zero order. This is, however, sufficient to solve the equation of motion in the first order. The answer is

$$X(1)_{mn} = -\frac{\hbar}{M^2\omega^3}\left(m + \frac{E_0}{\hbar\omega}\right)\delta_{mn} +$$

$$+ \frac{1}{3M\omega^2}\left(x_m x_{m+1}\delta_{m+2,n} + x_n x_{n+1}\delta_{m,n+2}\right),$$

$$\omega(1)_{m,m+1} = \omega(1)_{m+1,m} = 0, \qquad \omega(1)_{m+2,m} = 2\omega. \tag{3.12}$$

Having this we may calculate $:P^2:$ and $:X^2:$ in the first order in λ. From the requirement of commutativity of these matrices with P and X, respectively, we get the conditions

$$D_{m,m+3,m+1} + D_{m,m+3,m+2} = 0,$$

$$D_{m,m+1,m-1}x_{m-1}^2 + D_{m,m+1,m+2}x_{m+1}^2 = 0,$$

$$D_{m,m+1,m+1} = \frac{1}{2}\left(1 - \frac{2E_0}{\hbar\omega}\right). \tag{3.13}$$

Calculating now $:X^3:$ from (2.19) we get

$$:X^3: = X^3 + 3\langle X^2\rangle_0\left(\frac{2E_0}{\hbar\omega} - 1\right)X, \tag{3.14}$$

which again reduces to the Wick ordered product in the case of $E_0 = 0$.

Completing the result for $:X^3:$, we may find $:X^4:$ either from the second order perturbation theory for (3.11) or more simply the first order of approximation to the anharmonic oscillator

$$H = \frac{1}{2M} :P^2: + \frac{\omega^2 M}{2} :X^2: + \frac{\lambda}{4} :X^4:. \tag{3.15}$$

After a few tedious calculations we get then

$$:X^4: = X^4+6\langle X^2\rangle_0\left(\frac{2E_0}{\hbar\omega}-1\right)X^2+3\langle X^2\rangle_0^2\left(\frac{2E_0}{\hbar\omega}-1\right). \tag{3.16}$$

4. Application to physical systems with internal symmetries

It is obvious that for the validity of the multiplication law (2.2) an implicit assumption must be made. This assumption consists in admitting in the theory only such physical systems which are represented by some restricted class of sets (2.1) for which the right hand side of (2.2) is finite. For all known examples of quantum mechanical systems this assumption is satisfied *a posteriori*. It is not satisfied, however, if independently of any dynamics we consider a system having some internal symmetry properties manifesting themselves in the equality of an infinite number of the representatives X_{mn}. For such systems the Heisenberg law (2.2) cannot be applied nor can the problem be formulated directly as an operator formalism in a Hilbert space. Later on we shall show that quantum field theory belongs indeed to such a category and it is therefore necessary to construct it in an analogy to the situation described in this section.

It is, however, not difficult to give an example of the system with the above-described property in the framework of non-relativistic physics. In fact, let us imagine a system consisting of an infinite number of harmonic oscillators with different frequencies ω_j such that the ratio of any two of them is never a rational number and such that

$$M_j\omega_j = \text{const.} \tag{4.1}$$

Since the transition amplitudes for an oscillator depend only on the product $M\omega$, we see that an infinite number of them are equal.

To see how our generalized product works in such cases we shall start from the formula

$$(X\circ Y)_{mn} = \sum_{k\in Z}(A_{mnk}X_{mk}Y_{kn}+D_{mnk}Y_{mk}X_{kn}). \tag{4.2}$$

In general we cannot reduce this expression to the form (2.12), since in the present case the commutator algebra has a different meaning. We cannot here have the usual commutation relations owing simply to the fact that already the Thomas-Kuhn sum rule is not valid because it is meaningless.

Let us now assume that the set of all integers Z is divided into at most a denumerable number of disjoint infinite subsets Z_α and

$$X_{mn} = X_{\alpha\beta}, \qquad Y_{mn} = Y_{\alpha\beta}, \tag{4.3}$$

for $m\in Z_\alpha$ and $n\in Z_\beta$. Since the equality of representatives is a manifestation of some physical symmetries, the multiplication law should not violate such symmetries and therefore we assume here that

$$A_{mnk} = A_{\alpha\beta k}, \qquad D_{mnk} = D_{\alpha\beta k}. \tag{4.4}$$

It is then easy to see that

$$(X \circ Y)_{mn} = (X \circ Y)_{\alpha\beta}, \tag{4.5}$$

where

$$(X \circ Y)_{\alpha\beta} = \sum_{\gamma} (\tilde{A}_{\alpha\beta\gamma} X_{\alpha\gamma} Y_{\gamma\beta} + \tilde{D}_{\alpha\beta\gamma} Y_{\alpha\gamma} X_{\gamma\beta})$$

and

$$\tilde{A}_{\alpha\beta\gamma} = \sum_{k \in Z_\gamma} A_{\alpha\beta k}, \quad \tilde{D}_{\alpha\beta\gamma} = \sum_{k \in Z_\gamma} D_{\alpha\beta k}. \tag{4.6}$$

From these formulae we may extract the following method for treating systems with an infinite number of equal representatives:

First, we pass from the original set of representatives (2.1) to the set of equivalence classes where each equivalence class is specified by the pair (α, β); Second, we may apply the quantum multiplication rule to these sets of equivalence classes, and finally we enlarge the set of equivalence classes for the product to the originally indexed set of type (2.1). By this we see that it is not necessary to know all coefficients in the multiplication law but only the corresponding sums in (4.6). The values of these sums may be chosen from some additional requirements, *e. g.* the requirement of some form for the commutation relations.

It must be noted that although the representatives of X and Y do not correspond to operators in some Hilbert space, the smeared representatives

$$\int X_{mn} e^{i\omega_{mn} t} f(t) dt = X_{mn} \tilde{f}(\omega_{mn}) \tag{4.7}$$

owing to the choice of the test function $f(t)$, may do so. This is very similar to the situation in quantum field theory.

It must also be noted, however, that the above-described procedure is not quite unique. The non-uniqueness appears when the contracted set of equivalence classes of the representatives for the product contains non-zero diagonal elements. The passage to the original indexing may then be performed either in the way described above or simply by enlarging only the diagonal equivalence classes to a set of purely diagonal representatives in the original indexing. Such non-uniqueness is, however, always present in any quantum multiplication law, since we may always add to the diagonal terms in the product a term

$$\sum_{k,l \in Z} a_{mkl} X_{kl} Y_{lk}, \tag{4.8}$$

which is not exluded by the Rydberg-Ritz principle. We shall show in the second paper of this series that the above-mentioned non-uniqueness has an important application in the case of quantum field theory.

5. Application to systems with finite number of energy levels

It is known that the usual formalism of quantum mechanics can be applied to systems with a finite number of energy levels only for some special potentials. On the contrary, we shall briefly prove that our formalism can do this without any restriction on the forces.

We shall not, however, develop the general case but only illustrate this on the example of the harmonic oscillator having only two energy levels E_0 and E_1. From the equation of motion we get then

$$E_1 = E_0 + \hbar\omega. \tag{5.1}$$

The first obstacle to the application of the formalism of quantum mechanics to the present case is the commutation relation (2.9). It is, however, only superficial, since in the present case we have no commutation relation of that type as the analog to the Thomas-Kuhn sum rule reduces here simply to the normalization

$$|x_{01}|^2 = \frac{\hbar}{2M\omega} \tag{5.2}$$

and does not lead to (2.9). This change is not sufficient, however, for the applicability of the conventional formalism to the present case, since the Hamiltonian is conventionally proportional to the unit matrix and this implies that $\omega = 0$. It is easy to see that in our formulation of the multiplication law we may have a solution with arbitrary ω. The negative feature of such formulation is, however, that fact that the higher powers of a given matrix never commute with the original matrix but may anticommute. This obviously requires a change in the conventional measurement theory constructed on the basis of such formalism.

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