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A SEMICLASSICAL MODEL OF MULTIPARTICLE PRODUCTION

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A model of multiparticle production in high energy hadron collisions developed by Heisenberg more than 20 years ago is reexamined and compared to recent data on multiplicities, dispersion of multiplicities and partial cross-sections. It is found that, contrary to popular belief, the behaviour of the multiplicity predicted by his strong coupling model is still consistent with the data. Predictions about the intimate connection between the so--called KNO-scaling, and the constancy of the inelastic cross-section are discussed in the light of the present experimental results.

1. *Introduction*

Multiple particle production is one of the most interesting features and sources of new information which have come from high energy collision processes. A large number of models have been developed over the years in an attempt to analyze and explain the data. Some of the earliest models were those pioneered by Heisenberg [1–6], Fermi [7] and Landau [8]. Their semiclassical models, which are deeply rooted in physical intuition, were gradually discarded as more sophisticated models were developed. Recently Carruthers and co-workers have revived Landau's hydrodynamical model and have shown that, when recast in a more modern form, it describes the data consistently [9]. Fermi's simple phase-space model, on the other hand, does nor agree with the data.[1] Heisenberg's model which is based on a shock-wave-like expansion of very strongly interacting meson fields has fallen into disrepute, mainly because of its asymptotic prediction of a linear growth of the meson multiplicity with the energy in the center of mass system (cms). Continuing the reanalysis of the older models, we have taken another look at the physical ideas and intuition of Heisenberg, and have found that his strong-coupling model is still consistent with the present data.

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[1] There are two important consequences of Fermi's model which are not satisfied in nature. First the ratio of heavier secondaries to pions comes out too high; second, the prediction for the transverse momentum distribution of secondaries is wrong. The thermodynamic model of Hagedorn tries to improve on both points [10].

Heisenberg, over the 16-year period from 1936 to 1952, carried out what are probably the first attempts to describe a multiple particle production from high energy proton-proton collisions and finally developed a definite predictive model based on a combination of the classical and quantum mechanical ideas [5]. He assumed that the energy released during the collision process could be described in terms of the geometric overlap of the meson fields of the two nucleons, *i.e.* a space integral over the Fourier transforms of the hadronic form factors. The latter depend directly on the classical impact parameter. The energy goes into a meson field which propagates outwards as a shock wave whose amplitude, it is assumed, can be determined by a classical non-linear Lagrangian. As the field spreads out, its self-interaction dies away and particles (mesons) coalesce. From a description of the interaction, one is able to derive a meson spectrum which can be compared with the data.

The Landau model is similar to the Heisenberg model in many respects except that transport of the energy released in the collision occurs through the relativistic hydrodynamical expansion of a hadronic fluid. The energy dependence of the particle multiplicity is given by the square root of the cms energy. The Heisenberg model may be regarded, in some respects, as an extreme version of such a Landau fluid model. Adopting a "phenomenological" point of view, we shall discuss in Section 2 a class of models labeled by the strength of the mesons' self-interaction; at one extreme we have the weak coupling or bremsstrahlung model, at the other Heisenberg's strong coupling model [4]. We shall also revive the concept of the overlap function which allows for a natural explanation of the observed "leading particle" effect. In Section 3 the models are compared with the data on the average multiplicity, its dispersion and the n-particle cross sections; the latter are approached from the standpoint of the recently discovered KNO-scaling.

2. *Theoretical aspects of Heisenberg-like models*

The models of high energy collisions, which we are considering, exhibit two important features: The interaction dynamics determine the production mechanism of the secondaries from the energy ε, and the geometry of the collision determines what fraction $\gamma(b)$ this energy is of the maximum available cms energy Q,

$$\varepsilon = \gamma(b)Q, \tag{1a}$$

where

$$Q = \sqrt{s} - m_A - m_B. \tag{1b}$$

We have included the assumption that the incoming particles retain their identity (and masses m_A and m_B), except for a possible change of their charge states. The ansatz (1a) accounts for the empirical leading particle effect in high energy hadron collisions. We also assume that the amount of energy released can be described by the overlap of the two hadrons in the cms, so that the inelasticity is a function of the impact parameter b of the collision.

2.1. Dynamics and the energy spectrum

The main idea of Heisenberg's theory (1952) is that the energy ε is converted into a meson field which is described by a nonlinear (classical) Lagrangian. The field then expands as a shock wave from which, after it has dissipated, free mesons emerge. The number of produced mesons is proportional to the intensity of the wave front and, therefore, to the amount of energy deposited in the meon field. Thus by solving the equantions of motion for the nonlinear meson field, one is able to obtain the meson multiplicity.

Heisenberg treated two special cases: a weak coupling model (WCM), which is representative of a renormalizable theory with a small coupling constant, and a strong coupling model (SCM), in which the mesons have maximally strong interaction [5]. A particular example of an equation of motion which leads to the WCM is

$$\Box\varphi+\kappa^2\varphi+\eta\varphi^3 = 0, \tag{2}$$

where φ is the meson field, κ its mass and η a coupling parameter. When solved in one space dimension, Eq. (2) has the solution

$$\varphi(\xi) = a[1-0.25(\kappa^2+\eta a^2)\xi+\ldots] \tag{3}$$

with $\xi = (ct)^2-x^2$. In Eq. (3) a is an arbitrary constant because of the quasilinearity of Eq. (2). To leading order the field amplitude close to the light cone $\xi = 0$ is constant and identical to the free field case ($\eta = 0$).

As an example of strong coupling Heisenberg took Born's nonpolynomial Lagrangian

$$\mathscr{L} = l^{-4}\left[1-l^4\left(\sum \frac{\partial\varphi}{\partial x_\nu}\frac{\partial\varphi}{\partial x^\nu}-\kappa^2\varphi\right)\right]^{1/2}, \tag{4}$$

with a length parameter l [11]. Integrating the equations of motion in one space dimension yields

$$\varphi(\xi) = \sqrt{\xi}[1+a'\xi+\ldots], \tag{5}$$

where a' is an arbitrary constant [5]. The strong interaction of the meson field is reflected by the initial $\sqrt{\xi}$-growth of the field amplitude close to the light cone.

One is able to calculate the number of mesons n and their energy ε by taking the Fourier transform of the classical meson field amplitude $\varphi(\xi)$ [6]. The meson spectrum in the SCM is given by

$$\frac{dn}{dk_0} = \frac{C}{k_0^2}, \tag{6a}$$

and

$$\frac{d\varepsilon}{dk_0} = \frac{C}{k_0}, \tag{6b}$$

where k_0 is the energy of a meson and C a constant. Integrating the energy spectrum Eq. (6b) from the minimum rest mass κ up to the maximum energy fixes C; then Eq. (6a) is integrated to obtain the number of produced secondaries

$$n(Q) = \frac{x-1}{\log x}, \tag{7a}$$

where

$$x = \frac{\gamma Q}{\kappa}. \tag{8}$$

Asymptotically the multiplicity (7a) rises linearly with the reaction energy Q, but at present machine energies this behaviour is partly masked by the $\log x$ denominator. The average energy of the secondaries in the SCM is

$$\bar{\varepsilon} = \frac{x\kappa \log x}{x-1} \approx \kappa \log x \quad \text{for} \quad x \gg 1. \tag{7b}$$

In the WCM the spectra

$$\frac{dn}{dk_0} = \frac{C'}{k_0}, \tag{9a}$$

and

$$\frac{d\varepsilon}{dk_0} = C' \tag{9b}$$

are obtained. Hence we find for the multiplicity

$$n(Q) = \frac{x}{x-1} \log x. \tag{10a}$$

In the case of electromagnetic coupling this model corresponds to a bremstrahlung theory. Note that the bremsstrahlung spectrum Eq. (9b) determines in Eq. (10a) the factor in front of the logarithm which is essentially unity. The average energy of a secondary in the WCM becomes

$$\bar{\varepsilon} = \frac{\kappa(x-1)}{\log x}, \tag{10b}$$

i.e., it increases nearly linearly with the energy Q for $x \gg 1$.

These two examples represent limiting cases for the expected strength of meson field interactions. Since we expect to construct Lagrangians leading to a meson field of any intermediate self-coupling strength, we generalize Eqs (6) and (9) to [4]

$$\frac{dn}{dk_0} = \frac{C''}{k_0^{1+\alpha}} \tag{11a}$$

and

$$\frac{d\varepsilon}{dk_0} = \frac{C''}{k_0^{\alpha}}, \quad 0 < \alpha < 1, \tag{11b}$$

from which the multiplicities

$$n(Q) = x^{\alpha}\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1-x^{-\alpha}}{1-x^{-1+\alpha}}\right) \tag{12}$$

are derived. In a model with $\alpha = 1/2$, the "medium-strong" coupling model (MSCM), one finds

$$n(Q) = \sqrt{x} = (\gamma Q/\kappa)^{1/2}, \tag{12a}$$

and

$$\bar{\varepsilon} = (\kappa\gamma Q)^{1/2}. \tag{12b}$$

Formally this energy dependence of the multiplicity is the same as that in the Fermi-Landau hydrodynamical model. However, we note that the constant factor in Eq. (12a) is completely determined by the energy spectrum of Eq. (11b).

The mass κ determines the energy scale for meson production. Heisenberg took $\kappa = 0.14$ GeV. However, there are some empirical facts which are not taken into account by the theory. First, the production mechanism does not consider the different pion charge states. In addition to the obvious constraint of charge conservation, recent high energy data exhibit correlations between pions. One can roughly say that pions tend to be created in clusters, which contain on the average about two or three pions. Secondly, the derivation of the present theory is strictly one-dimensional so nothing can be deduced about the angular or transverse spectra of the secondaries. Taking advantage of the empirical fact that the average transverse momentum of the pion is approximately energy independent and of the order of 300 MeV, we use an effective (longitudinal) pair mass κ which is in the range of 0.5 to 1.0 GeV.[2]

2.2. Overlap function and inelasticity

Because quantum effects and other possible sources of fluctuations were not included in this semiclassical approach, the number of secondaries in a reaction with fixed (γQ) is uniquely determined. For collisions at a given energy Q, the variation in the multiplicity is obtained by equating γ with the overlap function, which is defined as the space integral over the product of the two nucleons' Fourier-transformed form factor. The physical picture is that the overlapping parts of the meson clouds are stripped off of the nucleons (Fig. 1). Thus the fraction of the total energy which goes into the meson production is given by the overlap function's dependence on the impact parameter b.

The average multiplicity is calculated by simply averaging Eq. (7a), (10a), or (12a) over all impact parameters, $0 \leqslant b \leqslant b_{max}$, where b_{max} is determined by requiring that there be just enough energy in the meson field to produce one pion or pion pair. This minimim multiplicity is obtained in all our models (WCM, MSCM, and SCM) by equating $x = \gamma Q/\kappa$ to unity.

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[2] Of course it is possible, in principle, to extend this one-dimensional analysis of the Heisenberg model to obtain the angular distribution of the secondaries. We expect that the result is, to the first approximation, described by a similar calculation in Landau's liquid drop model [8]. As has been demonstrated recently, the rapidity data can be fitted well with the old prediction [9].

The total inelastic cross-section, average multiplicity and dispersion are given by [3]

$$\sigma = \pi b_{max}^2, \tag{13}$$

$$\langle n \rangle = b_{max}^{-2} \int_0^{b_{max}} db^2 n(Q), \tag{14}$$

and

$$\langle n^2 \rangle = b_{max}^{-2} \int_0^{b_{max}} db^2 [n(Q)]^2. \tag{15}$$

We have considered three different kinds of hadronic density distributions for the overlap function which we feel illustrate different qualitative features in Heisenberg's semiclassical approach.

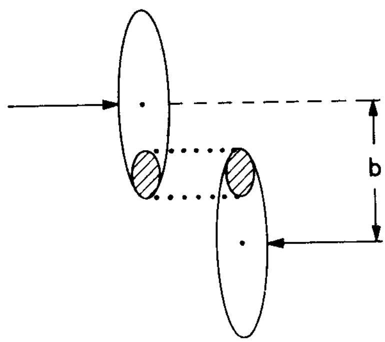

Fig. 1. Schematic illustration of two colliding hadrons: The region of maximal overlap, which is shaded, determines the amount of energy which is available for particle production

We assume first that the hadronic density is proportional to the strength of a static Yukawa meson field with mass μ. Intergration over the space at fixed b yields

$$\gamma_Y(b) = e^{-\mu b}, \tag{16a}$$

and

$$\sigma_Y = \frac{\pi}{\mu^2} (\log \hat{Q})^2, \quad (\hat{Q} = Q/\kappa). \tag{16b}$$

A Gaussian distribution is also reasonable since it is a good approximation to the proton's electromagnetic form factor [12]. This gives

$$\gamma_G(b) = e^{-\mu^2 b^2}, \tag{17a}$$

and

$$\sigma_G = \frac{\pi}{\mu^2} \log \hat{Q}. \tag{17b}$$

[3] Here we have assumed that all of the energy ε is used for particle production. It is straightforward to generalize this ansatz by introducing a new "overlap function" $\Gamma(b)$, $\Gamma(b) = \gamma(b)\, p(b)$, where $p(b)$ is the probability of an inelastic collision at an impact parameter b.

Although they decrease exponentially, the overlap functions in the above examples extend to arbitrarily large impact parameters. This leads to an increasing b_{max} and therefore to a total cross-section that grows with the energy.

Another possibility is to assume a finite extension for the nucleon so that

$$\gamma_F(b) = 0, \quad \text{for} \quad b \geqslant b_0, \tag{18a}$$

and

$$\sigma_F = \pi b_0^2, \quad b_0 = 2/\mu. \tag{18b}$$

Since b_0 is finite, the asymptotic cross-section is constant.

We have no convincing theoretical reasons to prefer one overlap function or the other. If one believed in the meson cloud structure of high energy hadrons one would prefer the model of Eqs (16), but measurement of the electromagnetic form factor yields a different distribution of the charged proton density. As long as there is no reliable theory we look to the data for suggestions as to the correct structure of the overlap integral. We only assume that it is a falling function of b.[4] The overlap functions lead directly to an average inelasticity, defined by

$$\langle\gamma(Q)\rangle = b_{max}^{-2} \int_0^{b_{max}} db^2 \gamma(b). \tag{19}$$

We obtain

$$\langle\gamma_Y\rangle = \frac{2}{(\log \hat{Q})^2}\,[1-(1+\log \hat{Q})/\hat{Q}] \xrightarrow[\text{large } \hat{Q}]{} \frac{2}{(\log \hat{Q})^2}, \tag{16c}$$

$$\langle\gamma_G\rangle = \frac{1}{\log \hat{Q}}\left[1-\frac{1}{\hat{Q}}\right] \xrightarrow[\text{large } \hat{Q}]{} \frac{1}{\log \hat{Q}}, \tag{17c}$$

$$\langle\gamma_F\rangle = f(b_{max}) \xrightarrow[\text{large } \hat{Q}]{} \text{const.}, \tag{18c}$$

where $f(b_{max})$ is an algebraic expression with no Q-dependence and Y, G, F refer to the Yukawa, Gaussian and finite overlap integrals, respectively. Combining Eqs (16b), (17b) and (18b) with (16c), (17c), and (18c) respectively, we find that the quantity $\langle\gamma\rangle\sigma$ is constant for not too low energies. Since the cosmic ray data always measures this product, information about the two quantities separately would be most useful [14].

3. *Data analysis of the models*

When comparing the models of Section 2 to the data we differentiate between those data for which one does or does not need to specify the detailed nature of the overlap integral. An example of the latter is the average multiplicity of the secondaries. Although

[4] This means that higher multiplicities are created in collisions with small impact parameters, contrary to the assumptions of the multiperipheral model. An indication in favour of the overlap integral hypothesis might be found in a recent experiment of A. Ramananskas *et al* [13] which gives higher multiplicities to impacts where the incident proton obtains a large transverse momentum; we interpret the latter fact as due to a more central collision.

the inelasticity function $\gamma(b)$ enters the calculation of the average multiplicity *via* Eq. (14), it can be approximated by a constant $\bar{\gamma}$ which is then absorbed into the mass factor $\kappa/\bar{\gamma}$.[5]. On the other hand, in calculating the n-particle cross-sections and the dispersion of the multiplicity, the detailed structure of the overlap integral enters crucially. Some illustrative examples for particular choices of $\gamma(b)$ are given in Section 3.2.

3.1. Average multiplicity of the secondaries

We deviate from the normal line of approach by attempting to remove the incident particles' final state parameters, *i.e.* mass and charge, from the data before comparison with the model predictions. Instead of the energy variable $\sqrt{s}$, we use Q which is defined

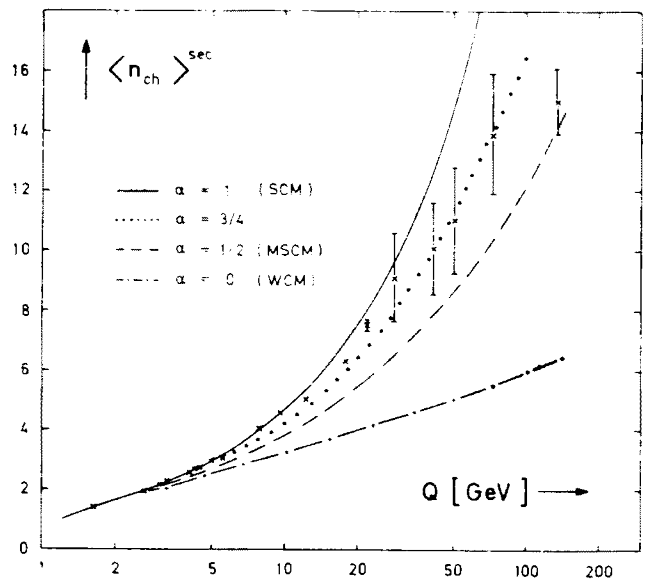

Fig. 2. Fits to the average charged multiplicity of the secondaries: The models and data analysis are described in Section 3.1 of the text: Q is the reaction energy, the parameter α is defined by Eq. (12). The data are from the complications of Ammosov *et al.* [24] and Ganguli and Malhotra [18]

in Eq. (1b). Wróblewski found that in the different reactions $\pi^{\pm}$p, $K^{\pm}$p and $\bar{\text{p}}$p the average multiplicity is a universal function of Q [15]. Since Q is the maximum energy available for the production of secondaries, their charged multiplicity is used rather than the total charged multiplicity. The experimental determination of the former is difficult, so we make a guess which seems to be consistent with the data. In pp collisions at 12 and

[5] Since this effective inelasticity $\bar{\gamma}$ differs from the average inelasticity $\langle\gamma\rangle$ in Eqs (16c), (17c) and (18c) by only a few percent, we use, with negligible error, $\langle\gamma\rangle$ in the following calculations. As a first approximation we choose $\bar{\gamma} = \langle\gamma\rangle = 0.5$ independent of the primary energy Q. This seems to agree with the results of the cosmic ray studies reported in Ref. [14] which show a roughly constant $\langle\gamma\rangle = 0.5\pm(0.10$ to $0.15)$ over a wide range of energies. However, one should note that in cosmic ray experiments, the product $\sigma\langle\gamma\rangle$ is observed rather than $\langle\gamma\rangle$ separately.

24 GeV/c lab momenta the average number of final state protons is 1.27 ± 0.02 and 1.24 ± 0.02 respectively [16]; at the ISR energy of 52 GeV, a typical event contains 1.5 protons and 0.2 antiprotons [17]. Thus we conclude, with necessary reservations, that a good energy-independent approximation to the average charged multiplicity of secondaries is

$$\langle n_{ch} \rangle^{sec} = \langle n_{ch} \rangle^{tot} - 1.3. \tag{20}$$

The (Wróblewski) universality of the dependence of $\langle n_{ch} \rangle^{sec}$ on Q indicates a production mechanism which is more or less independent of the incoming hadrons.

We make the standard assumptions that the produced particles are mostly pions and that their charge states are equally accessible to obtain

$$\langle n \rangle = 1.5 \langle n_{ch} \rangle^{sec} = 1.5(\langle n_{ch} \rangle^{tot} - 1.3). \tag{21}$$

TABLE I

Fits to the average charged multiplicity of the secondaries: Columns 6, 7, 8 parametrize results for the three models (WCM, MSCM, SCM) which are obtained from Eq. (7a), (12a), (10a), respectively. Column 5 gives the "effective" energy x scaled to a cluster mass of 0.6 GeV and average inelasticity of 1/2. The factor 4/3 appearing in the theoretical predictions accounts for the facts that only charged particles are considered (2/3) and that the secondaries occur in clusters of two pions. The data are from the compilations of Ammosov *et al.* [24] and Ganguli and Malhotra [18]

P_L[GeV/c]	Q[GeV]	$\langle n_{ch} \rangle$	$\langle n_{ch} \rangle - 1.3$	$x = \frac{Q}{1.2}$	$\frac{\langle n_{ch} \rangle - 1.3}{4/3 \frac{x \log x}{x-1}}$	$\frac{\langle n_{ch} \rangle - 1.3}{4/3\, x^{1/2}}$	$\frac{\langle n_{ch} \rangle - 1.3}{4/3 \frac{x-1}{\log x}}$
1	2	3	4	5	6	7	8
5.5	1.62	$2.76 \pm .01$	1.46	1.35	0.95	0.94	0.94
10.0	2.66	$3.22 \pm .05$	1.92	2.22	0.99	0.97	0.94
12.0	3.04	$3.43 \pm .03$	2.13	2.53	1.04	1.01	0.97
12.88	3.22	$3.57 \pm .06$	2.27	2.68	1.08	1.04	1.00
18.0	4.09	$3.85 \pm .07$	2.55	3.41	1.10	1.04	0.97
19.0	4.24	$4.02 \pm .02$	2.72	3.53	1.16	1.09	1.02
21.08	4.54	$4.02 \pm .07$	2.72	3.78	1.13	1.05	0.98
24.0	4.91	$4.25 \pm .03$	2.95	4.09	1.19	1.09	1.01
24.12	4.92	$4.15 \pm .07$	2.85	4.10	1.15	1.06	0.97
28.44	5.55	$4.33 \pm .08$	3.03	4.62	1.16	1.06	0.99
50.0	7.90	$5.35 \pm .11$	4.05	6.58	1.37	1.26	1.02
69.0	9.57	$5.89 \pm .07$	4.59	7.97	1.45	1.22	1.02
102.0	12.08	$6.34 \pm .14$	5.04	10.07	1 47	1.19	0.96
205.0	17.89	$7.65 \pm .17$	6.35	14.91	1.68	1.20	0.91
303.0	21.88	$8.86 \pm .16$	7.56	18.24	1.84	1.33	0.96
484	28.28	10.4 ± 1.5	9.10	23.6	2.07	1.40	0.95
1060	42.63	11.35 ± 1.6	10.05	35.6	2.05	1.26	0.78
1490	51.0	12.3 ± 1.8	11.0	42.5	2.15	1.26	0.74
3000	73.2	15.2 ± 2.0	13.9	61.0	2.56	1.34	0.70
10000	135.0	16.3 ± 1.1	15.0	112.5	2.36	1.06	0.46

Illustrative examples of model calculations are presented in Table I and Fig. 2. For the purpose of parametrization we have taken small clusters of two pions with an energy-independent scale mass of $\tilde{\kappa} = \kappa/\bar{y} = 1.2$ GeV.[6] None of the three models (WCM, MSCM, SCM) describes the data as well as previous empirical fits [18]. We have not made any attempt to find a "best fit" by varying the parameters, trying various models for the overlap function or considering an energy dependence for $\bar{y}$, but only present the curves to illustrate the models' qualitative features. Perhaps the best fit is shown by the dotted curve which is calculated from Eq. (12) with $\alpha = 3/4$, *i.e.* for a model intermediate to the Fermi-Landau and the Heisenberg ones.

(a) Weak Coupling or Bremsstrahlung Model (WCM):

The curve corresponding to the WCM in Fig. 2 describes the observed multiplicities rather poorly. Fitting the lower energy values exactly makes the curve fall below the ISR data points. The WCM can be brought into agreement with the higher energy observations by assuming that the mesons are produced in large clusters of, say, six pions. In this case one obtains

$$\langle n_{\mathrm{ch}} \rangle = 4\,\frac{x}{x-1}\,(\log x). \tag{22}$$

We feel that such massive clusters are not in the spirit of a bremsstrahlung type radiation model.

(b) Medium-Strong Coupling of Fermi-Landau-like Model (MSCM):

Column 7 of Table I gives the ratio of the observed multiplicities to those calculated in the MSCM, assuming clusters of two pions and an energy scale $\tilde{\kappa} = 1.2$ GeV. At the highest Batavia energies the theory gives values which are about 20% below the data points. The fit can be improved by making $\tilde{\kappa}$ energy dependent or taking clusters with more than two pions. Although the $s^{1/4}$ fits to the charged multiplicity given in the literature are quite good, we feel that our parametrization (the production of secondaries more or less independent of the through-going primaries) is more in the spirit of the Heisenberg and Landau models.

(c) Strong Coupling or Heisenberg Model (SCM):

As shown in Column 8 of Table I, a not unreasonable fit to the data up through Batavia energies is provided by the SCM. At the very large ISR and even higher cosmic ray energies the predicted multiplicities become too large. This can be interpreted either as a decrease of the effective average inelasticity $\bar{y}$ (from 50% to about 30%), as an increase of the longitudinal mass κ of the produced secondaries (from 0.3 to 0.5 GeV) or most probably as a combination of both effects. In any case we conclude that the Heisenberg model is still consistent with the present data. Clearly, a clean separation of the leading

[6] Taking clusters of three pions (preferably $\pi^+\pi^-\pi^0$) does not alter our conclusions very much. With a scale mass $\tilde{\kappa} = 2.4$ the WCM curve is raised slightly for higher energies. The SCM curve, on the other hand, does not rise as fast as in the fit in the text. Generally smaller clusters are more in the spirit of our models.

particles' charges from that of the seconadries, and determination of the energy dependence of the mean inelasticity would allow for a more definite statement. If it turns out that the average inelasticity remains constant at ISR and higher energies, an intermediate coupling model with the parameter $\alpha = 0.75$ in Eq. (12) yields a good fit to the data.

3.2. Partial cross-sections, KNO scaling and dispersion of multiplicities

Recently Koba, Nielsen and Olesen presented an analysis of the n-prong pp cross--sections, in which they noted that the data could be described by a "universal" energy--independent scaling function

$$\Psi(x_{\text{ch}}) = \langle n \rangle \frac{\sigma_n}{\sigma}$$

where

$$x_{\text{ch}} = \frac{n}{\langle n \rangle} \tag{23}$$

is the scaling variable [19]. Slattery made a through analysis of the data between 19 and 303 GeV/c and presented a good empirical scaling function [20]. Assuming that the inclusive cross-sections obey Feynman scaling directly leads to the requirement that $R = \langle n \rangle / D$, where D is the square root dispersion of the average charged multiplicity, be a constant and hence to the KNO-scaling [19]; however the question remains as to why Feynman scaling should extend down to such low cms energies ($\approx$ 6 GeV).

We have reanalyzed the data of Slattery in terms of $\langle n \rangle^{\text{sec}}$ and Q, the variables naturally preferred by the semiclassical models, and again obtain a scaling distribution.[7] In Fig. 3 we present the scaled data expressed in terms of the scaling variable

$$x = \frac{n_{\text{ch}} - 1.3}{\langle n_{\text{ch}} \rangle - 1.3}. \tag{24}$$

The success of this form for the scaling variable is perhaps not too surprising as purely empirical scaling functions have been found which use $n_{\text{ch}} - 0.9$ as the variable [21].

In the geometric models we invert the expression for $n(Q)$ to obtain the n-particle cross-sections,[8]

$$\sigma_n = \pi(b^2_{n-\frac{1}{2}} - b^2_{n+\frac{1}{2}}) \approx -\pi \frac{db^2}{dn}. \tag{25}$$

The scaling function becomes

$$\Psi\left(\frac{n}{\langle n \rangle}\right) \approx -\frac{\langle n \rangle}{b^2_{\max}} \frac{db^2}{dn}. \tag{26}$$

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[7] We mention that we have taken Slattery's data only foı convenience. The improvea analysis of the recent NAL data prefers a KNO scaling vaiiable x, Eq. (24) rather than the variable x_{ch}, Eq. (23) [20].

[8] A similar approach has been taken by J. Dias de Deus [22].

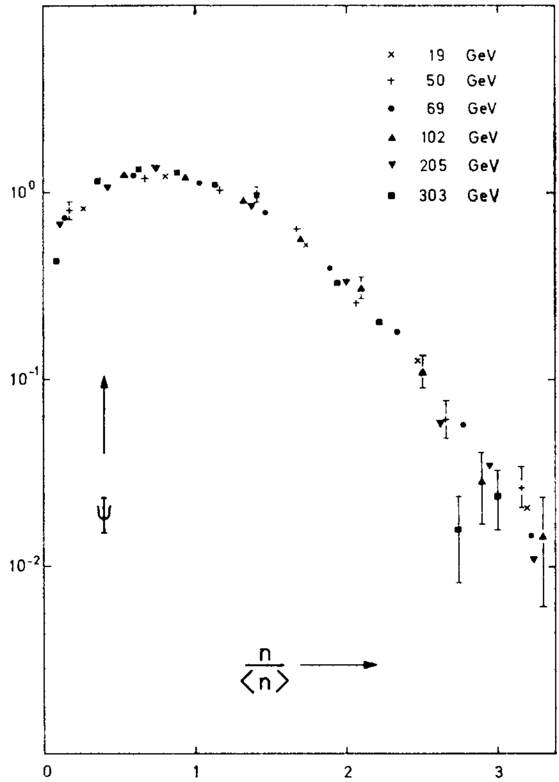

Fig. 3. The n-particle cross-sections: The energy-independent scaling variable defined by Eq. (24) in the energy range from $19 \leqslant P_L \leqslant 303$ GeV/c. Data are from the compilation of Slattery [20]

Since the conclusions about scaling in the MSCM and SCM are similar we shall only cite examples from the MSCM. The finite overlap function,

$$\gamma(b) = (1-\alpha\hat{b}^2-\beta\hat{b}^4)^2, \tag{27a}$$

with $\alpha+\beta = 1$ and $\hat{b} = b/b_{\max}$, yields the universal KNO-curve,

$$\Psi_F = \frac{C}{\beta}\left[\frac{\alpha^2}{\beta^2}+\frac{4}{\beta}\left(1-\frac{n}{\langle n\rangle}\right)C\right]^{-1/2}, \tag{27b}$$

where

$$C = \langle\gamma^{1/2}\rangle = \int_0^1 d\hat{b}^2[\gamma(b)]^{1/2}. \tag{27c}$$

The Gaussian structure for the hadronic density leads to

$$\Psi_G = \frac{2}{\hat{b}^2_{\max}}\frac{\langle n\rangle}{n} = \frac{2}{\log\hat{Q}}\frac{\langle n\rangle}{n}. \tag{28}$$

Note that in the first example Ψ_F scales since C is a constant. There is no scaling in the second one because the log Q-increase in the total inelastic cross-section causes Ψ_G to shrink with increasing Q. For the Yukawa potential, scaling fails by a factor of $(\log Q)^{-2}$.

Thus we predict that KNO-scaling breaks down if the hadronic radius grows at higher energies.

In order to qualitatively describe the scaling curve the inelasticity function must be specified more carefully. We note that a mixture of the two examples agrees qualitatively with the data. That is, for small b we use the Gaussian form factor to get Ψ decreasing with increasing n, whereas for large b we use the finite range form factor with its constant b_{max} to get a growing scaling function. In any case, we can state that KNO-scaling, if valid, is independent of Feynman scaling in our geometrical model [22].

A word should be said about the maximum multiplicity which, in our deterministic geometrical approach, is slightly less than $2\langle n\rangle$. Experiments give events with higher multiplicities, *e.g.* 22 prong events in 200 GeV/c pp collisions, which exceed $\langle n\rangle$ by nearly a factor of 3. However, since these events are fairly rare, — their cross-sections drop exponentially with prong number — we interpret them as statistical or quantum fluctuations which should be introduced into a more realistic calculation.

We add a few qualitative remarks on the dispersion of the average multiplicities. As was first pointed out by Czyżewski and Rybicki for pp collisions, the ratio of the average multiplicity to the mean square dispersion of the charged multiplicity (CR-ratio) tends to a constant for cms energies above 10 GeV, or [23]

$$R \equiv \langle n_{ch}\rangle/D_{ch} \approx 2.0, \qquad D_{ch}^2 = \langle n_{ch}^2\rangle - \langle n_{ch}\rangle^2. \tag{29}$$

The limit is reached from above. Similar features are shown by the $\pi^{\pm}$p, $K^{\pm}$p, γp and $\bar{\text{p}}$p data [24]. Since there is no explicit data for the dispersion of charged secondaries we assume that the CR-ratio in this case exhibits the same behaviour.

In our geometric models, the CR-ratio approaches its limit from above. Because there must be enough energy, in the meson field to produce at least one secondary, $b_{max} < b_0$ at lower energies. This removes the fairly flat tail of $\gamma(b)$ from the integration. For example, in the Fermi-Landau-like model with finite range overlap integral, Eq. (27a), at $\hat{Q} = 2$ ($\approx$ 3.5 GeV cms energy) one obtains $b_{max} = 0.75\ b_0$ and $R = 2.8$. In the bremsstrahlung model with Yukawa and Gaussian overlap functions the CR-ratios are $\sqrt{2}$ and $\sqrt{3}$, respectively. However, with a finite overlap function the dispersion D is constant so the CR-ratio increases with Q,

$$R = \frac{\log \hat{Q} + \langle \log \gamma\rangle}{[\langle(\log \gamma)^2\rangle + \langle \log \gamma\rangle^2]^{1/2}}\,. \tag{30}$$

Hence in the WCM the finite range overlap function does not agree with the data.

In both the SCM and the MSCM, the CR-ratio for the Yukawa and Gaussian overlap functions fall roughly as

$$R \propto (\log Q)^{-n}, \tag{31}$$

where $n = 1$, $1/2$, respectively. Although, R is not asymptotically constant in that case, the Gaussian cannot yet be ruled out. An example which agrees semiquantitatively with the data is obtained by using the finite range function for $\gamma(b)$ given by Eq. (27a) with $\alpha = 0.15$ and $\beta = 0.85$. (The same values for the parameters have been used above!)

For lab momenta up to 400 GeV/c the available data seem to prefer models with semistrong couplings and with a finite hadron radius. However, rising inelastic cross--sections at ISR and higher energies would demand in our geometric picture models with an increasing radius.

4. Conclusion

We conclude that, contrary to popular belief, Heisenberg's 1952 theory of strongly interacting mesons describes the present data on high energy hadron collisions rather well. The extra logarithmic factor in Eq. (7a) due to energy conservation softens the linear dependence on Q at present machine energies. Other power law dependencies of the multiplicity, *e.g.* Q^{α}, where $1/2 \leqslant \alpha < 1$, are also consistent with the data. An interesting feature is that, from the point of view of our model analysis, the logarithmic dependence of the multiplicity, *i.e.* the bremsstrahlung-like model, is rather strongly disfavoured. On the other hand, the SCM requires at ISR energies a decreasing inelasticity.[9]

In many respects our analysis is crude and qualitative. Because of the one-dimensional approximation used in solving the equations of motion, we are unable to say anything about the transverse momentum spectra of the secondaries. Also the production of other particles, *e.g.* kaons, nucleons, and hyperons, cannot be described by the nonlinear meson field; up to Batavia energies neglecting them seems reasonable. The theory can be extended to include non-pionic secondaries by distributing the reaction energy γQ among the higher mass modes,

$$\gamma Q = \varepsilon_{\pi} + \varepsilon_{K} + \varepsilon_{N} \ldots, \tag{32}$$

where ε_i is the energy of each mode which is to be described by its own equations of motion. Of course there is still the problem of developing a theory for the relative magnitude of the vatrious ε's as a function of the total available energy. The heavier mass modes can be incorporated by slowly increasing the scale mass κ with energy.

In these geometric models there is an intimate relationship between the finite hadronic radius and the quantities which either scale or approach a constant at higher energies, *e.g.* inelastic cross-section, average inelasticity, KNO-scaling and the CR-ratio. An increasing radius, which reflects a Gaussian or Yukawa type overlap function, results in a decreasing CR-ratio and average inelasticity. Since the present ISR data seem to indicate an increase in σ of about 10%, $\langle\gamma\rangle$ should, as is the case for the cosmic ray data and all forms of the geometric models, decrease at roughly the same rate so that $\langle\gamma\rangle\sigma$ remains constant. Also, since the observed KNO-scaling and CR-ratio depend on a finite hadronic radius, it would be interesting to see if these quantities continue to scale at higher energies. Thus data on the inelasticity parameter, its dispersion and energy dependence are crucial in helping to determine the validity of this approach. One might then be able to draw further conclusions about the precise structure of the overlap integral and the nature of the underlying dynamics.

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[9] An indication of such a behaviour was found earlier in a discussion of cosmic ray data [25].

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