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GENERALIZATION OF REGGE-BEHAVED DAMA TO PRODUCTION PROCESSES

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The properties of the generalization of a Regge-behaved dual amplitude with Mandelstam analyticity (DAMA) to production processes are studied. A procedure of analytic continuation of A_5 into the physical domain is suggested. The amplitude has necessary dual, asymptotic and resonance properties.

Recently an amplitude belonging to the class of dual models with Mandelstam analyticity (DAMA) [1] has been suggested [2]. This amplitude has a number of attractive features, which make it a good candidate for an input in the analytic S-matrix theory [3]. The amplitude [2]

$$A(s, t) = \int_0^1 dz(z/g)^{-\alpha(s,z)-1}[(1-z)/g]^{-\alpha(t,1-z)-1} \tag{1}$$

with $g = \text{const.} > 1$ and Regge trajectories depending on the Mandelstam and the integration variables multiplicatively, *i. e.*

$$\alpha(s, z) = \alpha(s(1-z)),$$
$$\alpha(s) = [\alpha(s, z)]_{z=0}, \tag{2}$$

has Regge asymptotic behaviour along any direction in the s-plane for all t if [2, 4]

$$|\alpha(s)/(\sqrt{s}\ln s)| \xrightarrow[|s|\to\infty]{} \text{const.} \geqslant 0. \tag{3}$$

Besides, if the condition

$$|\alpha(s)/\sqrt{s}| \xrightarrow[|s|\to\infty]{} \text{const.} \geqslant 0 \tag{4}$$

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is fulfilled, then the amplitude has a polynomially bounded double spectral function [5] and satisfies the Mandelstam representation with a final number of subtractions [2, 4, 5]. In the present paper we consider the generalization of the amplitude (1) to N-particle reactions [6–8] and study its properties. Preliminary results were published earlier [8].

We note that in the case of reasonable homotopies different from (2) all results presented below, possibly besides the factorization, will hold if $g = 1$ (and, of course, (3) and (4) hold). Since this case is technically more complicated we shall assume below that condition (2) is fulfilled.

1. N-Point amplitude

There is a straightforward generalization of DAMA to N-point functions [6–8]

$$A_N = \int_0^1 \prod_{j=2}^{N-2} dz_{1,j}(1/J) \prod_P (z_P/g)^{-\alpha(s_P, z_P)-1}, \tag{5}$$

where

$$J = \prod_{i<j} (z_{i,j})^{j-i-1}; \quad i = 2, 3, \ldots, N-2; \quad j = 3, \ldots, N-1.$$

Here P runs over all pairs of indices (i, j) of Mandelstam channels and

$$s_{ij} = (p_i + p_{i+1} + \ldots + p_j)^2.$$

The variables z_P satisfy dual conditions

$$z_P = 1 - \prod_{\bar{P}} z_{\bar{P}}, \tag{6}$$

where $\bar{P}$ is the channel dual to P. Dual conditions (6) for the independent variables $z_{1,j}$ read

$$z_{p,q} = \frac{(1-\omega_{p,q-1})(1-\omega_{p-1,q})}{(1-\omega_{p-1,q-1})(1-\omega_{p,q})}, \tag{7}$$

where $\omega_{p,q} = z_{1,p} z_{1,p+1} \ldots z_{1,q}$ and $z_{1,1} = z_{1,N-1} = 0$ by definition.

Amplitude (5) is 1) dual; 2) crossing-symmetric; 3) analytic. For $\mathrm{Re}\,\alpha(s_{ij}) < 0$ the integral (5) converges, *i. e.* it is an analytic function of the variables s_{ij}; 4) cyclically and anticyclically symmetric; 5) has poles on physical trajectories. At $z_P = 0$ the amplitude has a pole lying on the physical Regge trajectory, since $\alpha(s_P, 0) = \alpha(s_P)$; 6) is free of simultaneous poles in dual channels (due to condition (6)).

Other properties of the amplitude A_N will be demonstrated in the particular case, A_5.

For the five-point function we have

$$A_5 = \int_0^1 \int_0^1 dz_{12} dz_{13} (1/z_{24}) \cdot (z_{12}/g)^{-\alpha(s_{12},z_{12})-1} (z_{13}/g)^{-\alpha(s_{13},z_{13})-1} \times$$

$$\times (z_{23}/g)^{-\alpha(s_{23},z_{23})-1} (z_{24}/g)^{-\alpha(s_{24},z_{24})-1} (z_{34}/g)^{-\alpha(s_{34},z_{34})-1} \tag{8}$$

with dual conditions $z_{12} = 1-z_{23}z_{24}$, $z_{13} = 1-z_{34}z_{24}$, $z_{23} = 1-z_{12}z_{34}$, $z_{24} = 1-z_{12}z_{13}$, $z_{34} = 1-z_{13}z_{24}$.

By introducing cyclic variables $s_{i,i+1}$ and inserting $x = z_{12}$, $y = z_{13}$ we get

$$A_5(s_{i,i+1}) = (1/g)\int_0^1\int_0^1 dxdy \prod_{j=1}^{5} \varphi_j(x, y, s_{j,j+1}), \qquad s_{56} \equiv s_{51}, \tag{9}$$

where

$$\varphi_1(x, y, s) = (x/g)^{-\alpha(s,x)-1},$$
$$\varphi_2(x, y, s) = \{(1-x)/[g(1-xy)]\}^{-\alpha(s,(1-x)/(1-xy))-1},$$
$$\varphi_3(x, y, s) = \{(1-y)/[g(1-xy)]\}^{-\alpha(s,(1-y)/(1-xy))-1},$$
$$\varphi_4(x, y, s) = (y/g)^{-\alpha(s,y)-1},$$
$$\varphi_5(x, y, s) = [(1-xy)/g]^{-\alpha(s,1-xy)-2}. \tag{9a}$$

2. *Analytic continuation of A_5*

The integral representation (9) converges for all $\operatorname{Re}\alpha(s_{i,i+1}) < 0$. However, in the physical regions for the reactions shown in Fig. 1a, b, the conditions

$$\operatorname{Re}\alpha(s_{12}) > 0, \operatorname{Re}\alpha(s_{34}) > 0, \operatorname{Re}\alpha(s_{45}) > 0, \tag{10a}$$

and

$$\operatorname{Re}\alpha(s_{12}) > 0, \tag{10b}$$

hold respectively. Below, we present a procedure for analytic continuation of the amplitude (9) into the region (10a) (the analytic continuation into the region (10b) is a particular case of this procedure). Without loss of generality we shall assume that $\operatorname{Re}\alpha(s_{i,i+1}) < 0$ is equivalent to $s_{i,i+1} < 0$.

Let us write A_s in the form

$$A_5(s_{i,i+1}) = (1/g)\int_0^1 dx(x/g)^{-\alpha(s_{12},x)-1}F(x, s_{i,i+1}), \tag{11}$$

where $F(x, s_{i,i+1})$ is an integral over y from 0 to 1 with the integrand containing the remaining part of the integrand (9). Consider now the function

$$\tilde{F}(x, s_{i,i+1}) = \int_{C_1} dy \prod_{i=2}^{5} \varphi_i(x, y, s_{i,i+1})\times$$
$$\times\{1-\exp[-2\pi i\alpha(s_{34}, (1-y)/(1-xy)]\}^{-1}\times$$
$$\times\{1-\exp[-2\pi i\alpha(s_{45}, y)]\}^{-1}, \tag{12}$$

where the integration contour C_1 is shown in Fig. 2a. Now we show that this function is the analytic continuation of $F(x, s_{i,i+1})$ in s_{34} and s_{45} for every $x \in G$, where

$$G \equiv \{-\delta < \operatorname{Re} x \leqslant 1; |\operatorname{Im} x| < \varepsilon\}, \tag{13}$$

$\delta, \varepsilon \neq 0$ are positive numbers.

The integrand of (12) for $x \in G$ contains the following singularities in y:

1) fixed branch points $y = 0$, $y = 1$;

2) moving branch points

$$\tilde{y}_{23} = (s_{23}^0 - xs_{23})/[x(s_{23}^0 - s_{23})], \tag{14a}$$

$$\tilde{y}_{34} = s_{34}^0/[s_{34} + x(s_{34}^0 - s_{34})], \tag{14b}$$

$$\tilde{y}_{45} = 1 - s_{45}^0/s_{45}, \tag{14c}$$

$$\tilde{y}_{51} = s_{51}^0/(xs_{51}), \tag{14d}$$

defined by the relation $\alpha(s_{i,i+1}, z) = \alpha(s_{i,i+1}^0)$. In each variable $s_{i,i+1} < 0$ we can find restrictions on the domain G (13) such that for $x \in G$ the points (14) will not fall inside

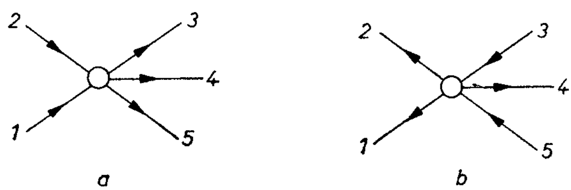

Fig. 1. Diagrams for the five-point function

the contour C_1 (alternativelly, that they will miss the segment y, $[0,1]$) being able only to press the points r_1 and r_2 of this contour to the point $y = 1$ from the right as $x \to 1$ along the real axis. The severest of these restrictions will define the domain G in which the procedure of the analytic continuation is valid.

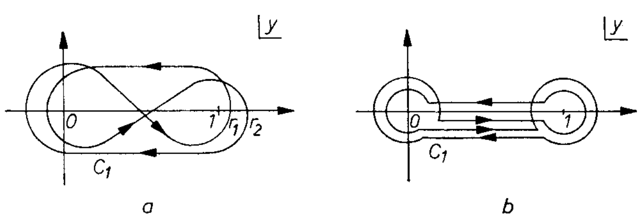

Fig. 2. Integration contour C_1 in variable y

Other branch points (as well as singularities of other type caused by the form of the trajectory $\alpha(s_{i,i+1})$) can appear for a more complicated parametrization of the trajectories only on the second (nonphysical) sheet in $s_{i,i+1}$. Therefore these points are complex, *i.e.* they can only impose additional restrictions on the domain G without affecting the procedure under consideration.

3) Simple moving poles at $y = y_m$ and $y = y_n$ defined by

$$\alpha(s_{34}, z_m) = m; \quad \alpha(s_{45}, z_n) = n; \quad m, n = 0, \pm 1, \pm 2, \ldots$$

Hence

$$y_m = [1 - z_m(s_{34})]/[1 - x z_m(s_{34})], \quad y_n = z_n(s_{45}).$$

The motion of these poles is determined by $z_{m,n}(s_{i,i+1})$, *i.e.* it depends on the choice of the trajectories. However, it is clear that when constructing the contour C_1 they can be encircled if there are no poles at $y = 0$ and $y = 1$. For y_n these poles will correspond to solutions $z_n(s_{45}) = 0$ and $z_n(s_{45}) = 1$, and for $y_m = 0$ to the solutions $z_m(s_{34}) = 1$. If $z = 1$ then $\alpha(s, z) = \alpha(0) = \text{const}$ *i.e.* there are no solutions. If $z_n = 0$ then $\alpha(s_{45}, z) = \alpha(s_{45})$ and we can find such a subdomain of the domain $s_{45} < 0$ in which the physical trajectory does not pass through integers. For poles at $y_m = 1$ the situation is more complicated: this condition is satisfied at $x = 1$ for any $z_m(s_{34})$. The study of the behaviour of such poles for $x \to 1$ from the left along the real axes shows that for $\operatorname{Im} z_m(s_{34}) \neq 0$ or $\operatorname{Re} z_m(s_{34}) \notin [0, 1]$ the poles miss the segment on the real axes $y \in [0, 1]$, they can only press the points r_1 and r_2 of the contour C_1 to the point $y = 1$. Actually, by putting $x = 1 - \lambda$, $\lambda > 0$ and $z_m = a_m + ib_m$ we get for $\lambda \to 0$

$$\operatorname{Re} y_m \simeq 1 + \lambda(a_m^2 + b_m^2 - a_m), \quad \operatorname{Im} y_m \simeq -b_m \lambda,$$

which gives the above-mentioned restrictions on the imaginary and real part of $z_m(s_{34})$. For the realization e.g. of the second of these restrictions at arbitrary m we can chose $-\mu < s_{34} < 0$, where $\mu < 0$ is a sufficiently small number. Here $\alpha(s_{34}, z_m)$ substantially differs from $\alpha(s_{34}, 1) = \alpha(0) = \text{const}$ only for sufficiently large $|z_m(s_{45})|$.

Thus, the study of singularities in y of the integrand in (12) shows that for a certain subdomain of negative $s_{i,i+1}$ we can draw a contour free of these singularities and then we can squeeze it as shown in Fig. 2b for every $x \neq 1$ belonging to the domain G. Then

$$\tilde{F}(x, s_{i,i+1}) = \int_{\varrho_1}^{1-\varrho_2} dy \cdot \prod_{i=2}^{5} \varphi_i(x, y, s_{i,i+1}) +$$

$$+ \int_{|y|=\varrho_1} dy \cdot \prod_{i=2}^{5} \varphi_i(x, y, s_{i,i+1}) \cdot \{1 - \exp[-2\pi i \alpha(s_{45}, y)]\}^{-1} +$$

$$+ \int_{|1-y|=\varrho_2} dy \cdot \prod_{i=2}^{5} \varphi_i(x, y, s_{i,i+1}) \cdot \{1 - \exp[-2\pi i \alpha(s_{34}, (1-y)/(1-xy))]\}^{-1} \tag{15}$$

and for $\varrho_1, \varrho_2 \to 0$

$$\int_{|y|=\varrho_1} \propto \varrho_1^{-\operatorname{Re}\alpha(s_{45})} \to 0, \qquad \int_{|1-y|=\varrho_2} \propto \varrho_2^{-\operatorname{Re}\alpha(s_{34})} \to 0.$$

Hence

$$\lim_{\substack{\varrho_1 \to 0 \\ \varrho_2 \to 0}} \tilde{F}(x, s_{i,i+1}) = F(x, s_{i,i+1}). \tag{16}$$

For $x = 1$ the integrand contains a complicated singularity at $y = 1$ (the contour C_1 in this case passes through $y = 1$ due to a pinch).

By setting $1-x = \lambda$ and extracting the leading term in λ in the expansion of (15) in ϱ_2 we get

$$\int\limits_{|1-y|=\varrho_2} \propto \lambda^{-\alpha(s_{51},\lambda)-1}(\varrho_2/\lambda)^{-\operatorname{Re}\alpha(s_{34})} \xrightarrow[\varrho_2=0(\lambda)]{\lambda\to 0} 0.$$

Thus

$$\lim_{\substack{\varrho_1\to 0\\ x\to 1\\ \varrho_2=o(1-x)}} \tilde{F}(x, s_{i,i+1}) = F(1, s_{i,i+1}). \tag{17}$$

Expressions (16) and (17) show that $\tilde{F}(x, s_{i,i+1})$ is the analytic continuation of $F(x, s_{i,i+1})$ in s_{34} and s_{45} for all $x \in G$.

Consider now the function

$$\tilde{A}_5(s_{i,i+1}) = (1/g) \int\limits_{C_2} dx(x/g)^{-\alpha(s_{12},x)-1}\{\exp[-2\pi i\alpha(s_{12}, x)]-1\}^{-1} \cdot \tilde{F}(x, s_{i,i+1}), \tag{18}$$

where C_2 is the contour shown in Fig. 3a, while $C_2 \in G$.

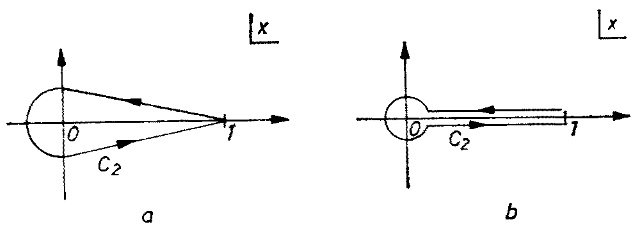

Fig. 3. Integration contour C_2 in variable x

This function is the analytic continuation of $A_5(s_{i,i+1})$ into the domain (10.a). The proof is similar to that for A_4 (in one variable) [9]. Now writing

$$\tilde{A}_5(s_{i,i+1}) = (1/g) \int\limits_{\sigma}^{1} dx(x/g)^{-\alpha(s_{12},x)-1}\tilde{F}(x, s_{i,i+1}) +$$

$$+(1/g) \int\limits_{|x|=\sigma} dx(x/g)^{-\alpha(s_{12},x)-1}\{\exp[-2\pi i\alpha(s_{12}, x)]-1\}^{-1} \cdot \tilde{F}(x, s_{i,i+1}) \tag{19}$$

and squeezing the contour C_2 as shown in Fig. 3b we get

$$\int\limits_{|x|=\sigma} \propto \sigma^{-\operatorname{Re}\alpha(s_{12})} \xrightarrow[\sigma\to 0]{} 0$$

and

$$A_5(s_{i,i+1}) = \lim_{\sigma\to 0} \tilde{A}_5(s_{i,i+1}).$$

3. Pole structure

The study of the pole structure shows that similarly to A_4 the amplitude A_5 is free from ancestors and has multifold poles.

Consider, for example, simultaneous poles in the channels (12) and (45)

$$\alpha(s_{12}, x_m) = m, \tag{20a}$$

$$\alpha(s_{45}, y_n) = n. \tag{20b}$$

The poles in the amplitude in s_{12} and s_{45} appear for $m, n \geqslant 0$ when the singularities x_m and y_n collide with the branch pooints $x = 0$ and $y = 0$. To determine the structure of these poles we consider only the integral along the contour $|y| = \varrho_1$ (which gives poles in s_{45}) in the expression (15) and the integral along $|x| = \sigma$ (which gives poles in s_{12}) in the expression (19). The residue in the simultaneous pole (20a, b) in variables x and y equals

$$R_{mn} = G_{mn}\varphi_2(x_m, y_n, s_{23})\varphi_3(x_m, y_n, s_{34}) \times$$
$$\times \varphi_5(x_m, y_n, s_{51})\psi(x_m, s_{12})\psi(y_n, s_{45})/(x_m^{m+1} y_n^{n+1}),$$

where

$$\psi(z, s) = [d\alpha(s, z)/dz]^{-1} \tag{21}$$

and the functions φ_i are given by (9a). By expanding $\psi(x_m, s_{12})\varphi_2$ in x_m, $\psi(y_n, s_{45})\varphi_3$ in y_n and φ_5 in $(x_m y_n)$ we get

$$R_{mn} = G_{mn} x_m^{-(m+1)} y_n^{-(n+1)} \sum_{k,l,t=0}^{\infty} c_{klt}(s_{12}, s_{45}) \left[\frac{\partial^k \varphi_2}{\partial x^k}\right]_{x=0} \times$$
$$\times \left[\frac{\partial^l \varphi_3}{\partial y^l}\right]_{y=0} \left[\frac{\partial^t \varphi_5}{\partial (xy)^t}\right]_{xy=0} x_m^{k+t} y_n^{l+t}.$$

A further expansion gives

$$R_{mn} = \frac{G_{mn}}{x_m^{m+1} y_n^{n+1}} \sum_{k,l,t=0}^{\infty} \left[\frac{\partial^t \varphi_5}{\partial (xy)^t}\right]_{\substack{x=0\\y=0}} \sum_{p=0}^{\infty} \left[\frac{\partial^{l+p} \varphi_3}{\partial x^p \partial y^l}\right]_{\substack{x=0\\y=0}} \times$$
$$\times \sum_{q=0}^{\infty} \left[\frac{\partial^{k+q} \varphi_2}{\partial x^k \partial y^q}\right]_{\substack{x=0\\y=0}} c_{klt}^{pq}(s_{12}, s_{45}) x_m^{k+t+p} y_n^{l+t+q}.$$

Computing the derivatives we get

$$R_{mn} = \frac{\tilde{G}_{mn}}{x_m^{m+1} y_n^{n+1}} \sum_{k,l,t=0}^{\infty} B_t(s_{12}) B_k(s_{23}) B_l(s_{34}) \sum_{p=0}^{l} \sum_{q=0}^{k} c_{klt}^{pq} x_m^{k+t+p} y_n^{l+t+q}, \tag{22}$$

where $B_n(s)$ is a polynomial of n-th degree in s.

Finally, by extracting the resonance contributions we get the residue at the pole $\alpha(s_{12}) = i$, $\alpha(s_{45}) = j$

$$\text{Res } A_5 \propto \sum_{\mu \to 0}^{\min(i,j)} c_\mu(s_{12}, s_{45})\,(s_{51})^\mu B_{i-\mu}(s_{23}) B_{j-\mu}(s_{34}), \tag{23}$$

which is a polynomial (the powers of the polynomial indicate the absence of ansestors). One can also see from (22) that the amplitude has multifold poles, the $(m+1)$-fold pole appearing first on the m-th daughter level.

4. *Factorization on the leading trajectory and threshold behaviour*

For the study of the factorization properties of A_5 we restrict ourself to the analytic continuation in s_{12} only:

$$A_5(s_{i,i+1}) = (1/g) \int_{C_2} dx(x/g)^{-\alpha(s_{12},x)-1}\{\exp[-2\pi i\alpha(s_{12}, x)]-1\}^{-1} \cdot F(x, s_{i,i+1}),$$

where F is defined in section 2.

Consider factorization of A_5 at the pole $\alpha(s_{12}) = m$ belonging to the leading trajectory. It is evident that only the pinch resulting from the collision of a moving pole at $x = x_m$ (see (20.a)) with a fixed branch point $x = 0$ contributes to the resonance term. The residue at the pole $x = x_m$ equals

$$R_m = G_m x_m^{-(m+1)} \Xi(x_m, s_{i,i+1}),$$

where

$$\Xi(x_m, s_{i,i+1}) = \psi(x_m, s_{12}) \int_0^1 dy \prod_{i=2}^{5} \varphi_i(x_m, y, s_{i,i+1}).$$

The resonance contribution corresponding to simpler poles in s_{12} is

$$R_m^{\text{s.p.}} = \frac{G_m}{m!} \cdot \left[\frac{\partial^m \Xi(x_m, s_{i,i+1})}{\partial x_m^m}\right]_{x_m=0} \tag{24}$$

or

$$R_m^{\text{s.p.}} = \frac{G_m}{m!}\left\{\sum_{k=0}^{m} \binom{k}{m} \frac{\partial^{m-k}\psi}{\partial x_m^{m-k}} \int_0^1 dy \varphi_4 \sum_{l=0}^{k} \binom{l}{k} \frac{\partial^{k-l}\varphi_2}{\partial x_m^{k-l}} \times\right.$$

$$\left. \times \sum_{n=0}^{l} \binom{n}{l} \frac{\partial^{l-n}\varphi_5}{\partial x_m^{l-n}} \frac{\partial^n \varphi_3}{\partial x_m^n}\right\}_{x_m=0}. \tag{24a}$$

Terms of this expansion containing the powers $(s_{51})^\mu (s_{23})^{m-\mu}$, $\mu = 0, 1, 2, \ldots, m$ correspond to the leading trajectory. Noting that

$$\left[\frac{\partial^{k-l}\varphi_2}{\partial x_m^{k-l}}\right]_{x_m=0} = B_{k-l}(s_{23}), \qquad \left[\frac{\partial^{l-n}\varphi_5}{\partial x_m^{l-n}}\right]_{x_m=0} = B_{l-n}(s_{51}),$$

we obtain the following relation for the summation indices

$$(k-l)+(l-n) = k-n = m.$$

Hence $n = 0$, $k = m$, since $n \leqslant k \leqslant m$. Thus, on the leading trajectory

$$\operatorname*{Res}_{\alpha(s_{12})=m} A_5 \propto G_m(s_{12}) \int_0^1 dy\varphi_4 \sum_{l=0}^{m} c_{lm}(s_{23}, s_{51}) \left\{\frac{\partial^{m-l}\varphi_2}{\partial x_m^{m-l}} \frac{\partial^l \varphi_5}{\partial x_m^l} \varphi_3\right\}_{x_m=0},$$

and after evaluating the derivatives

$$\operatorname*{Res}_{\alpha(s_{12})=m} A_5 \propto \sum_{n=0}^{m} \tilde{G}_{mn}(s_{12}, s_{23}, s_{51}) A_4(-\alpha(s_{34})-n, -\alpha(s_{45})). \tag{25}$$

In particular, for $\alpha(s_{12}) \approx 0$

$$A_5 \approx \frac{G_0}{\alpha(s_{12})} A_4(s_{34}, s_{45}). \tag{26}$$

Thus, on the leading trajectory the amplitude A_5 factorizes: the residue at the pole is expressed through a sum of amplitudes of lower multiplicity. To prove this we have used the analytic continuation in one variable which holds for A_N at arbitrary N. Therefore our results concerning the factorization properties hold for A_N as well. This is not the case on the daughter levels. Even for simple poles of A_5 we obtain by removing (24.a)

$$R_m^{\text{s.p.}} \propto \sum_{k=0}^{m} \Bigg\{ c_k(s_{i,i+1}) A_4(-\alpha(s_{34})+k, -\alpha(s_{45}))+$$

$$+\sum_{n=1}^{k} \tilde{c}_{kn}(s_{i,i+1}) \int_0^1 dy \left(\frac{y}{g}\right)^{-\alpha(s_{45},y)+k-n-1} \frac{\partial^n}{\partial x^n}\left[\frac{1-y}{g(1-xy)}\right]_{x=0}^{-\alpha(s_{34},(1-y)/(1-xy))-1} \Bigg\}.$$

Besides terms containing A_4 here appear also other terms, which can not be reduced to A_4. Their presence is associated with the introduction to the trajectory of the dependence on the integration variable (and, thus, with the appearance of multifold poles in the amplitude).

By using the factorization properties of the amplitude it is possible to reduce the threshold behaviour problem of A_N to that of A_4. Actually, the threshold behaviour in one variable is independent of other variables. Therefore by considering A_N near the poles on the leading trajectories in these variables we can reduce the amplitude and thus use the results obtained for A_4 (Ref. [10]).

5. *Asymptotic behaviour*

A five-point amplitude must have two asymptotic limits, namely, single (a) and double (b) Regge limits (Fig. 4):

$$\text{a)}\quad A_5 \sim f_1(s_{23}, s_{34}, s_{51}, \eta)\,(s_{45})^{\alpha(s_{51})}$$

$$\text{for}\quad s_{12}, s_{45} \to \infty, \qquad \eta = s_{45}/s_{12} = \text{fixed};$$

$$\text{b)}\quad A_5 \sim f_2(s_{23}, s_{51}, \zeta)\,(s_{34})^{\alpha(s_{23})}(s_{45})^{\alpha(s_{51})}$$

$$\text{for}\quad s_{12}, s_{34}, s_{45} \to \infty, \qquad \zeta = s_{34}\,s_{45}/s_{12} = \text{fixed}; \qquad s_{23}, s_{51} \;-\; \text{fixed}$$

The proof of the asymptotic behaviour of the amplitude (9) requires special attention (see a similar proof [11] for A_4). Here we present only arguments supporting the existence

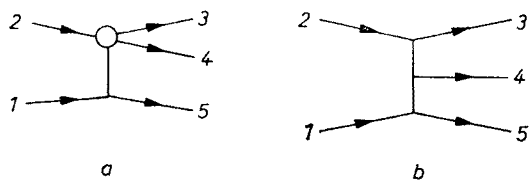

Fig. 4. Single (*a*) and double (*b*) Regge limits for a five-point function

of this behaviour for trajectories that increase not faster than the square root. For this purpose we rewrite (9) by using cyclic symmetry in the form

$$A_5 = (1/g) \int_0^1 \int_0^1 dx dy \prod_{i=1}^{5} \varphi_i(x, y, s_{i+1,i+2}), \qquad s_{67} \equiv s_{12}. \tag{27}$$

If for $|s| \to \infty$

$$|s^{-1/2}\alpha(s)| \to c_1 \geqslant 0,\ \operatorname{Re}\alpha(s) \to c_2 < -1 \qquad \text{or} \qquad \operatorname{Re}\alpha(s) \to -\infty \tag{28}$$

then the integrand in (27) will be polynomially bounded (for $|s_{i,i+1}| \to \infty$) along any ray on the physical sheet of these variables. Hence in this case the amplitude $A_5(s_{i,i+1})$ will be also polynomially bounded in the whole 5-dimensional complex space formed by $s_{i,i+1}$ (and will satisfy a dispersion relation). If the asymptotic condition (28) for $\operatorname{Re}\alpha(s)$ is not fulfilled we can arrive at the same result after a final number of subtractions.

In the double Regge limit the main contribution to the integral (27) comes from the region near the saddle points $x \approx 1/s_{34}$, $y \approx 1/s_{45}$. However, in this region the homotopies which depend on s_{23} and s_{51} reduce to physical trajectories and those which depend on s_{12}, s_{34} and s_{45} reduce to linear functions of its Mandelstam variables, *i.e.* the asymptotic behaviour of A_5 is similar to that of the Bardakci-Ruegg formula [12]

$$A_5 \sim G(s_{23}, s_{51}, \zeta)\,(-gs_{34})^{\alpha(s_{23})}(-gs_{45})^{\alpha(s_{11})},$$

where

$$G(s_{23}, s_{51}, \zeta) = g^7 \int_0^\infty \int_0^\infty dv dw\, w^{-\alpha(s_{51})-1} v^{-\alpha(s_{23})-1} g^{\alpha(-v)+\alpha(-w)+\alpha(-vm/\zeta)}.$$

The situation in the single Regge limit is similar.

6. *Summary and discussion*

The results presented above show that:

1. DAMA can be generalized without loosing its basic properties to processes involving N particles.

2. By using a Pochhamer-like method one can analytically continue the amplitude A_5 into the physical region of the Mandelstam variables s_{ij}. However, even for $N = 5$ the analytic continuation in all variables into the region $\mathrm{Re}\,\alpha(s_{ij}) \geqslant 0$ is cumbersome.

3. The analytic continuation suggested in sec. 2 can be used for applications.

4. The generalized amplitude has correct factorization properties on the leading trajectory. Factorization properties on the daughter trajectories require special consideration.

Finally, we mention the important problems, which require further investigation:

A detail study of asymptotic properties of the generalized amplitude is to be performed. This may yield new bounds on the asymptotic behaviour of the trajectories.

What is Mandelstam analyticity for an N-point function? Maybe the explicit example of DAMA will answer this question.

Another important problem of immediate interest is connected with the applications of DAMA to production processes.

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