# REDUCTION OF THE MODIFIED QUANTUM-MECHANICAL EQUATION FOR THE SYSTEM OF CHARGED DIRAC PARTICLES TO THE SUBSPACE OF POSITIVE ENERGY STATES. PART II

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(*Received November 21, 1972*)

The problem of reducing the new quasirelativistic equation (including radiative corrections to the order $c^{-3}$) proposed in the previous paper is presented. The well-known iterative Foldy-Wouthuysen method is simplified by introducing an intermediate scheme (generated by the generalized Case transformation) in the investigation of this equation. Supplementing this transformation by two simple unitary transformations, which have only an auxiliary meaning, brings this Hamiltonian to an even form. It is shown that the interpretation of this modified equation is free from difficulties and ambiguities of the Breit equation. All calculations in this paper are restricted to the $c^{-3}$ term.

## 1. *Introduction*

The presented paper is a continuation of investigations of the quantum-mechanical equations describing a system of charged Dirac particles (including radiative corrections), commenced in Part I (Janyszek [1]) where a modified quasirelativistic equation (extended by including interaction terms of the Pauli type) has been proposed. This equation is different from the modified Breit equation discussed widely by Chraplyvy [2], Hegstrom [3], [4] and is an extension of the generalized Bethe-Salpeter quantum-mechanical equation discussed in papers of Hanus and Janyszek [5]–[7]. This generalized Bethe-Salpeter equation differs form the Breit equation in the mutual particle interaction term (taking into account the hole theory postulate) and it appears to be more useful in the description of a system of charged Dirac particles than the Breit equation. Usually the starting point for practical applications is not the ordinary quasirelativistic equation (expressed in product space of Dirac spinors of the particles forming the system), but the reduced equation describing the system of particles in positive energy states by means of a respective effective Hamiltonian containing terms giving account of relativistic and radiative corrections. Such an effective Hamiltonian has been obtained in the quoted papers [2]–[4] using the iterative method of Chraplyvy (this being a generalization of the well known Foldy-Wouthuysen [8] iterative method for two particles). However, in order to obtain

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such a correct effective Hamiltonian in this manner, omission was required of the controversial term proportional to the fourth power in charge, which appears in the reduced Hamiltonian (widely discussed by Bethe [9] and the quoted papers [5]–[7]). This term stems from the Breit interaction structure (not taking into account the hole theory postulate). Moreover, as it is known, such an iterative method applied to the Breit equation cannot be generalized for the case of an arbitrary number of particles (if the Breit equation is generalized for a system of $N$-particles) Owing to this, in this paper we shall consider the question of reducing the equation proposed in I. Moreover, the iterative method will be considerably simplified by introducing an intermediate scheme generated by the generalized Case transformation (these being a product of the Case transformations for respective particles), which plays a considerable role in the investigation of this equation. This unitary transformation does not lead to a Hamiltonian of an even form but leads to a partially even form by introducing ordering of its terms with respect to their physical meaning. Now, in this Hamiltonian the odd terms are taken away by applying an additional unitary transformation according to the iterative FW procedure which has no influence on the, already correct, even part. Calculations carried out in [7] have been restricted to the order $c^{-2}$. As the required accuracy of the proposed equation is $c^{-3}$, derivation will be carried out up to this accuracy. Considerations of this kind have already been carried out for the case of one particle in an external electromagnetic field, by Hanus and Mrugała [10], [11].

## 2. *Transformation to the intermediate scheme*

As the starting point in our considerations we take the modified $N$-particle equation which, according to the notation assumed in paper I, has the following form

$$i\hbar\frac{\partial\Psi}{\partial t} = H'\Psi, \qquad H' = \sum_K H'_K + \sum_{K,L}{}' W'_{K,L}, \qquad (K, L = \mathrm{I}, \mathrm{II} \dots N), \qquad K \neq L, \tag{1}$$

where

$$H'_K = H_K - \varrho_{3,K} g_K^{(1)} \mu_K \mathscr{H}_K^{\mathrm{ex}} - \varrho_{2,K} g_K^{(1)} \mu_K E_K^{\mathrm{ex}} + \\ - g_K^{(2)} \frac{4\pi e_K \hbar^2}{m_K^2 c^2}\left(\varrho_K^{\mathrm{ex}} - \frac{1}{c}\varrho_{1,K}\sigma_K j_K^{\mathrm{ex}}\right), \tag{2}$$

$\mu_K = e_K\hbar\sigma_K/2m_K c$ stands for the magnetic moment of the $K$-th particle order ($c^{-1}$), $g_K^{(1)}$ and $g_K^{(2)}$ denote dimensionless constants in the coupling constants for the additional interactions (of the order 1 for proton and $c^{-1}$ for electron)

$$H_K = \varrho_{3,K} m_K c^2 + c\varrho_{1,K}\sigma_K\left(p_K - \frac{e_K}{c}A_K^{\mathrm{ex}}\right) + e_K\Phi_K^{\mathrm{ex}}, \tag{3}$$

$$\mathscr{H}_K^{\mathrm{ex}} = \mathrm{rot}_K A_K^{\mathrm{ex}}, \qquad E_K^{\mathrm{ex}} = -\mathrm{grad}_K \Phi_K^{\mathrm{ex}}, \tag{4}$$

$$\Delta_K A_K^{\mathrm{ex}} = -\frac{4\pi}{c} j_K^{\mathrm{ex}}, \qquad \Delta_K\Phi_K^{\mathrm{ex}} = -4\pi\varrho_K^{\mathrm{ex}}, \tag{5}$$

$\varrho_K^{\text{ex}}$ and $\boldsymbol{j}_K^{\text{ex}}$ are charge and current densities from sources of external field in the place of the $K$-th particle, respectively

$$W'_{K,L} = \frac{1}{4}\Big[\lambda_K + \lambda_L, \frac{1}{2} V_{K,L} - \varrho_{3,K} g_K^{(1)} \mu_K \mathscr{H}_{K,L} - \varrho_{2,K} g_K^{(1)} \mu_K \tilde{\boldsymbol{E}}_{K,L} + \\ - g_K^{(2)} \frac{4\pi e_K \hbar^2}{m_K^2 c^2}\Big(\varrho_{K,L} - \frac{1}{c}\,\varrho_{1,K}\boldsymbol{\sigma}_K \boldsymbol{j}_{K,L}\Big)\Big]_+, \tag{6}$$

$\varrho_{K,L}$ and $\boldsymbol{j}_{K,L}$ are charge and current densities of the $L$-th particle in the place of the $K$-th particle respectively. $V_{K,L}$ represents the Brait interaction of particles $K, L$ and $\lambda_K$ is the kinetical energy signum operator of the $K$-th particle

$$V_{K,L} = \frac{e_K e_L}{r_{K,L}} - \frac{1}{2}\, e_K e_L \varrho_{1,K}\varrho_{1,L} J_{K,L},\; J_{K,L} = \frac{\boldsymbol{\sigma}_K \cdot \boldsymbol{\sigma}_L}{r_{K,L}} + \frac{(\boldsymbol{\sigma}_K \cdot \boldsymbol{r}_{K,L})(\boldsymbol{\sigma}_L \cdot \boldsymbol{r}_{K,L})}{r_{K,L}^3}, \tag{7}$$

$$\lambda_K = T_K \cdot [(T_K^2)]^{-1/2}, \quad T_K = \varrho_{3,K} m_K c^2 + c\varrho_{1,K}\boldsymbol{\sigma}_K\Big(\boldsymbol{p}_K - \frac{e_K}{c}\, A_K^{\text{ex}}\Big). \tag{8}$$

$\mathscr{H}_{K,L}$ and $\tilde{\boldsymbol{E}}_{K,L}$ are fields produced by the $L$-th particle in the place of the $K$-th particle, $\boldsymbol{A}_{K,L}$ and $\Phi_{K,L}$ are operator generalizations of the potential for the case of mutual interaction terms

$$\mathscr{H}_{K,L} = \text{rot}_K\, A_{K,L}, \qquad \boldsymbol{E}_{K,L} = -\text{grad}_K\, \Phi_{K,L}, \tag{9}$$

$$\Delta_K \boldsymbol{A}_{K,L} = -\frac{4\pi}{c}\, \boldsymbol{j}_{K,L}, \qquad \Delta_K \Phi_{K,L} = -4\pi\varrho_{K,L}, \tag{10}$$

$$\boldsymbol{A}_{K,L} = \varrho_{1,L}\hat{\boldsymbol{a}}_{K,L} + \tfrac{1}{2}\,\varrho_{3,L}\boldsymbol{d}_{K,L}^{(m)}, \tag{11}$$

$$\Phi_{K,L} = \frac{e_L}{r_{K,L}} + \frac{1}{2}\,\varrho_{2,L} d_{K,L}^{(e)}, \tag{12}$$

$$\hat{\boldsymbol{a}}_{K,L} = \frac{e_L}{2}\left\{\frac{\boldsymbol{\sigma}_L}{r_{K,L}} + \frac{(\boldsymbol{\sigma}_L \cdot \boldsymbol{r}_{K,L}) \cdot \boldsymbol{r}_{K,L}}{r_{K,L}^3}\right\}, \qquad \boldsymbol{d}_{K,L}^{(m)} = g_L^{(1)}\Big(\boldsymbol{\mu}_L \times \frac{\boldsymbol{r}_{K,L}}{r_{K,L}^3}\Big),$$

$$d_{K,L}^{(e)} = g_L^{(1)}\boldsymbol{\mu}_L \frac{\boldsymbol{r}_{K,L}}{r_{K,L}^3}, \tag{13}$$

$d_{K,L}^{(e)}$ and $\boldsymbol{d}_{K,L}^{(m)}$ are the contributions to the potentials which have their source in additional "electric" and magnetic dipol moments of the $L$-th particle.

We construct the unitary operator

$$U = U_{\text{I}} \cdot U_{\text{II}} \ldots U_K, \tag{14}$$

where $U_K$ denotes the Case transformation operator depending on $K$-th particle variables

$$U_K = \exp(iS_K), \quad S_K = \frac{1}{2}\,\varrho_{2,K}\, \text{tg}^{-1}\Big(\frac{z_K}{m_K c}\Big), \quad z_K = \boldsymbol{\sigma}_K\Big(\boldsymbol{p}_K - \frac{e_K}{c}\, A_K^{\text{ex}}\Big). \tag{15}$$

Using the well-known expressions

$$H'^{U} = \exp(iS)H'\exp(-iS) = H'+i[S, H']_- + \frac{i^2}{2!}[S,[S,H']_-]_- + \dots \tag{16}$$

where $S = \sum_K S_K$ and expressing $S_K$ by the power series

$$S_K = \frac{1}{2}\varrho_{2,K}\,\mathrm{tg}^{-1}\left(\frac{z_K}{m_K c}\right) = \frac{1}{2}\varrho_{2,K}\left(\frac{z_K}{m_K c} - \frac{1}{3}\frac{z_K^3}{m_K^3 c^3} + \dots\right), \tag{17}$$

we may find an approximate form (with accuracy to $c^{-3}$) of the Hamiltonian (1) which results in the intermediate scheme $U$. Now we shall calculate successively terms of this Hamiltonian. According to (16) and (17), one-particle Hamiltonians $H'_K$ in the intermediate scheme have the following form

$$\begin{aligned} H'_K &= UH'_K U^+ = U_K H'_K U_K^+ = \varrho_{3,K}\left\{m_K c^2 + \frac{p_K^2}{2m_K} - \frac{p_K^4}{8m_K^3 c^2} + \frac{1}{c}p_K^{\mathrm{ex}} + \right. \\ &\quad \left. + \frac{1}{c^3}\tilde{P}_K^{\mathrm{ex}}\right\} + e_K \Phi_K^{\mathrm{ex}} + \frac{1}{c^2}F_K^{\mathrm{ex}} + \frac{1}{c^2}Q_K^{\mathrm{ex}} + \\ &\quad + \varrho_{2,K}\left(\frac{1}{c}Y_K^{\mathrm{ex}} + \frac{1}{c^3}\mathscr{E}_K^{\mathrm{ex}}\right) + \varrho_{1,K}\frac{1}{c^3}\mathscr{M}_K^{\mathrm{ex}}, \end{aligned} \tag{18}$$

where

$$\frac{1}{c}P_K^{\mathrm{ex}} = -\frac{e_K}{2m_K c}\left[\left(p_K - \frac{e_K}{c}A_K^{\mathrm{ex}}\right), A_K^{\mathrm{ex}}\right]_+ - (1+g_K^{(1)})\mu_K \mathscr{H}_K^{\mathrm{ex}}, \tag{19}$$

$$\frac{1}{c^3}\tilde{P}_K^{\mathrm{ex}} = \frac{e_K}{8m_K^3 c^3}[p_K^2, [p_K, A_K^{\mathrm{ex}}]_+]_+ + \frac{(1+g_K^{(1)})}{4m_K^2 c^2}[p_K^2, \mu_K \mathscr{H}_K^{\mathrm{ex}}]_+, \tag{20}$$

$$\frac{1}{c^2}F_K^{\mathrm{ex}} = -\frac{(1+2g_K^{(1)})}{4m_K c}\mu_K\left\{E_K^{\mathrm{ex}}\times\left(p_K - \frac{e_K}{c}A_K^{\mathrm{ex}}\right) - \left(p_K - \frac{e_K}{c}A_K^{\mathrm{ex}}\right)\times E_K^{\mathrm{ex}}\right\}, \tag{21}$$

$$\frac{1}{c^2}Q_K^{\mathrm{ex}} = -\frac{(1+2g_K^{(1)}+8g_K^{(2)})e_K\hbar^2}{8m_K^2 c^2}\,\mathrm{div}_K E_K^{\mathrm{ex}}, \qquad \mathrm{div}_K E_K^{\mathrm{ex}} = 4\pi\varrho_K^{\mathrm{ex}}, \tag{22}$$

$$\frac{1}{c}Y_K^{\mathrm{ex}} = -(1+g_K^{(1)})\mu_K E_K^{\mathrm{ex}}, \tag{23}$$

$$\frac{1}{c^2}\mathscr{E}_K^{\mathrm{ex}} = \frac{i}{6m_K c^3}[\sigma_K\cdot p_K, F_K^{\mathrm{ex}}+Q_K^{\mathrm{ex}}]_- - \frac{i}{6m_K^3 c^3}[(\sigma_K\cdot p_K)p_K^2, e_K\Phi_K^{\mathrm{ex}}]_-, \tag{24}$$

$$\frac{1}{c^3}\mathscr{M}_K^{\mathrm{ex}} = g_K^{(2)}\frac{4\pi e_K\hbar^2}{m_K^2 c^3}\sigma_K\cdot j_K^{\mathrm{ex}} - \frac{1}{4m_K c^3}\left[\sigma_K\left(p_K - \frac{e_K}{c}A_K^{\mathrm{ex}}\right), P_K^{\mathrm{ex}}\right]_+. \tag{25}$$

The obtained Hamiltonian $H_K'^U$ contains already all the even terms (proportional to $\varrho_{3,K}$ and free from $\varrho_K$) of the correct even one particle Hamiltonian obtained in the quoted papers [10] and [11]. Thus in the next step we shall try to find an additional unitary transformation (constructed according to the iterative FW procedure) removing from this Hamiltonian the needless odd terms which have the character of the interaction of a particle with the external field. Terms of the so-obtained even one-particle Hamiltonian have simple physical interpretation. $1/cP_K^{ex}$ is the Pauli-type terms expressing the interaction of a particle with external magnetic field. The term $1/c^3\tilde{P}_K^{ex}$ together with $-p_K^4/8m_K^3c^2$ stand for the correction to the kinetical energy (depending on the external magnetic field). The term $1/c^2F_K^{ex}$ is a modified spin-orbital interaction taking into account the anomalous magnetic moment of the particle and depending on the external magnetic field by means of kinetical momentum $p_K-\frac{e_K}{c}A_K^{ex}$. $1/c^2Q_K^{ex}$ is the modified Darwin correction.

Now we establish the form of operator $W'_{K,L}$ in the intermediate scheme. With respect to

$$U\lambda_K U^+ = U_K\lambda_K U_K^+ = \varrho_{3,K}, \tag{26}$$

we obtain

$$W'_{K,L} = \frac{1}{4}\Big[\varrho_{3,K}+\varrho_{3,L}, \frac{1}{2}V_{K,L}-U\Big\{\varrho_{3,K}g_K^{(1)}\mu_K\mathscr{H}_{K,L}+\varrho_{2,K}g_K^{(1)}\mu_K\tilde{E}_{K,L}+ + g_K^{(2)}\frac{4\pi e_K\hbar^2}{m_K^2c^2}\Big(\varrho_{K,L}-\frac{1}{c}\sigma_K j_{K,L}\Big)\Big\}U^+\Big]_+ \tag{27}$$

Calculation of the elements of this anticommutator consists in devoloping it into series of commutators and anticommutators (according to (16), (17)) and ordering respectively the components $\varrho_K$ and $\varrho_L$. One can see from (27) that the role of operators $\varrho_{3,K}+\varrho_{3,L}$ consists in removing from the interaction terms proportional to $\varrho_{1,K}\cdot\varrho_{1,K}$ and $\varrho_{2,K}\cdot\varrho_{2,L}$, which results instantly from

$$[\varrho_{3,K}+\varrho_{3,L}, \varrho_{1,K}\cdot\varrho_{1,L}]_+ = [\varrho_{3,K}+\varrho_{3,L}, \varrho_{2,K}\cdot\varrho_{2,L}]_+ = 0. \tag{28}$$

In this way we have taken away terms which have caused difficulties in the reduction of the Breit equation, especially the term $-1/2\,e_Ke_L\varrho_{1,K}\,\varrho_{1,L}\,J_{K,L}$ (see (7)), leading in the next step of iterative procedure to the appearance of a controversial contribution proportional to the fourth power in charge. After expanding operators present in (27) and calculating commutators and anticommutators, we obtain, together with (18),

$$H'^U = \sum_K\Big\{\varrho_{3,K}\Big(m_Kc^2+\frac{p_K^2}{2m_K}-\frac{p_K^4}{8m_K^3c^2}+\frac{1}{c}P_K^{ex}+\frac{1}{c^3}\tilde{P}_K^{ex}\Big)+ + e_K\Phi_K^{ex}+\frac{1}{c^2}(Q_K^{ex}+F_K^{ex})+\varrho_{2,K}\Big(\frac{1}{c}Y_K^{ex}+\frac{1}{c^3}\mathscr{E}_K^{ex}\Big)+$$

$$+\varrho_{1,K}\frac{1}{c^3}\mathscr{M}_K^{\mathrm{ex}}\Big\}+\sum_{K,L}{}'\Big\{\frac{1}{2}(\varrho_{3,K}+\varrho_{3,L})\Big[\frac{1}{2}\frac{e_Ke_L}{r_{K,L}}+$$

$$+\frac{1}{c^2}\Big(F_{K,L}+Q_{K,L}+\frac{1}{2}M_{K,L}+\frac{1}{2}L_{K,L}+G_{K,L}\Big)\Big]+$$

$$+\varrho_{2,K}\varrho_{3,L}\Big(\frac{1}{2c}Y_{K,L}+\frac{1}{c^3}\mathscr{E}_{K,L}\Big)+\varrho_{1,K}\Big(\frac{1}{2c}\hat{P}_{L,K}+\frac{1}{c^3}\mathscr{M}_{K,L}\Big)\Big\}, \tag{29}$$

where

$$\boldsymbol{E}_{K,L}=-\mathrm{grad}_K\frac{e_L}{r_{K,L}}, \tag{30}$$

$$\frac{1}{c}Y_{K,L}=-(1+g_K^{(1)})\boldsymbol{\mu}_K\boldsymbol{E}_{K,L} \tag{31}$$

$$\frac{1}{c^2}Q_{K,L}=-\frac{e_K\hbar^2(1+2g_K^{(1)}+8g_K^{(2)})}{8m_K^2c^2}\mathrm{div}_K\,\boldsymbol{E}_{K,L},\quad \mathrm{div}_K\,\boldsymbol{E}_{K,L}=4\pi\varrho_{K,L}, \tag{32}$$

$$\frac{1}{c_2^2}F_{K,L}=-\frac{(1+2g_K^{(1)})}{4m_Kc}\boldsymbol{\mu}_K\Big\{\boldsymbol{E}_{K,L}\times\Big(\boldsymbol{p}_K-\frac{e_K}{c}\boldsymbol{A}_K^{\mathrm{ex}}\Big)-\Big(\boldsymbol{p}_K-\frac{e_K}{c}\boldsymbol{A}_K^{\mathrm{ex}}\Big)\times\boldsymbol{E}_{K,L}\Big\}, \tag{33}$$

$$\frac{1}{c^2}G_{K,L}=-\frac{(1+g_K^{(1)})}{2m_Kc}\boldsymbol{\mu}_K\Big\{\boldsymbol{E}_{L,K}\times\Big(\boldsymbol{p}_L-\frac{e_L}{c}\boldsymbol{A}_K^{\mathrm{ex}}\Big)-\Big(\boldsymbol{p}_L-\frac{e_L}{c}\boldsymbol{A}_L^{\mathrm{ex}}\Big)\times\boldsymbol{E}_{L,K}\Big\}, \tag{34}$$

$$\frac{1}{c^2}M_{K,L}=(1+g_K^{(1)})(1+g_L^{(1)})\Big\{\frac{\boldsymbol{\mu}_K\boldsymbol{\mu}_L}{r_{K,L}^3}-3\frac{(\boldsymbol{\mu}_K\cdot\boldsymbol{r}_{K,L})(\boldsymbol{\mu}_L\cdot\boldsymbol{r}_{K,L})}{r_{K,L}^5}-\frac{8\pi}{3}\boldsymbol{\mu}_K\boldsymbol{\mu}_L\delta(\boldsymbol{r}_{K,L})\Big\}, \tag{35}$$

$$\frac{1}{c^2}L_{K,L}=-\frac{e_Ke_L}{2m_Km_Lc^2}\Big\{\frac{1}{r_{K,L}}\Big(\boldsymbol{p}_K-\frac{e_K}{c}\boldsymbol{A}_K^{\mathrm{ex}}\Big)\Big(\boldsymbol{p}_L-\frac{e_L}{c}\boldsymbol{A}_L^{\mathrm{ex}}\Big)+$$

$$+\frac{\boldsymbol{r}_{K,L}}{r_{K,L}^3}\Big[\boldsymbol{r}_{K,L}\Big(\boldsymbol{p}_K-\frac{e_K}{c}\boldsymbol{A}_K^{\mathrm{ex}}\Big)\Big]\Big(\boldsymbol{p}_L-\frac{e_L}{c}\boldsymbol{A}_L^{\mathrm{ex}}\Big)\Big\}, \tag{36}$$

$$\frac{1}{c}\hat{P}_{K,L}=-\frac{e_K}{2m_Kc}\Big[\Big(\boldsymbol{p}_K-\frac{e_K}{c}\boldsymbol{A}_K^{\mathrm{ex}}\Big),\ \hat{\boldsymbol{a}}_{K,L}\Big]\ -(1+g_K^{(1)})\boldsymbol{\mu}_K\cdot\mathrm{rot}_K\,\hat{\boldsymbol{a}}_{K,L}, \tag{37}$$

$$\frac{1}{c^3}\mathscr{M}_{K,L}=-\frac{e_Ke_L}{24m_L^3c^3}[z_L^3,J_{L,K}]_++\frac{e_Ke_L}{192m_L^3c^3}[z_L,[z_L,[z_L,J_{L,K}]_+]_+]_++$$

$$+\frac{e_K e_L}{96 m_K^2 m_L c^3}[z_L,[z_K,[z_K,J_{,LK}]_+]_+]_+ + \frac{e_K \hbar^2}{2m_K^2 c^2} g_K^{(2)} \sigma_K \Delta_K d_{K,L}^{(m)} +$$

$$+\frac{g_K^{(1)}}{8m_K c}[z_K, \mu_K \cdot \mathrm{rot}_K\, d_{K,L}^{(m)}]_+ + \frac{g_K^{(1)}}{2m_L c}[z_L, \mu_K \cdot \mathrm{rot}_K\, d_{K,L}^{(m)}]_+ +$$

$$+\frac{g_K^{(1)}}{8m_K m_L c^3}[z_L,[z_K,\mu_K \cdot \mathrm{rot}_K\, a_{K,L}]_+]_+ + \frac{g_L^{(1)}}{8m_K^2 c^2}[z_K,[z_K,\mu_L\, \mathrm{rot}_L\, \hat{a}_{L,K}]_+]_+, \tag{38}$$

$$\frac{1}{c^3}\mathscr{E}_{K,L} = -\frac{ie_K e_L}{12 m_K^3 c^3}\left[z_K^3, \frac{1}{r_{K,L}}\right]_- - \frac{ie_K e_L}{48 m_L^2 m_K c^3}\left[z_L,\left[z_L,\left[z_K,\frac{1}{r_{K,L}}\right]_-\right]_-\right]_- +$$

$$-\frac{ie_K e_L}{96 m_K^3 c^3}\left[z_K,\left[z_K,\left[z_K,\frac{1}{r_{K,L}}\right]_-\right]_-\right]_- - g_L^{(2)}\frac{e_L \hbar^2}{4m_L^2 c^2}\Delta_K d_{K,L}^{(e)} +$$

$$+\frac{ig_L^{(1)}}{8m_L c}[z_L, \mu_L \cdot \mathrm{grad}_L\, d_{K,L}^{(e)}]_- - \frac{ig_K^{(1)}}{8m_K^2 c^2}[z_K,[z_K,\mu_K E_{K,L}]_-]_- +$$

$$-\frac{ig_L^{(1)}}{8m_K m_L c^2}[z_K,[z_L,\mu_L E_{L,K}]_-]_- - \frac{ig_K^{(2)} e_K \hbar^2}{4m_K^3 c^3}[z_K, \mathrm{div}_K\, E_{K,L}]_- +$$

$$-\frac{ig_L^{(2)} e_L \hbar^2}{4m_L^2 m_K c^2}[z_K, \mathrm{div}_L\, E_{L,K}]_- + \frac{ig_L^{(1)}}{8m_L c}[z_L, \mu_K \cdot \mathrm{grad}_K\, d_{K,L}^{(e)}]_-, \tag{39}$$

The quantities designed by (36)–(37) have a simple physical interpretation. $1/cY_{K,L}$ represents the interaction of the anomalous "electric" moment with the electric field of the $L$-th particle. $1/c^2 Q_{K,L}$ and $1/c^2 F_{K,L}$ are counterparts of Darwin and spin-orbital corrections (occurring in the one-particle Hamiltonian) for the case of mutual interaction of the $K$-th and $L$-th particles. $1/c\hat{P}_{K,L}$ is equivalent to the Pauli term representing the interaction of the $K$-th particle with the magnetic field of the $L$-th particle. The term $1/c^2 G_{K,L}$ is the interaction of the spin of the $K$-th particle with the orbital moment of the $L$-th particle and $1/c^2 M_{K,L}$ is the known expression for the interaction of two anomalous magnetic moments of particles. The discussed terms represent, as in the one-particle problem, the dependence on external magnetic field by kinetical momentum. In accordance with what was expected, there are still odd terms in the Hamiltonian (29) which have similar character to terms (24) and (25). For this reason the above-mentioned scheme is called the intermediate scheme. Nevertheless, the transformation (similarly as the Case transformation for one particle) has already introduced a far-going arrangement of terms according to other physical meaning. It is interesting to investigate the even part of this Hamiltonian (i. e. terms proportional to $\varrho_{3,K}$ and free from operators $\varrho_K$). If one puts $\varrho_{3,K} = +1$ then the even part of this Hamiltonian represents the total reduced Hamiltonian to the subspace of positive energy states. Hence it can be seen that further transformation of this Hamiltonian would remove the still remaining odd terms, but which would not change the correct even part of the Hamiltonian.

*3. Transformation of the Hamiltonian to even form and its reduction to the subspace of positive energy states*

As it can be seen from the form of the Hamiltonian (29), it is sufficient to construct a simple unitary transformation according to the FW iterative procedure. Namely

$$U' = \exp(iS'), \tag{40}$$

$$S' = -\frac{1}{2c^2}\left\{\sum_K \frac{1}{m_K}\left[\varrho_{1,K}\left(\frac{1}{c}Y_K^{ex}+\frac{1}{c^3}\mathscr{E}_K^{ex}\right)-\varrho_{2,K}\frac{1}{c^3}\mathscr{M}_K^{ex}\right]+\right.$$

$$\left.+\sum_{K,L}{}' \frac{1}{m_K}\left[\varrho_{1,K}\varrho_{3,L}\left(\frac{1}{2c}Y_{K,L}+\frac{1}{c^3}\mathscr{E}_{K,L}\right)-\varrho_{2,K}\left(\frac{1}{2c}\hat{P}_{L,K}+\frac{1}{c^3}\mathscr{M}_{K,L}\right)\right]\right\}, \tag{41}$$

in order to eliminate from the Hamiltonian (29) the odd terms. However, this transformation introduces new odd terms to the Hamiltonian, but in order $c^{-3}$

$$\sum_K \varrho_{2,K}\frac{1}{c^3}B_K^{ex}+\sum_{K,L}{}'\left\{\varrho_{2,K}\varrho_{3,L}\frac{1}{c^3}C_{K,L}+\varrho_{1,K}\frac{1}{c^3}D_{K,L}\right\}, \tag{42}$$

where

$$\frac{1}{c^3}B_K^{ex} = \frac{1}{4m_K^2c^3}[p_K^2, Y_K^{ex}]_+, \tag{43}$$

$$\frac{1}{c^3}C_{K,L} = \frac{1}{8m_K^2c^3}[Y_{K,L}, p_K^2]_+ + \frac{2}{8m_K^2c^3}[p_L^2, \hat{P}_{L,K}]_- +$$

$$+\frac{ie_Ke_L}{8m_Lc^3}\left[\hat{P}_{L,K}, \frac{1}{r_{K,L}}\right] + \varrho_{3,L}\frac{ie_L}{4m_Kc^3}[\hat{P}_{L,K}, \Phi_K^{ex}]_-, \tag{44}$$

$$\frac{1}{c^3}D_{K,L} = \frac{1}{8m_L^2c^3}[Y_{K,L}, p_L^2]_+ - \frac{1}{8m_K^2c^3}[\hat{P}_{L,K}, p_K^2]_+ + \frac{e_Ke_L}{8m_Kc^3}\left[\hat{P}_{L,K}, \frac{1}{r_{K,L}}\right]_+. \tag{45}$$

Terms (43)–(45) can be easily eliminated by means of a simple unitary transformation (analogous in character to transformation $U'$) without introducing any additional terms

$$U'' = \exp(iS''), \tag{46}$$

$$S'' = -\frac{1}{2c^2}\left\{\sum_K \frac{1}{m_K}\varrho_{1,K}\frac{1}{c^3}B_K^{ex}+\sum{}' \frac{1}{m_K}\left[\varrho_{1,K}\varrho_{3,L}\frac{1}{c^3}C_{K,L}+\right.\right.$$

$$\left.\left.-\varrho_{2,K}\frac{1}{c^3}D_{K,L}\right]\right\}. \tag{47}$$

As a result of applying this transformation we obtain finally the $N$-particle modified even Hamiltonian

$$H'_{\text{even}} = \sum_{K} \Big\{ \varrho_{3,K} \Big( m_K c^2 + \frac{p_K^2}{2m_K} - \frac{p_K^4}{8m_K^3 c^2} + \frac{1}{c} P_K^{\text{ex}} + \frac{1}{c^3} \tilde{P}_K^{\text{ex}} \Big) + e_K \Phi_K^{\text{ex}} + $$
$$+ \frac{1}{c^2} (Q_K^{\text{ex}} + F_K^{\text{ex}}) \Big\} + \sum_{K,L}{}' \Big\{ \frac{1}{2} (\varrho_{3K} + \varrho_{3,L}) \Big[ \frac{1}{2} \frac{e_K e_L}{r_{K,L}} +$$
$$+ \frac{1}{c^2} \Big( F_{K,L} + Q_{K,L} + \frac{1}{2} M_{K,L} + \frac{1}{2} L_{K,L} + G_{K,L} \Big) \Big] \Big\}. \tag{48}$$

It can be seen from the form of this Hamiltonian that the complete separation of positive and negative energy states is possible. There exists $2^N$ possible states with definite sign of energy which belong to $2^N$ possible combinations of the eigenvalues $\varrho_{3,\text{I}} = \pm 1$, $\varrho_{3,\text{II}} = \pm 1, \ldots \varrho_{3,N} = \pm 1$. The choice of the eigenvalues $\varrho_{3,K}$ with opposite sign gives rise to vanishing mutual interaction of particles (in accordance with the assumption of the hole theory). In particular, if we select all $\varrho_{3,K} = +1$, we obtain the reduced effective $N$-fermion modified Hamiltonian

$$H_{\text{eff}} = \sum_{K} \Big\{ m_K c^2 + \frac{p_K^2}{2m_K} - \frac{p_K^4}{8m_K^3 c^2} + \frac{1}{c} P_K^{\text{ex}} + \frac{1}{c^3} \tilde{P}_K^{\text{ex}} + e_K \Phi_K^{\text{ex}} +$$
$$+ \frac{1}{c^2} (Q_K^{\text{ex}} + F_K^{\text{ex}}) \Big\} + \sum_{K,L}{}' \Big\{ \frac{1}{2} \frac{e_K e_L}{r_{K,L}} + \frac{1}{c^2} \Big( F_{K,L} + Q_{K,L} + \frac{1}{2} M_{K,L} + \frac{1}{2} L_{K,L} + G_{K,L} \Big) \Big\}. \tag{49}$$

## 4. *Discussion of the results*

The considerations presented in both parts of this paper concern two connected questions:

1. Establishment of the form of the modified quasirelativistic equation describing charged Dirac particles.

2. Approximate reduction of this equation to the subspace of positive energy states.

The form of approximate quantum-mechanical equation has been assumed by introducing in a phenomenological way additional interaction terms which take into account the radiative correction. Besides, the known additional interaction terms, i. e. additional magnetic and "electric" moment interactions with the external field as well as with the field produced by other particles, the terms with charge and current densities of the sources of external field and of respective particles have been introduced into considerations. In this way we have obtained a complete generalization of the Pauli procedure for the case of many Dirac particles. In the preceding part I, the internal agreement of such a modification of the quasirelativistic equation with accuracy to the order $c^{-3}$ has been justified. The correctness of the assumption of the presented modified quasirelativistic equation (1) has been entirely justified after its reduction. As a result, we have obtained a correct effective Hamiltonian containing terms which give account of all relativistic and radiative corrections with accuracy up to $c^{-3}$ (see, (49)). From the obtained form of the Hamiltonian (49) one can see that a consequently carried out calculation to the order $c^{-3}$ caused the appearance of relativistic corrections depending on the external magnetic field, i. e. corrections to the kinetical energy $1/c^3\hat{P}_K^{ex}$, spin-orbital $1/c^2F_K^{ex}1/c^2F_{K,L}$, $1/c^2G_{K,L}$ and orbital $1/c^2L_{K,L}$. These relativistic corrections have been already obtained by Hegstrom ([3], [4]) in his investigation of Zeeman-spliting in the hydrogen atom. The introduction in the initial Hamiltonian of terms of interaction of additional magnetic and "electric" particle moments with external field and field produced by other particles have lead to the appearance in the reduced Hamiltonian of terms containing anomalous magnetic moments of particles, a result also obtained in [3], [4]. The introduction in the presented paper of terms which give account of the remaining radiative correction, i. e. terms including charge and current densities has influenced the modification of the Darwin corrections $1/c^2Q_K^{ex}$ and $1/c^2Q_{K,L}$ (see (22), (23)).

The method of reduction used in our paper has been based on the quoted fundamental papers by Foldy and Wouthuysen, but the introduced generalization for the case of many particles is principally different from that used in papers [2]-[4], and refers, according to papers [5]-[7], to the Case transformation for one Dirac particle. The generalized Case transformation brought the Hamiltonian to the intermediate scheme which is of considerable importance for illustrative interpretation with respect to classification of several terms of the Hamiltonian according to their physical meaning. This transformation has separated from the Hamiltonian the even part consisting of a complete, correct and even Hamiltonian. Supplementing this transformation by two simple additional unitary transformations, which have only auxiliary meaning, has brought this Hamiltonian to an even form which permits the complete separation of the subspaces belonging to different signs of energy.

## REFERENCES


[1] H. Janyszek, *Acta Phys. Pol.* **B5**, 447 (1974).
[2] Z. V. Chraplyvy, Saint Louis University Missouri, *Technical Note*, August 16 (1954).
[3] H. A. Hegstrom, *Phys. Rev.* **184**, 17 (1969).
[4] H. Grotch, H. A. Hegstrom, *Phys. Rev.* **A4**, 59 (1971).
[5] W. Hanus, H. Janyszek, *Bull. Acad. Pol. Sci. Ser. Math. Astron. et Phys.* **19**, 659 (1971).
[6] W. Hanus, H. Janyszek, *Acta Phys. Pol.* **B2**, 695 (1971).
[7] H. Janyszek, *Acta Phys. Pol.* **B3**, 621 (1972).
[8] L. Foldy, S. Wouthuysen, *Phys. Rev.* **78**, 29 (1950).
[9] H. Bethe, *Handbuch der Physik*, Vol. 24, Part 1, p. 375, 1933.
[10] W. Hanus, R. Mrugała, *Acta Phys. Pol.* **B3**, 437 (1972).
[11] W. Hanus, R. Mrugała, *Acta Phys. Pol.* **B3**, 443 (1972).