# PASSAGE FROM THE FUNDAMENTAL TENSOR $g_{\mu\nu}$ OF THE GRAVITATIONAL THEORY TO THE FIELD STRUCTURE $g_{\mu\nu}$ OF THE UNIFIED THEORIES

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A theorem on obtaining exact solutions for a particular field structure from those of vacuum field equations of general theory as well as from some simpler solutions of unified theories is derived. With the help of this result the most general solution for the particular field structure is developed from the already known simpler solutions. The physical implications of this theorem in relation to some of the parallel work of other authors is discussed.

## 1. *Introduction*

In general theory of relativity certain problems which are connected with generating solutions from well-known simpler solutions have been tackled by various authors, such as Datta Majumdar (1947), Misra and Radhakrishna (1962), Harrison (1965), Buchdahl (1959), Janis *et al.* (1969), and others. From the point of view of obtaining the solutions, the problem seems to concern exploring the devices for tackling highly nonlinear equations

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{1}$$

for certain types of energy-momentum distributions. Ingenious attempts have been made by various workers over this problem. We now briefly review the work of the above-mentioned authors to highlight certain aspects providing the background of our present investigation.

Datta Majumdar (1947) has shown that if $T_{\mu\nu}$ represents Einstein-Maxwell source-free fields, the static line-element

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}+g_{44}(dt)^2, \quad (\mu, \nu = 1, 2, 3)$$

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with the help of (1), gives the relation

$$g_{44} = A + B\Phi + \tfrac{1}{2}\Phi^2,$$

where $\Phi$ is the electrostatic potential and, $A$ and $B$, are arbitrary constants. The problem has been further generalized by Misra and Radhakrishna (1962), and later by Harrison (1965) to the case of non-static Weyl-fields.

Buchdahl (1959) has obtained a similar result in the case of zero-rest-mass scalar meson fields. He has shown that given any static solution of Einstein's vacuum equations $R_{kl} = 0$, a one parameter family of pairs of solutions of the field equations with scalar field, *viz.*, $R_{\mu\nu} = -\kappa V_{;\mu} V_{;\nu}$, $g^{\mu\nu} V_{;\mu\nu} = 0$, where $V$ is the scalar potential of the field, can be constructed from the already known vacuum solutions. Thus, for every vacuum solution, with the help of this result, a solution can always be constructed which corresponds to the presence of some zero-rest-mass scalar field. This result has further been extended by Janis *et al.* (1969) to the case of static coupled electromagnetic and zero--rest-mass scalar meson fields. Here again, with the help of the result obtained by them, they have been able to generate a coupled system in two stages. In the first stage a zero--rest-mass scalar meson field is generated and then the coupled system is developed. This can also work *vice versa*, *i.e.*, in the first stage the electromagnetic field is generated and then the coupled field. It may be mentioned here that all the above physical situations have been developed primarily from the empty-space solutions of Einstein's field equations. Thus, given any static solution for the empty-space field equations $R_{\mu\nu} = 0$, with the help of these results, it is always possible to build up the solutions in the case of non--vacuum fields at least of the types mentioned above (*viz.*, zero-rest-mass scalar fields, source free electromagnetic fields, a coupled field). An analysis of these investigations shows that a physical situation described by a certain state of energy-momentum tensor on the right-hand side of equation (1) can be generated from a physical situation of some origin in which the present physical state (the one which is generated) is completely absent.

In unified theory (Einstein and Schrödinger) a similar result has been obtained by Bandyopadhyay (1963) in the case of static spherically symmetric fields given by

$$g_{\mu\nu} = \begin{bmatrix} -\alpha & 0 & 0 & \omega \\ 0 & -\beta & f\sin\theta & 0 \\ 0 & -f\sin\theta & -\beta\sin^2\theta & 0 \\ -\omega & 0 & 0 & \gamma \end{bmatrix}, \tag{2}$$

where $\alpha$, $\beta$, $\gamma$, $f$ and $\omega$ are functions of $r$ only. The result states that for the field structure (2), if a solution is given for the case $\omega = 0$ for the "para-form" field equations of Schrödinger's unified field theory, a corresponding solution for the case $\omega \neq 0$ can always be constructed from the above-mentioned solution for the case $\omega = 0$. This result holds well for Einstein's version of unified theories also. This theorem shows that a physical situation (solution) corresponding to the presence of electric field ($\omega \neq 0$) can be generated from a physical state when the said field is completely absent ($\omega = 0$). The result is true in the presence as well as in the absence of magnetic field (*i.e.*, the term $f$). The interesting feature of this theorem is that when the magnetic field is absent ($f = 0$), a physical state corresponding to the presence of electric field can be developed from gravitational mass (Schwarzschild's solution) of general theory of relativity (when the cosmological term $\lambda$ is present). It has been observed by one of the authors (Tiwari (1971)) that a similar conclusion can be drawn, under a special restriction, in the case of magnetic fields also.

In the present work the above result of Bandyopadhyay (1963) has been further extended to the case of the field structure (the plane-symmetric field) given by

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & I(x) \\ 0 & G(x) & K(x) & 0 \\ 0 & -K(x) & G(x) & 0 \\ -I(x) & 0 & 0 & H(x) \end{bmatrix}. \tag{3}$$

It has been shown that, given any solution for the case $I = 0$ for the "para-form" field equations of Schrödinger's unified theory, it is always possible to obtain the solution for the case $I \neq 0$. Further, the set of the solutions obtained for the case $I \neq 0$, when $K = 0$, can be generated from the already known empty-space solutions of general theory of relativity. The problem has a well-defined physical meaning, similar to the one given in the case of the results obtained by Bandyopadhyay (1963). Thus, it can be seen that the solutions of the field equations $R_{\mu\nu} = 0$ of general theory of relativity play a very important role in the construction of all the above solutions, *i.e.*, the solutions developed for certain types of energy-momentum distributions in general theory of relativity and the solutions of unified field theory, given above. The authors have also found these results to be true for more general types of field-structures considered by Ghosh (1956). Summarising, we may say that we can pass from a special empty-space solution of general theory of relativity to the solutions of unified field theory. It is indeed highly gratifying to be able to build physical solutions either in general theory of relativity or in unified theories from the empty-space solutions which form a solid base of Einstein's gravitational theory.

In Section 2, the necessary preliminaries, as required for the proof of the theorem, have been worked out. The statement of the theorem has also been given in the same section. Section 3, deals with the application of this theorem to the construction of the general solutions for the field structure (3).

## 2. *The field equations and the theorem*

The "para-form" field equations of Schrödinger's unified field theory are:

$$g_{ik,l} - g_{sk}\Gamma^{s}_{il} - g_{is}\Gamma^{s}_{lk} = 0, \tag{4}$$

$$\Gamma^{s}_{\underset{\vee}{is}} = 0, \tag{5}$$

$$R_{\underline{ik}} - \lambda g_{\underline{ik}} = 0 \tag{6}$$

and

$$(R_{\underset{\vee}{ik},l}+R_{\underset{\vee}{kl},i}+R_{\underset{\vee}{li},k})-\lambda(g_{\underset{\vee}{ik},l}+g_{\underset{\vee}{li},k}+g_{\underset{\vee}{kl},i}) = 0, \tag{7}$$

where

$$R_{\mu\nu} = -\Gamma^{\sigma}_{\mu\nu,\sigma}+\Gamma^{\sigma}_{\mu\sigma,\nu}+\Gamma^{\sigma}_{\mu\alpha}\Gamma^{\alpha}_{\sigma\nu}-\Gamma^{\sigma}_{\alpha\sigma}\Gamma^{\alpha}_{\mu\nu}. \tag{8}$$

The symbols — and ˇ below the indices denote symmetry and antisymmetry, respectively, and a comma followed by a subscript denotes partial differentiation. Obviously, the "weaker form" equations of Einstein's unified field theory can be obtained from the above set of equations when $\lambda \to 0$.

The only surviving field equations (4) to (7) for the field structure (3), are

$$R_{11}-\lambda = 0, \tag{9}$$

$$R_{22}-\lambda G = R_{33}-\lambda G = 0, \tag{10}$$

$$R_{23}-\lambda K = m, \tag{11}$$

$$R_{44}-\lambda H = 0 \tag{12}$$

and

$$\Gamma_4 = \Gamma^4_{\underset{\vee}{4s}} = \left(\frac{HI'-\frac{1}{2}H'I}{H+I^2}\right)+I\left(\frac{GG'+KK'}{G^2+K^2}\right)^2 = 0, \tag{13}$$

where $m$ in equation (11) is an arbitrary constant and an overhead dash denotes differentiation with respect to the $x$ only. The nonvanishing components of $R_{\mu\nu}$ are given in the Appendix.

Let $\overset{I}{R}_{\mu\nu}(G, K, H, I, \theta)$ denote the expression for $R_{\mu\nu}$ in terms of $G$, $K$, $H$, $I$ and their derivatives and $\overset{0}{R}_{\mu\nu}(G, K, H+I^2, 0, \theta)$ denote the same when $I$ is zero, $H$ replaced by $(H+I^2)$ and $H'$ by $(H+I^2)'$. The term $\theta$ denotes the derivatives of $G$, $H$, $K$ and $I$.

We have

$$R_{44}+I^2R_{11} = I^2\frac{d}{dx}\left(\frac{1}{I}\Gamma_4\right)+I\left(\frac{I'}{I}-\frac{H'+2II'}{H+I^2}\right)(\Gamma_4)+$$

$$+\frac{d}{dx}\left(\frac{H'}{2}+II'\right)+\frac{1}{2}\left(\frac{H'}{2}+II'\right)\left(\frac{GG'+KK'}{G^2+K^2}-\frac{\frac{H'}{2}+II'}{H+I^2}\right) =$$

$$= \overset{0}{R}_{44}(G, K, H+I^2, 0, \theta). \tag{14}$$

If the left-hand side is denoted by $\overset{I}{L}(G, K, H, I, \theta)$, we have

$$\overset{I}{L}(G, K, H, I, \theta) = \overset{0}{R}_{44}(G, K, H+I^2, 0, \theta). \tag{15}$$

The following observations are true as may be verified from the expressions given in the Appendix.

$$\overset{I}{R}_{11}(G, K, H, I, \theta) = \overset{0}{R}_{11}(G, K, H+I^2, 0, \theta), \tag{16}$$

$$\overset{I}{R}_{22}(G, K, H, I, \theta) = \overset{0}{R}_{22}(G, K, H+I^2, 0, \theta), \tag{17}$$

$$\overset{I}{R}_{33}(G, K, H, I, \theta) = \overset{0}{R}_{33}(G, K, H+I^2, 0, \theta), \tag{18}$$

$$\overset{I}{R}_{23}(G, K, H, I, \theta) = \overset{0}{R}_{23}(G, K, H+I^2, 0, \theta). \tag{19}$$

From (14), we have

$$\overset{I}{L}(G, K, H, I, \theta) = \overset{I}{R}_{44}(G, K, H, I, \theta)+I^2 R_{11}(G, K, H, I, \theta). \tag{20}$$

This, in view of (9) and (12) reduces to

$$\overset{I}{L}(G, K, H, I, \theta)-\lambda(H+I^2) = 0. \tag{21}$$

It may be seen that the equation (12) can be deduced from the set of equations (9) and (21). Hence the set of equations (9) and (12) can be replaced by the equivalent set of equations (9) and (21). The equations (9) to (13) are, therefore, replaced by the following equivalent set of equations

$$\overset{I}{R}_{11}-\lambda = 0, \tag{22}$$

$$\overset{I}{R}_{22}-\lambda G = \overset{I}{R}_{33}-\lambda G = 0, \tag{23}$$

$$\overset{I}{R}_{23}-\lambda K = m, \tag{24}$$

$$\overset{I}{L}(G, K, H, I, \theta)-\lambda(H+I^2) = 0 \tag{25}$$

and

$$\Gamma_4 = 0. \tag{26}$$

In view of the relations (15) to (19), the set of equations (22) to (26), is equivalent to the set

$$\overset{0}{R}_{11}-\lambda = 0. \tag{27}$$

$$\overset{0}{R}_{22}-\lambda G = \overset{0}{R}_{33}-\lambda G = 0, \tag{28}$$

$$\overset{0}{R}_{23}-\lambda K = 0, \tag{29}$$

$$\overset{0}{R}_{44}-\lambda(H+I^2) = 0 \tag{30}$$

and

$$\Gamma_4 = 0. \tag{31}$$

The equation (31) is identically satisfied when $I = 0$. The set of equations (27) to (31) are obtained for the structure (3) when $I = 0$ and $H$ is replaced by $(H+I^2)$ (also $H'$ by $(H+I^2)')$. When $I = 0$, let the values of $G, K, H$ be denoted by $\overset{0}{G}, \overset{0}{K}, \overset{0}{H}$. As the two sets of equations (27) to (31) and (22) to (26) are equivalent, a solution $\overset{0}{G}, \overset{0}{K}, \overset{0}{H}$ of the former set will determine a corresponding solution $\overset{I}{G}, \overset{I}{K}, \overset{I}{H}, I$ of the latter set of equations by means of the following theorem:

Theorem: If a solution $\overset{0}{G}, \overset{0}{K}, \overset{0}{H}$ (when $I = 0$) is known, a solution $\overset{I}{G}, \overset{I}{K}, \overset{I}{H}, I$ can be constructed by taking

$$I^2 = \frac{l\overset{0}{H}}{\overset{0}{G}{}^2+\overset{0}{K}{}^2}, \tag{32}$$

where $l$ is an arbitrary constant

$$\overset{I}{G} = \overset{0}{G}, \quad \overset{I}{K} = \overset{0}{K} \tag{33}$$

and

$$\overset{I}{H} = \overset{0}{H}-I^2. \tag{34}$$

The value of $I$ in (34) is determined from (32). The proof of this theorem runs exactly along the same lines as that given by Bandyopadhyay (1963) for the spherically symmetric fields.

### *3. Construction of general solutions for the field structure (3)*

The derivation of the field structure (3) by assuming certain symmetry conditions has been given by Rao (1972). Exact solutions have been obtained in Einstein's unified field theory for "weaker form" equations by Bandyopadhyay (1951) and Rao (1959) and in Schrödinger's unified field theory by Sarkar (1965, 1966) and Tiwari (1971). The solutions obtained by Sarkar (1965, 1966) are limited only to two particular cases, *viz.*, when (*i*) $K = 0, I \neq 0$ and (*ii*) when $K \neq 0, I = 0$. The solutions obtained by Tiwari (1971) are more general than those obtained by Sarkar, in the sense that they have been obtained for the case $K \neq 0$, $I \neq 0$. But these are not the most general solutions due to a certain restriction in the field variable $K(K = lG$, where $l$ is an arbitrary constant). The problem of obtaining a more general solution remains open. The theorem established in the preceding section gives the method of obtaining such a general solution from the known solutions of Sarkar (1966) for the case $K \neq 0$, $I = 0$. The other solution of Sarkar (1965) for the case $K = 0$, $I \neq 0$ (when electric term is present, but magnetic term is absent) has been directly generated from the external Schwarzschild's solution of general theory of relativity (when the cosmological term $\lambda$ is present).

We now establish the most general solution, so far known, for the field structure (3), with the help of the theorem stated in Section 2. For this we start with the solutions obtained by Sarkar (1966) for Schrödinger's para-form field equations for the case $K \neq 0$, $I = 0$, and build up the corresponding general solution for the case $K \neq 0$, $I \neq 0$.

(a) Case I: $K \neq 0$, $I = 0$.

Sarkar (1966) has obtained two solutions for the above case. These are

$$\overset{0}{G} = aX^{2/3}\cos\left(\frac{a_1}{n}Y - c\right)\exp\left(\frac{b}{n}Y\right),$$

$$\overset{0}{K} = aX^{2/3}\sin\left(\frac{a_1}{n}Y - c\right)\exp\left(\frac{b}{n}Y\right),$$

$$\overset{0}{H} = \frac{X^{2/3}}{a^2}\exp\left(-\frac{2b}{n}Y\right),$$

$$X = (c_1 e^{\sqrt{3\lambda}x} + c_2 e^{-\sqrt{3\lambda}x}), \qquad Y = \tan^{-1}\left(\frac{c_1}{c_2}\exp(\sqrt{3\lambda}\,x)\right)$$

$$\frac{1}{n} = \frac{1}{\sqrt{3\lambda c_1 c_2}}, \tag{35}$$

$$a_1^2 + 3b^2 + 16c_1c_2\lambda = 0, \tag{36}$$

and

$$\overset{0}{G} = aU^{2/3}V^{b/2p}\cos\left(\frac{a_1}{2p}\log V - c\right),$$

$$\overset{0}{K} = aU^{2/3}V^{b/2p}\sin\left(\frac{a_1}{2p}\log V - c\right),$$

$$\overset{0}{H} = (U^{2/3}/a^2)V^{-b/p},$$

$$U = (b_1 e^{\sqrt{3\lambda}x} - b_2 e^{-\sqrt{3\lambda}x}), \qquad V = \left(\frac{\sqrt{b_1}\,e^{\frac{\sqrt{3\lambda}}{2}x} - \sqrt{b_2}\,e^{-\frac{\sqrt{3\lambda}}{2}x}}{\sqrt{b_1}\,e^{\frac{\sqrt{3\lambda}}{2}x} + \sqrt{b_2}\,e^{-\frac{\sqrt{3\lambda}}{2}x}}\right)$$

$$\frac{1}{p} = \frac{1}{\sqrt{3\lambda b_1 b_2}}, \tag{37}$$

$$a_1^2 + 3b^2 - 16\lambda b_1 b_2 = 0. \tag{38}$$

Here and in what follows, the small Latin alphabets with or without suffixes, will denote arbitrary constants.

By making use of the theorem, the following two solutions are obtained. The first solution, which is based on the solution (35), is

$$\overset{I}{G} = \overset{0}{G} = aX^{2/3}\cos\left(\frac{a_1}{n}Y-c\right)\exp\left(\frac{b}{n}Y\right),$$

$$\overset{I}{K} = \overset{0}{K} = aX^{2/3}\sin\left(\frac{a_1}{n}Y-c\right)\exp\left(\frac{b}{n}Y\right),$$

$$\overset{I}{H} = \frac{X^{2/3}}{a^2}\exp\left(-\frac{2b}{n}Y\right) - \frac{l^2}{a^2}X^{-2/3}\exp\left(-\frac{4b}{n}Y\right),$$

$$I = \frac{l}{a}X^{-1/3}\exp\left(-\frac{2b}{n}Y\right). \tag{39}$$

The other solution based on (37) is given by

$$\overset{I}{G} = \overset{0}{G} = aU^{2/3}V^{b/2p}\cos\left(\frac{a_1}{2p}\log V-c\right),$$

$$\overset{I}{K} = \overset{0}{K} = aU^{2/3}V^{b/2p}\sin\left(\frac{a_1}{2p}\log V-c\right),$$

$$\overset{I}{H} = \frac{U^{2/3}}{a^2}V^{-b/2p} - \frac{l^2}{a^2}U^{-2/3}V^{-2b/p},$$

$$I = \frac{l}{a^2}U^{-1/3}V^{-b/p}. \tag{40}$$

Both the above solutions, (39) and (40), are subject to the conditions (36) and (38), respectively.

(b) Case II: $K = 0,\ I \neq 0$.

The solution obtained by Sarkar (1965) for the second case (*viz.*, $K = 0,\ I \neq 0$) can be constructed, with the help of the theorem, from the solution for the case $K = 0,\ I = 0$. For this case, field equations (9) to (13), take the form

$$R_{11}-\lambda g_{11} = \frac{d}{dx}\left(\frac{G''}{G'} + \frac{H'/2}{H}\right) + \frac{1}{2}\left(\frac{G'}{G}\right)^2 + \left(\frac{H'/2}{H}\right)^2 - \lambda = 0, \tag{41}$$

$$R_{22}-\lambda g_{22} = R_{33}-\lambda g_{33} = \frac{1}{2}G'' + \frac{1}{2}G'\left(\frac{H'/2}{H}\right) - \lambda G = 0, \tag{42}$$

$$R_{44}-\lambda g_{44} = \frac{H''}{2} + \frac{1}{2}\left(\frac{H'G'}{G}\right) - \frac{1}{4}\frac{H'^2}{H} - \lambda H = 0. \tag{43}$$

Equation (42), in view of equations (41) and (43) after eliminating $H$, gives

$$2G''-\frac{1}{2}\frac{G'^2}{G}-2\lambda G=0, \tag{44}$$

which yields the solution

$$G=\left(a\cosh\frac{\sqrt{3\lambda}}{2}x+\frac{2}{\sqrt{3\lambda}}b\sinh\frac{\sqrt{3\lambda}}{2}x\right)^{4/3}. \tag{45}$$

From (42) and (44), on eliminating $\lambda$, we get

$$G''-G'\left(\frac{H'/2}{H}\right)-\frac{1}{2}\frac{G'^2}{G}=0, \tag{46}$$

which, with the help of (45), gives the solution

$$H=\frac{f\left(a\frac{\sqrt{3\lambda}}{2}\sinh\frac{\sqrt{3\lambda}}{2}x+b\cosh\frac{\sqrt{3\lambda}}{2}x\right)^2}{\left(a\cosh\frac{\sqrt{3\lambda}}{2}x+\frac{2}{\sqrt{3\lambda}}b\sinh\frac{\sqrt{3\lambda}}{2}x\right)^2}. \tag{47}$$

The solutions (45) and (47) correspond to the solution of

$$\underline{R}_{ik}=\lambda\underline{g}_{ik}$$

for plane-symmetric field in general theory of relativity. These solutions, with the help of the theorem, determine the value of $I$ as

$$I=\frac{e\left(a\frac{\sqrt{3\lambda}}{2}\sinh\frac{\sqrt{3\lambda}}{2}x+b\cosh\frac{\sqrt{3\lambda}}{2}x\right)}{\left(a\cosh\frac{\sqrt{3\lambda}}{2}x+\frac{2}{\sqrt{3\lambda}}b\sinh\frac{\sqrt{3\lambda}}{2}x\right)^{5/3}} \tag{48}$$

and the value of $H$ as

$$H=\frac{\left(a\frac{\sqrt{3\lambda}}{2}\sinh\frac{\sqrt{3\lambda}}{2}x+b\cosh\frac{\sqrt{3\lambda}}{2}x\right)^2}{\left(a\cosh\frac{\sqrt{3\lambda}}{2}x+\frac{2}{\sqrt{3\lambda}}b\sinh\frac{\sqrt{3\lambda}}{2}x\right)^{5/3}}\times$$

$$\times\left\{f^2-\frac{e^2}{\left(a\cosh\frac{\sqrt{3\lambda}}{2}x+\frac{2}{\sqrt{3\lambda}}b\sinh\frac{\sqrt{3\lambda}}{2}x\right)^{8/3}}\right\}. \tag{49}$$

The solutions (45), (48) and (49) are the same as obtained by Sarkar (1965) for the case $K=0$, $I\neq 0$.

**(b)** Bandyopadhyay (1951) has obtained a solution in Einstein's unified field theory for the field structure (3), for the case $K = 0$, $I \neq 0$, as follows

$$g_{11} = 1,$$
$$G = (k + \tfrac{3}{4}\sqrt{b}\,x)^{4/3},$$
$$H = \frac{16}{b(k + \frac{3}{4}\sqrt{b}\,x)^{2/3}}\left\{a - \frac{d^2}{(k + \frac{3}{4}\sqrt{b}\,x)^{8/3}}\right\},$$
$$I = \frac{4d}{3\sqrt{b}\,(k + \frac{3}{4}\sqrt{b}\,x)^{5/3}}. \tag{50}$$

This solution, with the help of the theorem, can be constructed from the solutions of the field equations $R_{ik} = 0$ of general theory of relativity. Similarly, the most general solution ($K \neq 0$, $I \neq 0$) obtained by Rao (1959) in the case of Einstein's unified theory can also be generated from the solution for the case $K \neq 0$, $I = 0$.

Remark (1): As everywhere regular solutions, if any, are of paramount importance in unified field theories, this theorem affords a powerful technique in tackling the highly complicated and non-linear differential equations of the unified theories of Einstein and Schrödinger. Of course, this technique is limited to static fields. It still remains to be verified whether such a result will hold for non-static fields.

Remark (2): The dissatisfaction of Einstein with the artificial concept of energy-momentum tensor of general theory of relativity and his consequent elaboration of unified theories is, in a way, justified since from the theorem we are able to pass from gravitational situation (solutions of $R_{ik} = 0$) to coupled fields as in unified theories without the help of the energy-momentum tensor.

## APPENDIX

The non-vanishing components of $R_{\mu\nu}$ for (3) are as follows:

$$R_{11} = \frac{d}{dx}\left(\frac{GG' + KK'}{G^2 + K^2} + \frac{\frac{1}{2}H' + II'}{H + I^2}\right) + \frac{1}{2}\left[\left(\frac{GG' + KK'}{G^2 + K^2}\right)^2 + \left(\frac{KG' - GK'}{G^2 + K^2}\right)^2\right] + \left(\frac{\frac{H'}{2} + II'}{H + I^2}\right)^2,$$

$$R_{22} = R_{33} = \frac{1}{2}\frac{d}{dx}\left(\frac{G^2G' + 2GKK' - K^2G'}{G^2 + K^2}\right) - \frac{1}{2}\left(\frac{KG' - GK'}{G^2 + K^2}\right) \times \left(\frac{K^2K' + 2KGG' - K'G^2}{G^2 + K^2}\right) + \frac{1}{2}\left(\frac{H'/2 + II'}{H + I^2}\right)\left(\frac{G^2G' + 2GKK' - K^2G'}{G^2 + K^2}\right),$$

$$R_{44} = \frac{d}{dx}\left(\frac{H'}{2} + 2I\left(\frac{HI' - \frac{1}{2}H'I}{H+I^2}\right)\right) - \left(\frac{HI' - \frac{1}{2}H'I}{H+I^2}\right)^2 -$$

$$-\frac{1}{2}I^2\left[\left(\frac{GG'+KK'}{G^2+K^2}\right)^2 - \left(\frac{KG'-GK'}{G^2+K^2}\right)^2\right] +$$

$$+\left(\frac{H'}{2} + 2I\left(\frac{HI' - \frac{1}{2}H'I}{H+I^2}\right)\right)\left(\frac{GG'+KK'}{G^2+K^2} - \frac{H'/2+II'}{H+I^2}\right),$$

$$R_{23} = -R_{32} = \frac{d}{dx}\left(\frac{K^2K' + 2KGG' - K'G^2}{G^2+K^2}\right) + \frac{1}{2}\left(\frac{KG'-GK'}{G^2+K^2}\right) \times$$

$$\times\left(\frac{G^2G' + 2GKK' - K^2G'}{G^2+K^2}\right) + \frac{1}{2}\left(\frac{H'/2+II'}{H+I^2}\right)\left(\frac{K^2K' + 2GG'K - K'G^2}{G^2+K^2}\right),$$

$$R_{14} = -R_{41} = -\frac{d}{dx}\left(\frac{HI' - \frac{1}{2}H'I}{H+I^2}\right) + \frac{1}{2}I\left[\left(\frac{GG'+KK'}{G^2+K^2}\right)^2 - \left(\frac{KG'-GK'}{G^2+K^2}\right)\right] -$$

$$-\left(\frac{HI' - \frac{1}{2}H'I}{H+I^2}\right)\left(\frac{GG'+KK'}{G^2+K^2}\right).$$

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