GENERALIZED MECHANICS AS A REPRESENTATION OF THE ORDINARY MECHANICS

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It is shown that the generalized mechanics of one masspoint may be interpreted as a special representation of the ordinary mechanics of a system of masspoints. The homomorphism of both representations is shown in the case of two masspoints coupled by a harmonic force. The new representation is applied in the special relativistic mechanics of masspoints.

1. Introduction

1. Weyssenhoff (1951) and recently Riewe (1971, 1972) have interpreted the generalized mechanics of Ostrogradsky (Borneas, 1959) as the mechanics of a spinning particle, *i. e.* of a particle with inner structure. In the present paper we show that the mechanics operating with higher derivatives of the coordinates of one masspoint may be interpreted as another representation (or description) of the ordinary mechanics of a system of masspoints. This interpretation is a generalization of the interpretation of the above-mentioned authors.

2. The indicator representation

2. Let us consider a system of N interacting masspoints in the one-dimensional case (for the sake of simplicity). The equations of motion of the system (which may or may not be deduced from a Lagrangian) constitute a system of coupled differential equations of the following form (dot $= d/dt$):

$$\ddot{x}_i = f_i(x_1, \ldots, x_N, \dot{x}_1, \ldots, \dot{x}_N), \quad i = 1, \ldots, N. \tag{1}$$

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Let us extend the system (1) by its derivatives up to orders $2N$:

$$\dddot{x}_i = \dot{f}_i(x_1, \ldots, \dot{x}_N),$$

$$\ldots$$

$$x_i^{(2N-1)} = f_i^{(2N-3)}(x_1, \ldots, \dot{x}_N), \tag{2}$$

and

$$x_1^{(2N)} = f_1^{(2N-2)}(x_1, \ldots, \dot{x}_N), \tag{3}$$

where in the last equation (3) the label of f is "one".

Among all possible mechanical systems (1) there is a subclass ("class I") where $x_2, \ldots, x_N, \dot{x}_2, \ldots, \dot{x}_N, \ldots, x_2^{(2N-1)}, \ldots x_N^{(2N-1)}$ can be eliminated from the system (1)–(2) and can be explicitly expressed as functions of $x_1, \dot{x}_1, \ddot{x}_1, \ldots, x_1^{(2N)}$. Substituting these functions into (3) one obtains a single equation of a single variable $x_1(t)$ of order $2N$

$$x_1^{(2N)} = F(x_1, \dot{x}_1, \ddot{x}_1, \ldots, x_1^{(2N-1)}) \tag{4}$$

which has the form of the equation of motion of one masspoint in the generalized mechanics[1]. It can be seen from the above derivation that in many cases the mechanics of one masspoint with higher derivatives is nothing else than the description of the motion of a single member of system of coupled masspoints belonging to the class I.

Besides the equation of motion (4) the linear momentum, the energy, and other properties of the systems of class I can be expressed by the single variable $x_1(t)$, as it will be shown below.

3. Within the class I there is a subclass ("class IL") where the equation of motion (4) can be derived from a Lagrangian. Let us see whether this one-masspoint-Lagrangian contains any information about the system of masspoints too. This will be studied by an example of a system of the class IL.

Let us consider two coupled masspoints in one dimension. The equations of motion are

$$m_1\ddot{x}_1 + k(x_1 - x_2) = 0, \qquad m_2\ddot{x}_2 + k(x_2 - x_1) = 0, \tag{5}$$

whose Lagrangian is

$$L_{12} = \tfrac{1}{2}\left[m_1\dot{x}_1^2 + m_2\dot{x}_2^2 - k(x_1 - x_2)^2\right]. \tag{6}$$

Let us eliminate the second masspoint, *i. e.* $x_2(t)$. Then (4) becomes

$$\frac{m_1 m_2}{m_1 + m_2}\, x_1^{(4)} + k x_1^{(2)} = 0, \tag{7}$$

for which one of the possible Lagrangians is (the Lagrangian is always determined only up to some additive and multiplicative entities):

$$L_1 = \frac{C}{2}\left[(m_1 + m_2)\dot{x}_1^2 - \frac{m_1 m_2}{k}\,\ddot{x}_1^2\right], \tag{8}$$

where C is a dimensionless constant, which will be "adjusted" later.

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[1] If the elimination of coordinates is stopped at $x_1, x_2, \ldots, x_n$ ($n < N$) then one arrives at the generalized mechanics of n masspoints. This case belongs to another subclass ("class n") which will not be treated here.

Let us now see what information is contained in (8).

First of all, (8) gives the equation of motion (7). Secondly, according to the Noether theorem two conserved quantities exist:

$$Q_x = C\left[(m_1+m_2)\dot{x}_1 + \frac{m_1 m_2}{k}\dddot{x}_1\right], \tag{9}$$

and

$$Q_t = \frac{C}{2}\left[(m_1+m_2)\dot{x}_1^2 + \frac{m_1 m_2}{k}\ddot{x}_1^2\right] - Q_x \dot{x}_1. \tag{10}$$

Q_x has the dimension of the linear momentum, Q_t that of the energy. Let us compare these quantities with the properties of the original system (5). The solution of (7) is

$$x_1(t) = a + bt + c\sin(\omega t + \varphi), \tag{11}$$

where a, b, c and φ are arbitrary constants. According to (11) the linear momentum of the masspoint "one" is

$$P_1 = m_1 b + m_1 c\omega \cos(\omega t + \varphi). \tag{12}$$

If we compare (9) and (12), we see that they are different. (9) is *conserved*, (12) is *not*. The non-conservation of P_1 and the conservation of Q_x are the proof that Q_x *does not belong to the masspoint* "*one*", although Q_x is expressed by $x_1(t)$ and by its derivatives. Substituting $x_2(t)$ from (5) into (9) we get

$$Q_x = C\left[m_1\dot{x}_1 + m_2\left(x_1 + \frac{m_1}{k}\ddot{x}_1\right)\right] = C(m_1\dot{x}_1 + m_2\dot{x}_2). \tag{13}$$

This means that in spite of the fact that (9) is expressed by $x_1(t)$ and $\dddot{x}_1(t)$, the quantity Q_x belongs to the system. Taking $C = 1$, Q_x is the linear momentum of the system.

Similarly, it may be shown that Q_t is proportional to the energy of the system (5). Therefore the Lagrangian (8) besides the description of the masspoint "one" *describes properties of the system* too.

4. Thus the generalized mechanics of one masspoint may be interpreted as a special representation, or description of a system of masspoints of class I. In this description the higher derivatives of the coordinate $x_1(t)$ of the "distinguished" masspoint appear instead of the coordinates of the remaining masspoints of the system. Since the properties of the system are described by $x_1(t)$, $\dot{x}_1(t)$, ..., the masspoint "one" may be called the "indicator" of the system. According to this interpretation the generalized mechanics of one masspoint is the mechanics of the indicator masspoint of a system. It may also be said that the generalized mechanics is the indicator representation of the ordinary one.

The examples of Weyssenhoff and Riewe belong to the class IL. It is understandable why their indicator masspoint rotates. In these examples at least one further masspoint is included implicitly, and they both (or all) rotate about the centre of mass of the system.

3. The relativistic case

5. The relativistic many-masspoint problem may be treated by means of the Fokker action principle. In the $1+1$ dimensional spacetime in the case of two coupled masspoints this action reads (Staruszkiewicz, 1967)

$$A = -m_{10}\int (dx_1^s dx_{1s})^{1/2} - m_{20}\int (dx_2^s dx_{2s})^{1/2} + \iint G[(x_1^s - x_2^s)(x_{1s} - x_{2s})]dx_1^u dx_{2u}, \tag{14}$$

where G represents the force by which the two masspoints are coupled. However, the functions G are known in a few cases only (Ramond 1973). *E. g.*, G is not known in the case of the harmonic force, which will be treated below. In addition to this, the Euler equations for this type of actions are not solvable in general, but in very exceptional cases only (Chern and Havas, 1973). Therefore there is a need for new methods.

In what follows we will treat the relativistic many-masspoint problem by the indicator representation, which leads to a solvable Euler equation. The prescriptions for the application of this method are the following:

(*i*) One takes the non-relativistic Lagrangian, or the equations of motion (1) of the system in question.

(*ii*) From (1) the equation (4) should be derived, which is the non-relativistic indicator representation of (1). If the Eq. (4) can be derived from a Lagrangian, then this non--relativistic indicator Lagrangian should be constructed.

(*iii*) The next step is the relativization of this Lagrangian, or of equations of motion (4).

(*iv*) Thereafter, the usual steps of the mechanics should be applied, which yield the motion of the indicator masspoint, and some information about the dynamical properties of the system.

How this can be performed will be illustrated by an example of a system of the class IL.

6. Besides the introduction of the indicator representation the relativization of the Lagrangian is the crucial point of the method presented here. Therefore we will shortly look over the relativistic theory of Lagrangian functions.

According to the basic principle of the relativistic mechanics the action A_1 of the indicator is an invariant, *i. e.* the Lagrangian L_1 obeys the equation

$$\bar{L}_1 d\bar{t} = L_1 dt, \tag{15}$$

where $\bar{L}_1, \bar{t}, \bar{x}_1(\bar{t}), \dot{\bar{x}}_1(\bar{t}), \ldots$ denote the Poincaré transformed entities. The explicit functional form of (15) is

$$f\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2}\left(t-\frac{vx_1}{c^2}\right)+a, \left(1-\frac{v^2}{c^2}\right)^{-1/2}(x_1-vt)+b, \frac{\dot{x}_1-v}{1-\frac{v\dot{x}_1}{c^2}}, \right.$$

$$\left. \ddot{x}_1\left(1-\frac{v^2}{c^2}\right)^{3/2}\left(1-\frac{v\dot{x}_1}{c^2}\right)^{-3}, \ldots\right] = f(t, x_1, \dot{x}_1, \ddot{x}_1, \ldots)\left(1-\frac{v^2}{c^2}\right)^{1/2}\left(1-\frac{v\dot{x}_1}{c^2}\right)^{-1} \tag{16}$$

whose general solution containing t, x_1, $\dot{x}_1$, $\ddot{x}_1$ only (two masspoint case) is

$$L_1 = \left(1-\frac{\dot{x}_1^2}{c^2}\right)^{1/2} g\left[\ddot{x}_1\left(1-\frac{\dot{x}_1^2}{c^2}\right)^{-3/2}\right], \tag{17}$$

where g is an arbitrary function of its invariant argument.

7. Let us now consider the relativistic motion of two masspoints coupled by a harmonic force. The non-relativistic indicator Lagrangian L_1 of this system is (8):

$$L_1 = \frac{1}{2}\left[(m_1+m_2)\dot{x}_1^2 - \frac{m_1 m_2}{k}\ddot{x}_1^2\right]. \tag{18}$$

The relativistic Lagrangian (17), which for $c \to \infty$ goes over into (8), will be

$$L_{15} = -c^2(m_{10}+m_{20})\left(1-\frac{\dot{x}_1^2}{c^2}\right)^{1/2} - \frac{m_{10}m_{20}}{2k}\ddot{x}_1^2\left(1-\frac{\dot{x}_1^2}{c^2}\right)^{-5/2} \tag{19}$$

The Eulerian of (19) is

$$\frac{\partial L_{15}}{\partial x_1} - \frac{d}{dt}\frac{\partial L_{15}}{\partial \dot{x}_1} + \frac{d^2}{dt^2}\frac{\partial L_{15}}{\partial \ddot{x}_1} = 0. \tag{20}$$

Since it does not contain x_1, we can operate with its first integral

$$\frac{\partial L_{15}}{\partial \dot{x}_1} - \frac{d}{dt}\frac{\partial L_{15}}{\partial \ddot{x}_1} = Q_x, \tag{21}$$

where Q_x = const is proportional to the linear momentum of the system. The explicit form of (21) is

$$(m_{10}+m_{20})B^{-1/2}\dot{x}_1 + \frac{5m_{10}m_{20}}{2kc^2}B^{-7/2}\dot{x}_1\ddot{x}_1^2 + \frac{m_{10}m_{10}}{k}B^{-5/2}\dddot{x}_1 = Q_x,$$

$$B \equiv 1-\dot{x}_1^2/c^2, \tag{22}$$

where the first term $m_{10}\dot{x}_1 B^{-1/2}$ is the linear momentum of the indicator masspoint.

The second conserved quantity of the system is

$$Q_t = \left(\frac{\partial L_{15}}{\partial \dot{x}_1} - \frac{d}{dt}\frac{\partial L_{15}}{\partial \ddot{x}_1}\right)\dot{x}_1 + \frac{\partial L_{15}}{\partial \ddot{x}_1}\ddot{x}_1 - L_{15} \tag{23}$$

whose explicit form is

$$Q_t = c^2(m_{10}+m_{20})B^{-1/2} + \frac{5m_{10}m_{20}}{2k}B^{-7/2} + \frac{m_{10}m_{20}}{k}(\dot{x}_1\dddot{x}_1 - 3\ddot{x}_1^2)B^{-5/2}; \tag{24}$$

it is connected with the energy of the system. The energy of the indicator is $c^2 m_{10} B^{-1/2}$. Q_t and Q_x constitute a Poincaré vector.

8. A particular solution of (22) is

$$x_1 = w_0 t + x_{10} \tag{25}$$

where w_0 = const. In this case (22) is

$$Q_x = (m_{10}+m_{20})w_0 \Big/ \left(1-\frac{w_0^2}{c^2}\right)^{1/2} \tag{26}$$

and (23) is

$$Q_t = c^2(m_{10}+m_{20}) \Big/ \left(1-\frac{w_0^2}{c^2}\right)^{1/2}. \tag{27}$$

One can conclude that both masspoints move together, and the interaction energy disappears.

9. Let us now consider the equation of motion in the coordinate system, where $Q_x = 0$. Then the first integral (22) becomes

$$(m_{10}+m_{20})B^{-1/2}\dot{x}_1 + \frac{5m_{10}m_{20}}{2kc^2} B^{-7/2}\dot{x}_1\ddot{x}_1^2 + \frac{m_{10}m_{20}}{k} B^{-5/2}\dddot{x}_1 = 0. \tag{28}$$

Since $\ddot{x}_1^2 \geqslant 0$, from (28) we have

$$\operatorname{sign} \dddot{x}_1 = -\operatorname{sign} \dot{x}_1 \tag{29}$$

which means that the indicator oscillates, as in the non-relativistic case.

The second integral of (20) is

$$\ddot{x}_1^2 = \left[a_0^2 - \frac{2c^2k(m_{10}+m_{20})}{m_{10}m_{20}}\right] B^{5/2} + \frac{2c^2(m_{10}+m_{20})k}{m_{10}m_{20}} B^3, \tag{30}$$

where a_0 is an arbitrary constant which is just the value of $\ddot{x}_1(t)$ when $\dot{x}_1(t) = 0$, *i. e.* at the turning point of the indicator. Denoting the right-hand side of (30) by $h_0^2(\dot{x}_1)$, and introducing the parameter p, the solution of the equation of motion in parametric form reads

$$t = \int_0^p h_0^{-1}(p)dp + t_0, \tag{31}$$

$$x_1 = \int_0^p ph_0^{-1}(p)dp + x_{10}. \tag{32}$$

10. In the case $Q_x \neq 0$ one gets the general solution of the equation of motion. In parametric form it reads

$$t = \int_0^p h^{-1}(p)dp + t_0, \tag{33}$$

$$x_1 = \int_0^p ph^{-1}(p)dp + x_{10}, \tag{34}$$

where

$$h^2(p) = \left(1-\frac{p^2}{c^2}\right)^{5/2} \left\{h^2(0) - \frac{2c^2k(m_{10}+m_{20})}{m_{10}m_{20}}\left[1-\left(1-\frac{p^2}{c^2}\right)^{1/2}\right] + \frac{2Q_xk}{m_{10}m_{20}}\, p\right\}, \tag{35}$$

and $h(0)$, Q_x, t_0, and x_{10} are the arbitrary constants.

11. It is seen from this example that in the relativistic mechanics the indicator representation may be useful, since (*i*) the relativization of the equation of motion, or that of the Lagrangian L_1 of the indicator $x_1(t)$ seems to be an easier method than the ortodox one (Van Dam and Wigner, 1965), and (*ii*) the indicator representation gives the full information at least for one masspoint of the system.

The motion of the remaining members of the system, and some other questions will be treated in another paper.

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