AN APPROXIMATIVE METHOD FOR COMPUTING CROSS-SECTIONS

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An approximative method consisting in combining the procedure of iteration with the best fit of a free parameter is discussed. This method is shown to be particularly simple and exact in a wide range of values of the coupling constant. It deserves to be called a "Refined Born Approximation". The problem of transition from the case of scattering on an external potential to the case of a mutual scattering of two particles is discussed anew.

1. General procedure

A viable method of computing cross-sections from field equations or from the Schrödinger equation in the strong coupling case is unknown except for the phase shift analysis. This method is, however, cumbersome except for the limits of very low and very high energies where one may either restrict the investigation to the first few terms of the expansion into the Legendre polynomials or use the asymptotic expansions.

We shall discuss some alternative methods of approximative calculations valid for arbitrary values of the energy, and having nothing to do with the assumption of a weak coupling. Generally speaking, these methods consist in a best fit of some free parameters introduced into the expression for the scattering amplitude. The general idea of such procedures will be explained on a simple example of elastic scattering of particles within the framework of non-relativistic quantum mechanics, but these methods may be also extended for the case of relativistic field theories.

Let us look for a stationary solution of the Schrödinger equation

$$\left(-\frac{1}{2m}\nabla^2+V\right)\psi(\vec{r}) = \omega\psi(\vec{r}). \tag{1}$$

Assume, as usual, the wave function to be of the form

$$\psi(\vec{r}) = e^{i\vec{p}\vec{r}}+\chi(\vec{r}), \tag{2}$$

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with

$$\frac{p^2}{2m} = \omega. \tag{3}$$

The lack of a (dimensional) factor in front of the exponential function means that the plane wave has been normalized so as to describe one particle per cm^3 in the incident beam.

It will be convenient for our purposes to find an integral equation for the scattered part $\chi(\vec{r})$ (but not for the total wave function ψ nor for the scattering amplitude f). Introducing (2) with (3) into (1) we get

$$(\nabla^2 + p^2)\chi = 2mV(e^{i\vec{p}\vec{r}} + \chi). \tag{4}$$

The use of the Green functions

$$G^{a/r}(\vec{r}) = \frac{1}{4\pi} \frac{e^{\pm ipr}}{r} \tag{5}$$

or, in the Fourier representation,

$$G^{r/a}(\vec{r}) = -\frac{1}{(2\pi)^3} \int d^3k \, \frac{e^{i\vec{k}\vec{r}}}{p^2 - k^2 \pm i\varepsilon}, \tag{6}$$

enables one to replace the differential equation (4) by an integral equation. If χ is to represent the outgoing wave, we have to use the retarded Green function G^r. Thus,

$$\chi(\vec{r}) = -\frac{1}{4\pi} \int d^3r' \, \frac{e^{ip|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \, U(\vec{r}') \, [e^{i\vec{p}\vec{r}'} + \chi(\vec{r}')], \tag{7}$$

where

$$U = 2m\,V. \tag{8}$$

Writing the asymptotic solution for $r \to \infty$ in the form

$$\chi(\vec{r}) \to \frac{e^{ipr}}{r} f, \tag{9}$$

one obtains the scattering amplitude f.

Putting the problem in the form of an integral equation for χ in the x-space has considerable advantages: If the potential is short-ranged then it is seen that the right--hand side of (7) depends only upon the values assumed by χ for small r. Therefore we have only to make a proper fit of $\chi(\vec{r})$ for small values of r so that, introducing this χ into the integrand to the right and performing the integrations, we shall get a correct result for the left-hand side of (7) for any r, small or large. Hence, the scattering amplitude and cross--section can be inferred.

By putting to the right-hand side of (7) $\chi^{(0)} = 0$ one obtains Born's approximation which, however, is legitimate only if the interaction is weak. A possibility of improving the approximation consists in introducing to the right-hand side of (7), instead of zero, a function dependent on a certain number of parameters

$$\chi^{(0)} = \xi(\vec{r}, \vec{p}, \alpha_1, \ldots, \alpha_N) \tag{10}$$

and trying to make a possibly best fit of the parameters.

The equation (7) is of the form

$$\chi = F(\chi), \tag{11}$$

where F means a linear functional. Let us consider, more generally, the linear relation

$$\eta = F(\xi). \tag{12}$$

The function η may be regarded as an approximate solution of the equation (11) if it differs little from ξ. Of course, one has to define properly the sense of "η differing little from ξ". If these functions were square integrable, then we could use, as a criterion, a small quadratic deviation in the whole space but, unfortunately, the outgoing wave, being a solution of (11), is not square integrable. The way out of this difficulty is possible due to the short range of the interaction. In fact, η depends only upon the values of ξ in the region of small r, of the order of magnitude of the range of interaction (unless one assumes ξ to increase unreasonably with r). Therefore it is sufficient to secure

$$\Delta(\xi, \eta) = \int_{r \leqslant R} d^3r |\eta - \xi|^2 \tag{13}$$

to be small, where R denotes the range of interaction, in order to guarantee η to be an approximative solution of (11), provided ξ has been chosen to be a decreasing, or at least, not increasing function for $r > R$.

Thus, a multi-parametric function of the type (10) should be introduced and the parameters extremalized from the equations

$$\frac{\partial \Delta^{(1)}}{\partial \alpha_j} = 0 \quad \text{where} \quad \Delta^{(1)} = \int_{r \leqslant R} dr |\chi^{(1)} - \chi^{(0)}|^2, \tag{14}$$

with $j = 1, 2, 3\ldots, N$ and

$$\chi^{(0)} = \xi, \quad \chi^{(1)} = \eta. \tag{15}$$

This procedure may be still refined by combining it with higher orders of the procedure of iteration, i. e. performing n iterations and requiring

$$\frac{\partial \Delta^{(n)}}{\partial \alpha_j} = 0, \quad \text{where} \quad \Delta^{(n)} = \int_{r \leqslant R} d^3r |\chi^{(n)} - \chi^{(n-1)}|^2. \tag{16}$$

Alternatively, starting with an N-parametric function (10) the parameters may be fitted at the point $r = 0$ by performing N iterations and demanding

$$\chi^{(0)}(0) = \chi^{(1)}(0) = \ldots = \chi^{(N)}(0). \tag{17}$$

It should be stressed that for large values of the coupling constant the convergence of the iterations to a solution is not guaranteed. Nevertheless, the conditions (16) or (17) do provide us with approximative solutions. The question whether this approximation is good or poor depends essentially upon the proper choice of the starting-point function (10). This choice is limited by the requirement of obtaining a viable procedure. To this end the function (10) must be chosen sufficiently simple for the integrations to be performed effectively. Moreover, in order to get simple conditions (16) or (17) for the parameters it is advisable to introduce (10) in the form of a linear function of the parameters. In the case of spherically symmetric potentials a plausible and sufficiently flexible form of the function (10) seems to be

$$\chi^{(0)}(r, \cos \vartheta) = \sum_{\mu} \sum_{\nu} \alpha_{\mu\nu} e^{i\mu \vec{p} \cdot \vec{r}} r^{\nu}, \tag{18}$$

where $\mu = 0, \pm 1, \dots \pm M$, $\nu = 0, 1, \dots N$.

A drawback of the above described procedures is that they do not provide one with estimates of the limits of accuracy of the approximations but, at any rate, they yield a criterion enabling one to estimate which one of a set of approximations is the best. Considering two starting-point functions ξ and $\bar{\xi}$ of, say, quite different form and computing η and $\bar{\eta}$ form the (general) formula (12) it may be claimed $\bar{\eta}$ to constitute a better approximation than η if

$$\Delta(\bar{\xi}, \bar{\eta}) < \Delta(\xi, \eta), \tag{19}$$

with $\Delta(\xi, \eta)$ defined by (13). Then also the value of the cross-section computed with the help of $\bar{\eta}$ will be more reliable than that computed with the help of η. Thus, $\Delta(\xi, \eta)$ may be called the "index of reliability". The existence of a criterion (19) enables one to undertake a more systematic search for suitable forms of starting-point functions.

2. *One-parametric fits*

With the lack of any better guess of the function (10) or (18) we recall that the first Born approximation yields, in several cases, surprisingly good results, especially as regards the angular dependence of the scattering amplitude. Therefore, we may introduce into the right-hand side of (7) the first Born approximation multiplied by an, at first, arbitrary coefficient

$$\chi^{(1)} = \beta \chi_B^{(1)} \tag{20}$$

and compute the left-hand side, to be called $\chi^{(2)}$. The easiest way to fit the parameter β is to equate the resulting function $\chi^{(2)}$ with the starting-point function $\chi^{(1)}$ at the point $r = 0$, i.e. to require

$$-\frac{1}{4\pi} \int d^3r \frac{e^{ipr}}{r} U(r) [e^{i\vec{p}\vec{r}} + \beta \chi_B^{(1)}(\vec{r})] = \beta \chi_B^{(1)}(0) \tag{21}$$

as a condition upon β.

Having computed the parameter β from (21) we may use again the equation (7) and compute its left-hand side for $r \to \infty$ with the known term (20) substituted for χ into the right-hand side. The result may be called the "Refined" (second) Born Approximation" (RBA).

The above described procedure may be still simplified by limiting oneself to something that may be called the "Refined (first) Born Approximation" consisting in the following: we may introduce into the right-hand side of (7) instead of χ, the plane wave

$$\chi^{(0)}(\vec{r}) = \alpha e^{i\vec{p}\vec{r}}. \tag{22}$$

The result, i.e. the left-hand side, becomes in this case nothing else but the first Born approximation multiplied by the factor $\beta = \alpha + 1$. The parameter α may be adjusted by equating both, the zero-order, and the first-order approximation at the origin $r = 0$

$$\chi^{(1)}(0) = \chi^{(0)}(0), \tag{23}$$

whence

$$(\alpha+1)\chi_B^{(1)}(0) = \alpha. \tag{23'}$$

In this case the angular dependence of the scattering amplitude is the same as that known from the first order Born approximation, but the amplitude appears to be $(\alpha+1)$-times larger.

3. *Scattering of extremely slow particles on the square well potential*

We shall discuss this simple example in detail because in this case the exact solution is known so that the reliability of our method may be checked and compared with other methods of approximations.

Let

$$V = \begin{cases} \gamma \cdot a \\ 0 \end{cases} \quad \text{for} \quad \begin{cases} a \cdot r < 1 \\ a \cdot r \geqslant 1. \end{cases} \tag{24}$$

For negative values of the constant γ it represents a well of radius a^{-1} and depth $a|\gamma|$. For large positive values of γ it represents a repulsive hard core.

Denoting

$$G = -\frac{2m\gamma}{a} \tag{25}$$

and going over to the limit $p \to 0$, the equation (7) simplifies considerably

$$\chi(\vec{r}) = \frac{Ga^2}{4\pi} \int_{(ar<1)} d^3r' \frac{1}{|\vec{r}-\vec{r}'|} \{1+\chi(\vec{r}')\}. \tag{26}$$

Introducing a plane wave $\alpha \exp i\,\vec{p}\vec{r}$ into the right-hand side of (26) we get, always in the limit $p \to 0$,

$$\chi^{(1)}(\vec{r}) = \frac{Ga^2}{4\pi}(1+\alpha) \int\limits_{(ar<1)} d^3r' \frac{1}{|\vec{r}-\vec{r}'|} \tag{26'}$$

whence

$$\chi^{(1)}(\vec{r}) = \begin{cases} \frac{G}{2}(1+\alpha)\left(1-\frac{(ar)^2}{3}\right) & \text{for } ar < 1 \\ \frac{G}{3}\frac{1+\alpha}{ar} & \text{for } ar > 1. \end{cases} \tag{27}$$

The self-consistency requirement (23) for α yields

$$\frac{G}{2}(1+\alpha) = \alpha \quad \text{or} \quad \alpha = \frac{G}{2-G}. \tag{28}$$

The asymptotic form of χ for $ar \gg 1$ yields the following expression for the scattering amplitude

$$f^{(1)} = \frac{1}{3}\frac{G}{a}(1+\alpha) = \frac{G}{3a}\frac{1}{1-\frac{1}{2}G}. \tag{29}$$

Thus, for $p \to 0$, the scattering amplitude becomes real, independent of the scattering angle, and possesses a pole for $G = 2$.

Let us compute the second iterative approximation. By introducing (27) into the right-hand side of (26) we find the following asymptotic value for $r \to 0$

$$\chi^{(2)}(0) = \frac{G}{2}\left[1+\frac{5}{12}(1+\alpha)G\right] \tag{30}$$

and for $a \cdot r \gg 1$

$$f^{(2)} = \frac{G}{3a}\left[1+\frac{2}{5}(1+\alpha)G\right] \tag{31}$$

respectively. The self-consistent fit

$$\chi^{(2)}(0) = \chi^{(1)}(0) \tag{32}$$

of the parameter α yields

$$\alpha = \frac{G}{\frac{12}{5} - G}, \tag{33}$$

a result very similar to (28). By introducing this value into (31) we get

$$f^{(2)} = \frac{G}{3a} \frac{1 - \frac{1}{60} G}{1 - \frac{5}{12} G}. \tag{34}$$

Comparing this result with (29) it is seen that both are very similar to each other provided $G < 25$. But more interesting will be the comparison of our result (34), or equivalently, of the scattering length, with the exact result as well as with the results of the usual Born approximations and with Schwinger's variational method. This comparison is given in Figure 1. It is seen that our refined (second) Born approximation agrees exceedingly with

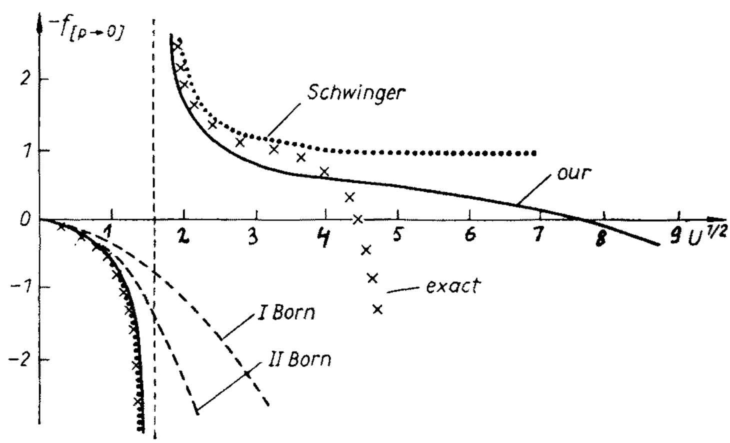

Fig. 1. Scattering length as a function of the depth of a square well U

the exact result, up to the values of about $G = 18$. In particular, it describes correctly the first pole appearing in the exact solution. The same advantages are exhibited by Schwinger's variational result (up to the values of about $G = 16$). However, Schwinger's variational computations are much more difficult and must be performed separately for each partial wave whereas our method, similarly as Born's procedure applies to the full amplitude without necessity of performing any expansion into Legendre polynomials.

4. *Scattering on the Yukawa potential*

Let us discuss briefly the scattering of a "nucleon" with mass m on a Yukawa potential

$$U = -Ga\frac{e^{-ar}}{r}, \qquad G = \frac{2m}{a}\frac{g^2}{4\pi}, \tag{35}$$

where a denotes the pion mass. Introducing this potential into the integral equation (7) and assuming, as a zero-order approximation, the function (22) it is immediately seen that the first order function $\chi^{(1)}$ is nothing else but the function resulting in the first Born approximation but multiplied by the factor $1+\alpha$. Consequently, the scattering amplitude will be also proportional to that of Born with the same factor of proportionality

$$f^{(1)} = (1+\alpha)f_{\mathrm{B}}^{(1)}. \tag{36}$$

The unknown parameter α may be fitted again by the condition (23). This yields

$$\chi^{(1)}(0) = (1+\alpha)\frac{Ga}{4\pi}\int d^3r\,\frac{e^{(ipr-ar+i\vec{p}\vec{r})}}{r^2} = \alpha, \tag{37}$$

whence

$$1+\alpha = \left[1-\frac{iG}{x}\ln(1-ix)\right]^{-1}, \tag{38}$$

where x means essentially the momentum of the incident nucleon

$$x = \frac{2p}{a}. \tag{39}$$

The relation between Born's and our (differential or total) cross-section is

$$\sigma^{(1)} = |1+\alpha|^2\sigma_{\mathrm{B}}^{(1)}. \tag{40}$$

In Fig. 2 the total elastic cross-section for a fixed value $x = 0.4$ is plotted as a function of G. It is seen that for such a comparatively small value of momentum, Born's result is correct only for G much smaller than unity. On the other hand, our result for the cross--section is by far better and exhibits correctly the resonant character of the scattering for G of the order of magnitude of unity.

Fig. 3 describes the dependence of the cross-section on the momentum for a fixed value $G = 10$. Our result coincides with that of Born for sufficiently high momenta ($x > 25$) where Born's approximation is known to be quite good. On the other hand, for strong coupling but comparatively small momenta there appears a great discrepancy between the two results.

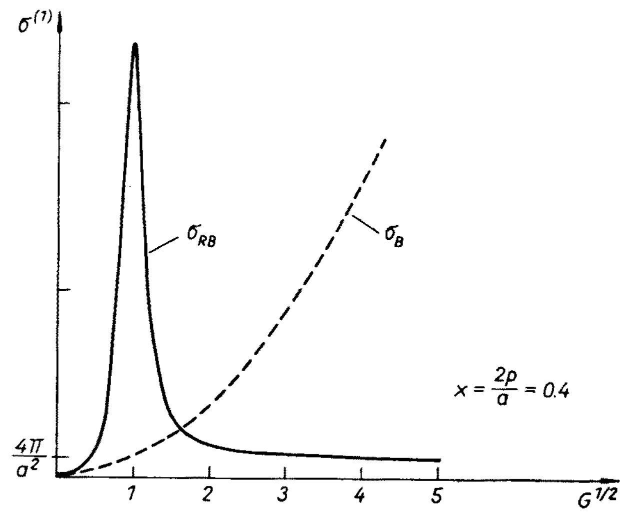

Fig. 2. Total cross-section for the Yukawa potential as a function of the coupling constant at a fixed value of momentum

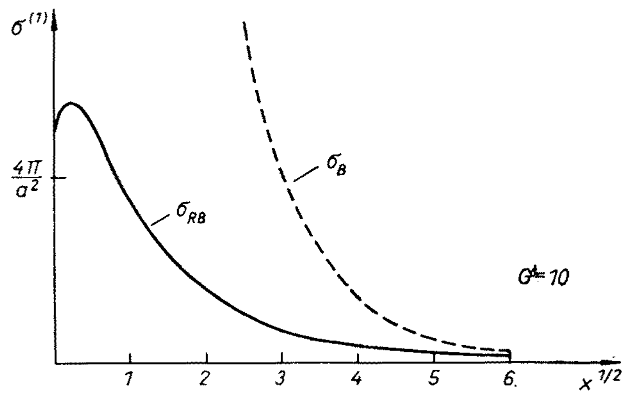

Fig. 3. Total cross-section for the Yukawa potential as a function of momentum at a fixed value of the coupling constant

5. Transition to the c. m. s.

In all text-books about scattering problems it is stated that the transition from the case of scattering on an external potential to the case of scattering of two particles (interacting by means of a potential of the same form) may be achieved simply by replacing the mass m by the reduced mass. The proof of this fact has been achieved by separating the motion of the centre of mass from the relative motion. However, an objection may be raised that in this case one has to do with a description of the motion of one of the two colliding particles in a non-inertial frame of reference whose origin is fixed at the position of the other particle which gives rise to doubts as to whether this procedure remains valid

also for the case of highly energetic collisions in the relativistic theory. Besi
approach is certainly unphysical since one never measures the relative coordin
always coordinates of the particle with respect to an inertial frame of referen

To meet such doubts and objections we shall discuss the problem of transiti
the case of scattering on an external potential to the case of a mutual scatterin
colliding particles anew in a more methodological fassion. To do so we may st
the Schrödinger equation for two particles

$$\left(-\frac{1}{2m_1}\nabla_1^2-\frac{1}{2m_2}\nabla_2^2+V(|\vec{r}_1-\vec{r}_2|)\right)\Psi(\vec{r}_1,\vec{r}_2,t) = i\frac{\partial}{\partial t}\Psi,$$

and look for a stationary solution

$$\Psi(\vec{r}_1,\vec{r}_2,t) = \psi(\vec{r}_1,\vec{r}_2)e^{-i\omega t}.$$

Let us perform the separation of variables by going over from $\vec{r}_1$, $\vec{r}_2$ to $\vec{r}$ and $\vec{R}$,
describes, as usual, the coordinates of the mass centre but $\vec{r}$ does not mean the
coordinates but is defined as follows

$$\vec{r} = \frac{m_2}{M}(\vec{r}_1-\vec{r}_2), \quad \vec{R} = \frac{m_1\vec{r}_1+m_2\vec{r}_2}{M},$$

where M means the total mass of the system

$$M = m_1+m_2.$$

The physical meaning of the new coordinates $\vec{r}$ becomes clear by remarking t

$$\vec{r} = \vec{r}_1-\vec{R}(\vec{r}_1,\vec{r}_2).$$

Thus, $\vec{r}$ denotes the coordinates of the first particle with respect to an *inertial* co
system whose origin coincides with the centre of mass of the system. This has me
gical advantages because one can measure directly positions, momenta, etc. of a
with respect to an inertial frame of reference but not with respect to the other
inasmuch as — in the latter case — we should "sit on the other particle" toget
our apparatus of measurement while this particle itself undergoes an accelerat
motion of the centre of mass being uniform, it may be brought to rest in an inertia
of reference. If the centre of mass coincided with the origin of this inertial frame
rence, then the coordinates $\vec{r}$ denote those of the first particle in this system of r

From (42) we get

$$\frac{\partial}{\partial x_1} = \frac{1}{M}\left(m_1\frac{\partial}{\partial X}+m_2\frac{\partial}{\partial x}\right), \quad \frac{\partial}{\partial x_2} = \frac{m_2}{M}\left(\frac{\partial}{\partial X}-\frac{\partial}{\partial x}\right)$$

and

$$\frac{1}{2m_1}\frac{\partial^2}{\partial x_1^2}+\frac{1}{2m_2}\frac{\partial^2}{\partial x_2^2} = \frac{1}{2M}\frac{\partial^2}{\partial X^2}+\frac{1}{2M}\frac{m_2}{m_1}\frac{\partial^2}{\partial x^2},$$

whence

$$\frac{1}{2m_1}\nabla_1^2+\frac{1}{2m_2}\nabla_2^2=\frac{1}{2M}\nabla_{\vec{R}}^2+\frac{1}{2M}\frac{m_2}{m_1}\nabla_{\vec{r}}^2, \tag{45}$$

while

$$\vec{r}_1-\vec{r}_2=\frac{M}{m_2}\vec{r}, \tag{46}$$

so that the stationary equation assumes the form

$$\left[-\frac{1}{2M}\nabla_{\vec{R}}^2-\frac{1}{2M}\frac{m_2}{m_1}\nabla_{\vec{r}}^2+V\left(\frac{M}{m_2}r\right)\right]\psi(\vec{R},\vec{r})=\omega\psi(\vec{R},\vec{r}) \tag{47}$$

and is separable.

At the first sight the above equation may look strange but it is certainly correct. In particular, if $m_2\to\infty$ then $M/m_2\to 1$ and the above equation turns over into the usual equation for the first particle in an external potential

$$\left[-\frac{1}{2m_1}\Delta_{\vec{r}}^2+V(\vec{r})\right]\psi(\vec{r},\vec{R})=\omega\psi(\vec{r},\vec{R}). \tag{47'}$$

The position of the centre of mass coincides with that of the second particle in this limit, and we may simply put into the above equation $\vec{R}=0$.

By performing the separation of variables in (47) and assuming the centre of mass to be at rest ($\vec{P}=0$) we are left with the equation

$$\left[-\frac{1}{2M}\frac{m_2}{m_1}\nabla^2+V\left(\frac{M}{m_2}r\right)\right]\psi(\vec{r})=\omega\psi(\vec{r}) \tag{48}$$

describing the motion of the first particle, or

$$\left[-\frac{1}{2m_1}\nabla^2-\omega_1\right]\psi=-\tilde{V}(r)\psi, \tag{49}$$

where

$$\omega_1=\frac{M}{m_2}\omega,\qquad \tilde{V}(r)=\frac{M}{m_2}V\left(\frac{M}{m_2}r\right). \tag{50}$$

In order to describe the scattering phenomena we assume a solution of (49) in the usual form

$$\psi(\vec{r})=e^{i\vec{p}\vec{r}}+\chi(\vec{r}). \tag{51}$$

Inasmuch as the total energy is only determined up to an arbitrary constant, we may choose this constant so that

$$\frac{p^2}{2m_1}=\omega_1. \tag{52}$$

By exchanging the indices $1 \leftrightarrow 2$ we find the following relation

$$\omega_1+\omega_2 = \frac{M}{\mu}\,\omega, \tag{52'}$$

where μ is the reduced mass. Introducing (51) and (52) into (49) we obtain an equation for χ

$$(\nabla^2+p^2)\chi = \tilde{U}\cdot(e^{i\vec{p}\vec{r}}+\chi), \tag{53}$$

where

$$\tilde{U} = 2m_1\tilde{V}. \tag{54}$$

It is seen that the equation (53) is identical in form with the equation (4) for the scattering on an external potential, the only difference being that V is to be replaced by $\tilde{V}$. Thus, the rule for going over from the scattering on an external potential to a mutual scattering of two particles is

$$V \to \tilde{V} \tag{55}$$

or

$$U = 2m_1 V \to \tilde{U} = 2m_1 \tilde{V}. \tag{55'}$$

Now, in the special case of Yukawa potential we have

$$U = -2m_1\frac{g^2}{4\pi}\frac{e^{-ar}}{r} \to \tilde{U} =$$

$$= -2m_1\frac{g^2}{4\pi}\frac{M}{m_2}\frac{e^{-a\frac{M}{m_2}r}}{\frac{M}{m_2}r} = -2m_1\frac{g^2}{4\pi}\frac{e^{-\tilde{a}r}}{r}, \tag{56}$$

where

$$\tilde{a} = \frac{M}{m_2}a. \tag{57}$$

Thus, the transition from the case of scattering on an external Yukawa potential to the mutual scattering of two particles consists, originally, in a replacement

$$a \to \tilde{a} = \frac{M}{m_2}a. \tag{58}$$

In view of the definition (35) of G we have

$$G = \frac{2m_1}{a}\frac{g^2}{4\pi} \to \tilde{G} = \frac{2m_1}{\tilde{a}}\frac{g^2}{4\pi} = \frac{2m_1 m_2}{aM}\frac{g^2}{4\pi} = \frac{2\mu}{a}\frac{g^2}{4\pi} \tag{59}$$

so that, indeed, the change of G may be viewed upon as a replacement of the actual mass of the scattered particle by the reduced mass. Moreover, the scattering amplitude involves the parameter a also through the variable x which undergoes a change

$$x = \frac{2p}{a} \to \tilde{x} = \frac{2p}{\tilde{a}} = \frac{2m_2 p}{Ma}. \tag{60}$$

Inasmuch as (in the non-relativistic theory) $p_1 = m_1 v$, the above replacement is identical with

$$\frac{m_1}{a} v \to \frac{m_1 m_2}{Ma} v = \frac{\mu}{a} v \tag{61}$$

which again may be interpreted as a replacement of the actual mass m_1 by the reduced mass μ while keeping the actual velocity of the particle and the value of a unchanged. However, the change of mass may be regarded as apparent while the genuine physical effect consists in the change (57), i. e. in a contraction of the range $R = 1/a$ of nuclear forces by the factor m_2/M for the first particle

$$\tilde{R}_1 = \frac{m_2}{M} R \tag{62}$$

and, similarly, by the factor m_1/M for the second particle

$$\tilde{R}_2 = \frac{m_1}{M} R. \tag{62'}$$