VERTEX FUNCTION IN QUANTUM ELECTRODYNAMICS WITH COMPENSATING CURRENT

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Quantum electrodynamics with compensating current, described in previous publica tions is further investigated. This paper is devoted to the problem of the definition of the vertex function in this theory. The main difficulty is contained in the fact that the gauge independent and compensating current dependent photon propagator is, in fact, a projector and therefore its inversion does not exist. Thus, the external photon line in the three-point fermion-fermion-photon propagator cannot be cut in an unambiguous way. It is shown, however, that the vertex function can be defined analytically in an unambiguous way by means of the effective external field analogous to the one used in the usual version of the theory. The problem of the renormalization of the gauge-independent vertex function is also considered and it is shown that this quantity obeys the integral equation analogous to one obtained by Brandt with the help of the Wilson expansion. In the Appendices expressions are given for the electron self-energy part and the second integral equation fulfilled by the vertex function.

1. *Introduction*

This paper is the last of the series of three papers devoted to the problem of the formulation of quantum electrodynamics with compensating current. In two previous papers of this series [2, 3] we have described the general outline of this theory [2] and the problems connected with the electromagnetic current definition and the vacuum polarization tensor [3]. Generalizing the ideas of Heisenberg and Euler [1], Zumino [10], Białynicki-Birula [9, 12], Johnson [6] and Mandelstam [11], we have shown in [2] that due to the compensating, nonconserved current a^λ, all the propagators and field operators can be defined in a gauge-invariant manner and, at least for a certain class of compensating currents, the theory can be formulated in the good Hilbert space with positive norm squared. Moreover, it turned out that quantum electrodynamics with compensating current is equivalent to the theory formulated in a certain gauge, namely the gauge described by

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this current [9, 10]. This means that, for instance, the compensating current-dependent photon propagator $\mathscr{D}^F_{\mu\nu}$ (formula (2.14b) in [2]) is equal to the propagator evaluated in the corresponding gauge and all gauge-dependent objects have the same form, as if they were calculated in this gauge.

The problems connected with the definition of the current operator have been investigated in [3]. It has been shown that the limiting procedure proposed by Heisenberg and Euler [1], Brandt [4] and Schwinger [13] can be generalized to the case of an arbitrary compensating current and gauge invariance of this definition is assured by the presence of the expression of the following type:

$$\exp\left[-ie\int d^4z(a^\lambda(z-x-\xi)-a^\lambda(z-x+\xi))A_\lambda(z)\right]. \tag{1.1}$$

This is an obvious generalization of the linear integral of the potential firstly used by Heisenberg and Euler [1]. Our definition of the current operator is consistent with the gauge invariance of this observable, since we have shown that it is independent on the choice of the compensating current.

This paper is mainly devoted to the problem of the definition of the vertex function in quantum electrodynamics with compensating current. It is not a trivial problem, since the gauge-independent and compensating current-dependent photon propagator is a projector and its inversion does not exist. Therefore, the effective external photon line expression in the propagator $G_\nu[x, y, z|a]$ (formula (2.13) in [3]) cannot be cut in an unambiguous way. We show, however, that the compensating current-dependent gauge-invariant vertex function $\Gamma^\mu[a]$ can be defined unambiguously as the functional derivative of the inverse fermion propagator with respect to the effective external electromagnetic field. This field can be introduced in a natural way into the fermion propagator equation, and therefore the next Section is devoted to the description of this object in quantum electrodynamics with compensating current. After introducing the effective external field we define the vertex function in Section 3. We show that its perturbation expansion can be constructed with the help of exactly the same Feynman diagrams as in the usual version of the theory, in which the expression corresponding to the internal photon line is equal to $\mathscr{D}^F_{\mu\nu}$. This result is consistent with the equivalence of the gauge dependence and the compensating current dependence, mentioned above. In Section 4 we obtain the integral equation fulfilled by the vertex function and give a detailed discussion of the renormalization of this object. The second integral equation satisfied by the vertex function is given in Appendix B. In Appendix A we discuss briefly the compensating current-dependent electron self-energy part and its renormalization.

2. *Electron propagator and effective electromagnetic field*

Similarly as in [3], we start from the expression for an arbitrary propagator in quantum electrodynamics with compensating current:

$$G_{\mu_1\cdots\mu_k}[x_1, \ldots, x_n, y_n, \ldots, y_1, z_1, \ldots, z_k|\mathscr{A}a] = \exp\left(-ie\int a\mathscr{A}\right)\times$$

$$\times i^k \frac{\delta^k}{\delta I^{\mu_1}(z_1) \dots \delta I^{\mu_k}(z_k)} \Big\{ (V[\mathscr{A}])^{-1} \exp\Big(-\frac{i}{2} \int I \Delta_F I \Big) \exp\Big(-\int I D_F^{(a)} \frac{\delta}{\delta \mathscr{A}} \Big) \times$$

$$\times \exp\Big(-\frac{i}{2} \int \frac{\delta}{\delta \mathscr{A}} \mathscr{D}^F \frac{\delta}{\delta \mathscr{A}} \Big) C[\mathscr{A}] K_F[x_1, \dots, x_n, y_n, \dots, y_1 | \mathscr{A}] \Big\}\Big|_{I=0} \tag{2.1}$$

It follows that the electron propagator can be written as:

$$G[x, y|\mathscr{A}a] = \exp\left[-ie \int d^4z (a^\lambda(z-x) - a^\lambda(z-y)) \mathscr{A}_\lambda(z)\right] \times$$

$$\times \Big\{ (V[\mathscr{A}])^{-1} \exp\Big(-\frac{i}{2} \int \frac{\delta}{\delta \mathscr{A}} \mathscr{D}^F \frac{\delta}{\delta \mathscr{A}} \Big) C[\mathscr{A}] K_F[x, y|\mathscr{A}] \Big\}. \tag{2.2}$$

One can easily show that this quantity satisfies the following integral-differential equation:

$$\Big(-i\gamma^\mu \frac{\partial}{\partial x^\mu} + m \Big) G[x, y|\mathscr{A}a] = \delta^{(4)}(x-y) - e\gamma^\nu \int d^4z \lambda_{\mu\nu}(z-x) \mathscr{A}^\mu(z) G[x, y|\mathscr{A}a] -$$

$$- e\gamma^\nu \int d^4z \lambda_{\nu\mu}(z-x) G^\mu[x, y, z|\mathscr{A}a], \tag{2.3}$$

where $\lambda_{\mu\nu}$ is the projective operator defined already in [3]:

$$\lambda_{\mu\nu}(z-x) = \delta^{(4)}(z-x) g_{\mu\nu} - \frac{\partial}{\partial z^\mu} a_\nu(z-x). \tag{2.4}$$

The three-point function appearing in the last term on the right-hand side of (2.3) can be written as the vacuum expectation value of the time ordered product containing the gauge-invariant vector field operator (formula (2.9) in [3]):

$$\int d^4z \lambda_{\nu\mu}(z-x) G^\mu[x, y, z|\mathscr{A}a] =$$

$$= i\langle 0| T[\psi(x) \exp(-ie \int d^4z' \{a^\lambda(z'-x) - a^\lambda(z'-y)\} A_\lambda(z')) \bar{\psi}(y) B_\nu(x)] |0\rangle, \tag{2.5}$$

where

$$B_\nu(x) = \int d^4z \lambda_{\nu\mu}(z-x) A^\mu(z). \tag{2.6}$$

Thus, similarly as in the case of the photon propagator, the right-hand side of (2.5) is expressed in terms of the propagator with two arguments equal. A formal quantity of this type must be carefully defined with the help of a limiting procedure; this problem will be briefly discussed in Appendix A. It follows from (2.1) that the three-point function can be formally given by the following formula:

$$G^\mu[x, y, z|\mathscr{A}a] = \exp\left[-ie \int d^4z (a^\lambda(w-x) - a^\lambda(w-y)) \mathscr{A}_\lambda(w)\right] \times$$

$$\times \Big\{ (V[\mathscr{A}])^{-1} \Big[-i \int d^4v D_F^{(a)\mu\lambda}(z-v) \frac{\delta}{\delta \mathscr{A}^\lambda(v)} \Big] \exp\Big(-\frac{i}{2} \int \frac{\delta}{\delta \mathscr{A}} \mathscr{D}^F \frac{\delta}{\delta \mathscr{A}} \Big) \times$$

$$\times C[\mathscr{A}] K_F[x, y|\mathscr{A}] \Big\}. \tag{2.7}$$

Using now the equations:

$$\int d^4v\lambda^{\mu\varrho}(v-w)a_\mu(v-z) = 0, \tag{2.8a}$$

$$\frac{\delta}{\delta\mathscr{A}^\mu(z)} C[\mathscr{A}] = e \operatorname{Tr}(\gamma_\mu K_F[z, z|\mathscr{A}])C[\mathscr{A}] \tag{2.8b}$$

we obtain after simple algebraic rearrangement:

$$G^\mu[x, y, z|\mathscr{A}a] = -i\int d^4vD_F^{(a)\mu\lambda}(z-v)\frac{\delta}{\delta\mathscr{A}^\lambda(v)} G[x, y|\mathscr{A}a] -$$

$$-ie\int d^4vD_F^{(a)\mu\lambda}(z-v) \operatorname{Tr}(\gamma_\lambda G[v, v|\mathscr{A}a])G[x, y|\mathscr{A}a]. \tag{2.9}$$

It follows from (2.9) that the electron propagator equation (2.3) can be written as:

$$\Bigg[-i\gamma^\mu\frac{\partial}{\partial x^\mu} + m + e\gamma^\nu\mathscr{B}_\nu(x) -$$

$$-ie\int d^4z\mathscr{D}^F_{\nu\lambda}(x-z)\frac{\delta}{\delta\mathscr{A}_\lambda(z)}\gamma^\nu\Bigg] G[x, y|\mathscr{A}a] = \delta^{(4)}(x-y), \tag{2.10}$$

where the c-number field $\mathscr{B}_\nu$ plays the role of the effective external electromagnetic field:

$$\mathscr{B}_\nu(x) = \int d^4z\lambda_{\nu\mu}(z-x)\mathscr{A}^\mu(z) - ie\int d^4z\mathscr{D}^F_{\nu\mu}(x-z) \operatorname{Tr}(\gamma^\mu G[z, z|\mathscr{A}a]). \tag{2.11}$$

This vector field is constructed of two terms, of which the first is given as the gauge transformation of the primary field $\mathscr{A}_\mu$ and the second describes the contribution from creation and annihilation of the electron-positon pairs in the vacuum. It follows from (2.8a) that the components of the effective field $\mathscr{B}_\nu$ satisfy:

$$\int d^4za^\lambda(z-x)\mathscr{B}_\lambda(z) = 0 \tag{2.12}$$

and therefore they are not independent. However, it is clear from (2.11) that this field is equal to the gauge transform of another effective field:

$$\mathfrak{a}_\nu(x) = \mathscr{A}_\nu(x) - ie\int d^4z\mathscr{D}^F_{\nu\mu}(x-z) \operatorname{Tr}(\gamma^\mu G[z, z|\mathscr{A}a]), \tag{2.13}$$

which has independent components. The equation satisfied by the electron propagator can be written in terms of this new field as:

$$\Bigg[-i\gamma^\nu\frac{\partial}{\partial x^\nu} + m + e\gamma^\nu\mathfrak{a}_\nu(x) - e\gamma^\nu\int d^4z\frac{\partial}{\partial z^\nu}a^\mu(z-x)\mathscr{A}_\mu(z) -$$

$$-ie\gamma^\nu\int d^4z\mathscr{D}^F_{\nu\lambda}(x-z)\frac{\delta}{\delta\mathscr{A}_\lambda(z)}\Bigg] G[x, y|\mathscr{A}a] = \delta^{(4)}(x-y). \tag{2.14}$$

The next step consists in elimination of the primary field $\mathscr{A}_\mu$ by the effective field $\mathfrak{a}_\mu$. To achieve this, we must find the derivative of $\mathfrak{a}_\mu$ with respect to $\mathscr{A}_\mu$:

$$\frac{\delta \mathfrak{a}_\nu(z)}{\delta \mathscr{A}_\lambda(z')} = g^\nu_\lambda \delta^{(4)}(z-z') - ie \int d^4z'' \mathscr{D}^F_{\nu\sigma}(z-z'') \frac{\delta}{\delta \mathscr{A}_\lambda(z')} \operatorname{Tr}(\gamma^\sigma G[z'', z''|\mathscr{A}a]). \quad (2.15)$$

It is easy to express the right-hand side of this equation by the connected part of the three point function (formula (2.5)), which we will denote by $\mathscr{G}^c_\mu$:

$$\mathscr{G}^c_\mu[x, y, z|\mathscr{A}a] = -i \int d^4v \mathscr{D}^F_{\mu\lambda}(z-v) \frac{\delta}{\delta \mathscr{A}_\lambda(v)} G[x, y|\mathscr{A}a]. \quad (2.16)$$

Thus:

$$\int d^4z'' \mathscr{D}^F_{\mu\lambda}(z-z'') \frac{\delta \mathfrak{a}_\nu(z')}{\delta \mathscr{A}_\lambda(z'')} = \mathscr{D}^F_{\mu\nu}(z-z') +$$

$$+ e \int d^4z'' \mathscr{D}^F_{\mu\lambda}(z-z'') \operatorname{Tr}(\gamma^\lambda \mathscr{G}^c_\nu[z'', z'', z'|\mathscr{A}a]). \quad (2.17)$$

The right-hand side of this equation can be easily recognized as the gauge-invariant photon propagator $\mathscr{G}^{(B)}_{\mu\nu}$ (formula (2.5) in [3]):

$$\mathscr{G}^{(B)}_{\mu\nu}[z_1, z_2|\mathscr{A}a] = i\langle 0; \text{out}\,|T(B_\mu(z_1)B_\nu(z_2))\,|0; \text{in}\rangle^c =$$

$$= \int d^4z d^4z' \lambda_{\mu\varrho}(z-z_1)\lambda_{\nu\sigma}(z'-z_2)\mathscr{G}^{\varrho\sigma}[z, z'|\mathscr{A}a]. \quad (2.18)$$

Finally:

$$\int d^4z'' \mathscr{D}^F_{\mu\lambda}(z-z'') \frac{\delta \mathfrak{a}_\nu(z')}{\delta \mathscr{A}_\lambda(z'')} = \mathscr{G}^{(B)}_{\mu\nu}[z, z'|\mathscr{A}a]. \quad (2.19)$$

This formula is of exactly the same form as in quantum electrodynamics without compensating current [7], in which the photon propagators $\Delta_{F\mu\nu}$ and $G_{\mu\nu}$ have been replaced by the gauge-invariant and compensating current-dependent quantities. However, in quantum electrodynamics with compensating current we have also another effective field $\mathscr{B}_\nu$, subject to the constraint equation (2.12). The external field $\mathscr{A}_\mu$ can now be eliminated from (2.14):

$$\left[-i\gamma^\nu \frac{\partial}{\partial x^\nu} + m + e\gamma^\nu \mathscr{B}_\nu(x) - \right.$$

$$\left. - ie\gamma^\nu \int d^4z \mathscr{G}^{(B)}_{\nu\lambda}[x, z|\mathfrak{a}a] \frac{\delta}{\delta \mathfrak{a}_\lambda(z)}\right] G[x, y|\mathfrak{a}a] = \delta^{(4)}(x-y) \quad (2.20)$$

and the connected part of the three point function $\mathscr{G}_\mu$ (formula (2.16)) can be rewritten as:

$$\mathscr{G}^c_\mu[x, y, z|\mathfrak{a}a] = -i \int d^4v \mathscr{G}^{(B)}_{\mu\lambda}[z, v|\mathfrak{a}a] \frac{\delta}{\delta \mathfrak{a}_\lambda(v)} G[x, y|\mathfrak{a}a]. \quad (2.21)$$

Functional differentiation with respect to the effective field $\mathfrak{a}_\mu$ can be performed since, as we have mentioned above, its four components are independent. The formulae obtained in this Section have a form analogous to the equations in the usual formulation of quantum electrodynamics. The only difference is contained in (2.20), where we have two kinds of effective external fields $\mathscr{B}_\nu$ and $\mathfrak{a}_\nu$. The basic equation for the definition of the vertex function will be (2.21).

3. Definition of the vertex function in quantum electrodynamics with compensating current

As we have shown in [3], the three-point fermion-fermion-photon propagator can be expressed with the help of the electron and photon propagators in the following form:

$$G^c_\mu[x, y, z|\mathscr{A}a] = ie \int d^4x'd^4y'd^4z'\mathscr{G}^{(a)}_{\mu\varrho}[z, z'|\mathscr{A}a]G[x, x'|\mathscr{A}a] \times$$
$$\times \Gamma^\varrho[x', y', z'|\mathscr{A}a]G[y', y|\mathscr{A}a], \tag{3.1}$$

where:

$$\mathscr{G}^{(a)}_{\mu\varrho}[z, z'|\mathscr{A}a] = \int d^4w\lambda_{\varrho\sigma}(w-z')\mathscr{G}^\sigma_\mu[z, w|\mathscr{A}a]. \tag{3.2}$$

Obviously, in the case of a non-zero external electromagnetic field, the three-point function will also contain the disconnected part, but we will be interested only in the connected one. Decomposition of G_μ into its connected and disconnected parts can be easily seen from (2.9). Since the vacuum current is equal to zero for vanishing external field, it follows from this formula that G_μ is connected in this case.

It will be more convenient to deal further with the propagator $\mathscr{G}_\mu$ defined in the previous Section by formula (2.5). It can be written with the help of the vertex function as:

$$\mathscr{G}^c_\mu[x, y, z|\mathfrak{a}a] = ie \int d^4x'd^4y'd^4z'\mathscr{G}^{(B)}_{\mu\lambda}[z, z'|\mathfrak{a}a]G[x, x'|\mathfrak{a}a] \times$$
$$\times \Gamma^\lambda[x', y', z'|\mathfrak{a}a]G[y', y|\mathfrak{a}a]. \tag{3.3}$$

We used here the effective external field as an independent variable instead of the primary field $\mathscr{A}_\mu$. Since the gauge-invariant photon propagator $\mathscr{G}^{(B)}_{\mu\nu}$, as well as $\mathscr{G}^{(a)}_{\mu\nu}$, is a projective operator, equations (3.1) and (3.3) do not define the vertex function unambiguously. This ambiguity is in a most straightforward way connected with the equation (2.8) from which it follows that any quantity proportional to the compensating current a^λ can be added to Γ^λ, leaving the propagator $\mathscr{G}_\mu$ unchanged. In spite of that it is possible to define the vertex part unambiguously and this possibility can be seen from the formula (2.21). Namely, we will use the following expression as our definition of the vertex function:

$$e\Gamma^\lambda[x, y, z|\mathfrak{a}a] = \frac{\delta}{\delta\mathfrak{a}_\lambda(z)} G^{-1}[x, y|\mathfrak{a}a]. \tag{3.4}$$

This expression has an analogous form as that in quantum electrodynamics without compensating current, namely, the vertex function is defined as the functional derivative of the inverse electron propagator with respect to the effective external field. Moreover, it is easy to check that formula (3.4) is consistent with (3.1) and (3.3). As we have already mentioned above, any object proportional to the compensating current can be added to Γ^{λ} but, nevertheless, formula (3.4) defines the vertex function in an unambiguous way. It is obvious that the perturbation expansion of this function can be obtained with the help of the same Feynman diagrams as in the conventional theory, in which the internal photon lines are connected with the gauge-independent photon propagator $\mathscr{D}^{F}_{\mu\nu}$. Thus, the Ward identity in the usual form is satisfied:

$$\frac{1}{i}\frac{\partial}{\partial z^{\lambda}}\Gamma^{\lambda}[x, y, z|\mathfrak{a}a] = [\delta^{(4)}(x-z)-\delta^{(4)}(y-z)]G^{-1}[x, y|\mathfrak{a}a], \tag{3.5a}$$

$$k_{\lambda}\tilde{\Gamma}^{\lambda}[p, k|a] = \tilde{G}^{-1}[p|a]-\tilde{G}^{-1}[p+k|a]. \tag{3.5b}$$

It should be noticed that adding to Γ^{λ} a four-vector proportional to a^{λ} we could obtain the "Ward identity" with vanishing right-hand side: $k_{\lambda}\Gamma^{\lambda} = 0$, and the propagator $\mathscr{G}_{\mu}$ would not change.

To end this Section we will obtain the integral equations satisfied by the three-point propagators G_{μ} and $\mathscr{G}_{\mu}$. It follows from (2.7) and from the following formula:

$$\frac{\delta}{\delta\mathscr{A}_{\mu}(z)}(C[\mathscr{A}]K_{F}[x, y|\mathscr{A}]) = e\,\mathrm{Tr}\,(\gamma^{\mu}_{z}K_{F}[x, z, z, y|\mathscr{A}])C[\mathscr{A}], \tag{3.6}$$

that:

$$G_{\mu}[x, y, z|\mathfrak{a}a] = -ie\int d^{4}wD^{(a)}_{F\mu\lambda}(z-w)\,\mathrm{Tr}\,(\gamma^{\lambda}_{w}G[x, w, w, y|\mathfrak{a}a]), \tag{3.7}$$

and therefore:

$$\mathscr{G}_{\mu}[x, y, z|\mathfrak{a}a] = i\int d^{4}w\mathscr{D}^{F}_{\mu\lambda}(z-w)\mathscr{H}^{\lambda}[x, y, w|\mathfrak{a}a], \tag{3.8}$$

where the propagator $\mathscr{H}^{\lambda}$ is defined as:

$$\mathscr{H}^{\lambda}[x, y, w|\mathfrak{a}a] = i^{2}\langle 0;\text{ out }|T[\psi(x)\exp(-ie\int d^{4}v\{a^{\nu}(v-x)- -a^{\nu}(v-y)\}A_{\nu}(v))\bar{\psi}(y)j^{\lambda}(v)]\,|0;\text{ in}\rangle. \tag{3.9}$$

We will further assume that $\mathscr{A}_{\mu}$ and therefore also $\mathfrak{a}_{\mu}$ are equal to zero. Choosing then x–y, x–z in $\mathscr{G}_{\mu}$ and x–y, x–w in $\mathscr{H}^{\lambda}$ as independent variables, we obtain in the momentum space:

$$\tilde{\mathscr{G}}_{\mu}[p, k|a] = i\tilde{\mathscr{D}}^{F}_{\mu\lambda}(-k)\tilde{\mathscr{H}}^{\lambda}[p, k|a]. \tag{3.10}$$

4. Renormalization of the vertex function

In this Section we will discuss problems connected with the renormalization of the vertex function in quantum electrodynamics with compensating current. The starting point of our considerations will be the momentum space equation (3.10). Since the calculations leading to the renormalized integral equation for the vertex function are similar to analogous calculations in the usual theory, we will not describe the details but rather sketch briefly the method.

In is clear from (3.9) that the propagator $\mathscr{H}^{\lambda}$ should be defined by means of the same limiting procedure which has already been described in [3]. Thus:

$$\mathscr{H}^{\lambda}[x, y, w|a] = \lim_{\xi\to 0}\Big\{-\frac{e}{2}\operatorname{Tr}(\gamma_w^{\lambda}G[x, w, w+\xi, y|a]+$$
$$+\gamma_w^{\lambda}G[x, w, w-\xi, y|a])-iC_1^{\lambda}(\xi)G[x, y|a]-iC_2^{\lambda\varrho}(\xi)\mathscr{G}_{\varrho}[x, y, w|a]-$$
$$-iC_3^{\lambda\varrho\sigma}(\xi)\frac{\partial}{\partial w^{\sigma}}\mathscr{G}_{\varrho}[x, y, w|a]-iC_4^{\lambda\varrho\sigma\kappa}(\xi)\frac{\partial}{\partial w^{\sigma}}\frac{\partial}{\partial w^{\kappa}}\mathscr{G}_{\varrho}[x, y, w|a]-$$
$$-C_6(\xi)\mathscr{H}^{\lambda}[x, y, w|a]\Big\}. \tag{4.1}$$

Choosing x–y, x–w and w–w' as independent variables in $G[x, w, w', y|a]$, we obtain in the momentum space:

$$\tilde{\mathscr{H}}^{\lambda}_{ab}[p, q|a] = \lim_{\xi\to 0}\Big\{-\frac{e}{2}\int\frac{d^4k}{(2\pi)^4}(e^{ik\xi}+e^{-ik\xi})\gamma^{\lambda}_{cd}\tilde{G}_a{}^{d;c}{}_b[p, q, k|a]-$$
$$-iC_1^{\lambda}(\xi)\tilde{G}_{ab}[p|a]\delta^{(4)}(q)-iC_2^{\lambda\varrho}(\xi)\tilde{\mathscr{G}}_{\varrho ab}[p, q|a]+$$
$$+C_3^{\lambda\varrho\sigma}(\xi)q_{\sigma}\tilde{\mathscr{G}}_{\varrho ab}[p, q|a]+iC_4^{\lambda\varrho\sigma\kappa}(\xi)q_{\sigma}q_{\kappa}\tilde{\mathscr{G}}_{\varrho ab}[p, q|a]-$$
$$-C_6(\xi)\tilde{\mathscr{H}}^{\lambda}_{ab}[p, q|a]\Big\}. \tag{4.2}$$

Dependence on the spinor indices has been explicitly pointed out.

In order to obtain the integral equation for the vertex function we must split the four point function $G[x, w, w', y|a]$ into its strongly connected, weakly connected and disconnected parts. In the momentum space we have therefore:

$$\tilde{G}_a{}^{d;c}{}_b[p, q, k|a] =$$
$$= (\tilde{G}[p+q|a]\tilde{G}[k-q|a]\tilde{E}[p, q, k|a]\tilde{G}[k|a]\tilde{G}[p|a])_a{}^{d;c}{}_b+\ldots \tag{4.3}$$

In this formula only the strongly connected part of $\tilde{G}$ has been written explicitly. The weakly connected part of this propagator can be easily constructed with the help of the Feynman diagrams depicted in Fig. 1.

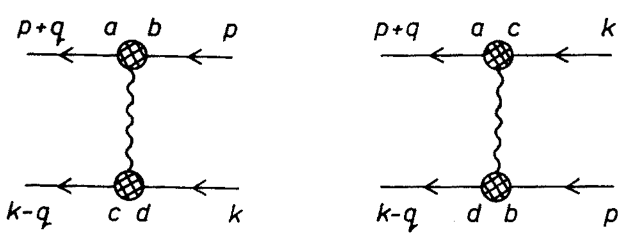

Fig. 1

All lines and vertices in these diagrams are effective lines and vertices. One must also remember the change of sign when the fermion indices are permuted. The disconnected part consists of two terms:

$$\tilde{G}_{ab}[p+q|a]\tilde{G}^{dc}[k|a]\delta^{(4)}(q)-\tilde{G}_{a}{}^{c}[p+q|a]\tilde{G}^{d}{}_{b}[p|a]\delta^{(4)}(p+q-k). \tag{4.4}$$

Equation (3.10) can be written with the help of the vertex function as:

$$\begin{aligned} ie\tilde{G}[p+q|a]\tilde{\Gamma}^{\lambda}[p,q|a]\tilde{G}[p|a]\tilde{\mathscr{G}}^{(B)}_{\mu\lambda}[q|a] &= \\ = i\tilde{\mathscr{D}}^{F}_{\mu\lambda}(-q)\tilde{\mathscr{H}}^{\lambda}[p,q|a], \end{aligned} \tag{4.5}$$

where as the independent variables in $\mathscr{G}_{\mu}[x,y,z|a]$ we choose x–y and x–z.

The photon propagator is given by:

$$\tilde{\mathscr{G}}^{(B)}_{\mu\nu} = \tilde{\mathscr{D}}^{F}_{\mu\nu}+\tilde{\mathscr{D}}^{F}_{\mu\lambda}\Pi^{\lambda\sigma}\tilde{\mathscr{G}}^{(B)}_{\sigma\nu}. \tag{4.6}$$

Renormalized expression for the vacuum polarization tensor is given by the formula (4.11) in Ref. [3]:

$$\begin{aligned} \tilde{\Pi}^{\lambda\sigma}[q|a] = \lim_{\xi\to 0}\Big\{\frac{ie^{2}}{2}\int\frac{d^{4}k}{(2\pi)^{4}}(e^{ik\xi}+e^{-ik\xi})\times \\ \times\operatorname{Tr}(\gamma^{\lambda}\tilde{G}[k+q|a]\tilde{\Gamma}^{\sigma}[k,q|a]\tilde{G}[k|a])+C_{2}^{\lambda\sigma}(\xi)-iC_{3}^{\lambda\sigma\alpha}(\xi)q_{\alpha}- \\ -C_{4}^{\lambda\sigma\alpha\beta}(\xi)q_{\alpha}q_{\beta}-C_{6}(\xi)\tilde{\Pi}^{\lambda\sigma}[q|a]\Big\}. \end{aligned} \tag{4.7}$$

Substituting now (4.7), (4.3), (4.4) and (4.2) into (4.5) and splitting the vertex function in the usual way:

$$\tilde{\Gamma}_{\mu}[p,q|a] = \gamma_{\mu}+\Lambda_{\mu}[p,q|a], \tag{4.8}$$

we obtain the desired integral equation:

$$\begin{aligned} \Lambda^{\mu}_{ab}[p,q|a] = \lim_{\xi\to 0}\Big\{\frac{1}{2}\int\frac{d^{4}k}{(2\pi)^{4}}(e^{ik\xi}+e^{ik\xi})\times \\ \times[\gamma^{\mu}_{cd}(\tilde{G}[p-k|a]\tilde{E}[p,q,p+q-k|a]\tilde{G}[p+q-k|a])_{a}{}^{d;c}{}_{b}+ \\ +ie^{2}(\tilde{\Gamma}^{\varrho}[p+q-k,k|a]\tilde{G}[p+q-k|a]\gamma^{\mu}\tilde{G}[p-k|a]\tilde{\Gamma}^{\sigma}(p,-k|a])_{ab}\times \\ \times\tilde{\mathscr{G}}^{(B)}_{\varrho\sigma}[k|a]+C_{6}(\xi)\tilde{\Gamma}^{\mu}_{ab}[p,q|a]\Big\}. \end{aligned} \tag{4.9}$$

The renormalization function $C_6(\xi)$ can be obtained from the following requirement:

$$\Lambda^{\mu}[p,0|a]|_{\gamma p=m} = 0. \tag{4.10}$$

Therefore:

$$\gamma^{\mu}_{ab}C_6(\xi) = -\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}(e^{ik\xi}+e^{-ik\xi})\times$$
$$\times\{\gamma^{\mu}_{cd}(\tilde{G}[p-k|a]\tilde{E}[p,0,p-k|a]\tilde{G}[p-k|a])_a{}^{d;c}{}_b+$$
$$+ie^2(\tilde{\Gamma}^{\varrho}[p-k,k|a]\tilde{G}[p-k|a]\gamma^{\mu}\tilde{G}[p-k|a]\tilde{\Gamma}^{\sigma}[p,-k|a])_{ab}\times$$
$$\times\tilde{\mathscr{G}}^{(B)}_{\varrho\sigma}[k|a]\}|_{\gamma p=m}. \tag{4.11}$$

In the second order of perturbation we obtain from (4.11):

$$\gamma^{\mu}C_6^{(2)}(\xi) = -\frac{ie^2}{2}\int\frac{d^4k}{(2\pi)^4}(e^{ik\xi}+e^{-ik\xi})\times$$
$$\times\gamma^{\varrho}\tilde{S}_F(p-k)\gamma^{\mu}\tilde{S}_F(p-k)\gamma^{\sigma}\tilde{\mathscr{D}}^F_{\varrho\sigma}(k)|_{\gamma p=m}, \tag{4.12}$$

in accordance with the formula (5.27) in [3].

5. *Conclusions and summary*

This paper completes the series of three publications devoted to the problem of the gauge-invariant formulation of quantum electrodynamics. Results presented here can be treated as the summary and development of the methods and results which have already been used before. The first publication, in which the "compensating current" was introduced, was the paper by Zumino [10]. This current was there used for the description of the gauge. This method was further developed by Johnson [6] and Białynicki-Birula [9] and it could be treated as the generalization of the gauge-invariant formulation of quantum electrodynamics proposed by Mandelstam [11].

In the present papers we mainly developed the compensating current method outlined in the preprint by Białynicki-Birula [12]. This method consists in the introduction of the operator phase factor:

$$\exp\left[-ie\int d^4z a^{\lambda}(z;x_1,\ldots,x_n,y_n,\ldots,y_1)A_{\lambda}(z)\right],$$

which describes, in a sense, the experimental device used to the production and detection of charged particles. Since this phase factor guarantees local charge conservation, all the propagators and field operators are gauge-invariant. On the other hand, it can be shown [2, 12] that the choice of a certain kind of the compensating current is equivalent to the choice of a gauge and therefore quantum electrodynamics with compensating current is equivalent to the theory formulated in a certain gauge.

Our papers give nothing new as far as all the well established results of quantum electrodynamics are concerned. However, we have shown that quantum electrodynamics can be formulated in a fully gauge-invariant manner in the Hilbert space with positive-definite scalar product, at least for a certain class of the compensating currents. We have also obtained renormalized expressions for the vacuum polarization tensor, vertex function and electron self-energy part and it turned out that they have the same form as in the usual formulation of quantum electrodynamics [4]. This result is, of course, not surprising and it is in a complete accordance with the equivalence theorem formulated in [12]. The presence of the compensating current simplifies particularly the calculations connected with the vacuum polarization tensor. Since gauge invariance is satisfied at every step of calculations, there is no need for any artificial method which would ensure the transversity of the vacuum polarization tensor. Due to the compensating current this tensor is transverse from the very beginning and it is regularized in a natural way with no need for any "ad hoc" regularization. It has also been shown that the vacuum polarization tensor, as well as the current operator, are independent on the form of the compensating current in accordance with the gauge invariance of these objects in the usual version of quantum electrodynamics.

One can hope that the compensating current method could be applied also in the gauge theories of the Yang-Mills type. Its application in the theory of such fields will not be, however, so straightforward as in quantum electrodynamics, since one will have to overcome difficulties connected with the presence of the isotopic degrees of freedom and with the noncommutativity of the τ-matrices.

The author is greatly indebted to Professor I. Białynicki-Birula for suggesting this investigation and for many enlightening discussions.

APPENDIX A

In this Appendix we investigate the electron self-energy function and its renormalization in quantum electrodynamics with compensating current. In the absence of the external electromagnetic field the differential equation satisfied by the electron propagator can be written as:

$$\left(-i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}+m\right)G[x, y|a] = \delta^{(4)}(x-y)-e\gamma^{\mu}\mathscr{G}_{\mu}[x, y, x|a], \tag{A1}$$

where the propagator $\mathscr{G}_{\nu}$ is defined by (2.5). The right-hand side of this equation contains the propagator with two arguments equal. This quantity, similarly as in the case of the current operator, must be carefully defined. Generalizing to the case of a non-zero compensating curıent the definition given by Brandt [4], we assume:

$$\begin{aligned} &T(\gamma^{\nu}\psi(x)\Phi[a]\bar{\psi}(y)\mathscr{B}_{\nu}(x)) = \\ &= \lim_{\xi\to 0}\Big\{\frac{1}{2}T(\gamma^{\nu}\psi(x)\Phi[a]\bar{\psi}(y)\mathscr{B}_{\nu}(w+\xi)+\gamma^{\nu}\psi(x)\Phi[a]\bar{\psi}(y)\mathscr{B}_{\nu}(x-\xi))- \\ &-D_{1}(\xi)T(\psi(x)\Phi[a]\bar{\psi}(y))-D_{2\mu}(\xi)\frac{\partial}{\partial x_{\mu}}T(\psi(x)\Phi[a]\bar{\psi}(y))- \\ &-D_{3}(\xi)T(\gamma^{\nu}\psi(x)\Phi[a]\bar{\psi}(y)B_{\nu}(x))\Big\}, \end{aligned} \tag{A2}$$

where:

$$\Phi[a] = \exp\left[-ie\int d^4z(a^\lambda(z-x)-a^\lambda(z-y))A_\lambda(z)\right]. \tag{A3}$$

Equation (A1) can be easily written in the integral form:

$$G[x, y|a] = S_F(x-y)-e\int d^4zS_F(x-z)\gamma^\nu\mathscr{G}_\nu[z, y, z|a]. \tag{A4}$$

It follows further from (A.2) that:

$$\gamma^\nu\mathscr{G}_\nu[z, y, z|a] = \lim_{\xi\to 0}\left\{\frac{1}{2}(\gamma^\nu\mathscr{G}_\nu[z, y, z+\xi|a]+\gamma^\nu\mathscr{G}_\nu[z, y, z-\xi|a])-\right.$$

$$\left.-D_1(\xi)G[z, y|a]-D_{2\mu}(\xi)\frac{\partial}{\partial z_\mu}G[z, y|a]-D_3(\xi)\gamma^\nu\mathscr{G}_\nu[z, y, z|a]\right\}. \tag{A5}$$

The self-energy function Σ will be defined in exactly the same way as in quantum electrodynamics without compensating current:

$$G[x-y|a] = S_F(x-y)+\int d^4wd^4y'S_F(x-w)\Sigma[w-y'|a]G[y'-y|a]. \tag{A6}$$

Expressing the three-point function in (A5) with the help of the vertex function (formula (3.3)) and then substituting (A5) and (A6) into (A4) we obtain in the momentum space:

$$\tilde{\Sigma}[p|a] = \lim_{\xi\to 0}\left\{-\frac{ie^2}{2}\int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\times\right.$$

$$\times\gamma^\nu\tilde{\mathscr{G}}^{(B)}_{\nu\lambda}[q|a]\tilde{G}[p+q|a]\tilde{\Gamma}^\lambda[p, q|a]+eD_1(\xi)-$$

$$\left.-iep^\mu D_{2\mu}(\xi)-D_3(\xi)\tilde{\Sigma}[p|a]\right\}. \tag{A7}$$

The renormalized self-energy part should satisfy the following conditions:

$$\tilde{\Sigma}[p|a]|_{\gamma p=m} = 0, \qquad \frac{\partial}{\partial\gamma p}\tilde{\Sigma}[p|a]|_{\gamma p=m} = 0, \tag{A8}$$

and therefore:

$$D_1(\xi) = \frac{ie}{2}\int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\times$$

$$\times\gamma^\nu\tilde{\mathscr{G}}^{(B)}_{\nu\lambda}[q|a]\tilde{G}[p+q|a]\tilde{\Gamma}^\lambda[p, q|a]|_{\gamma p=m}, \tag{A9a}$$

$$D_2(\xi) = -\frac{e}{2}\int\frac{d^4q}{(2\pi)^4}(e^{iq\xi}+e^{-iq\xi})\times$$

$$\times\frac{\partial}{\partial\gamma p}\{\gamma^\nu\tilde{\mathscr{G}}^{(B)}_{\nu\lambda}[q|a]\tilde{G}[p+q|a]\tilde{\Gamma}^\lambda[p, q|a]\}|_{\gamma p=m}, \tag{A9b}$$

where $p^\mu D_{2\mu}(\xi) = (\gamma p-m)D_2(\xi)$. The function $D_3(\xi)$ cannot be found with the help of the self-energy part and will be calculated in Appendix B.

APPENDIX B

We will now find the second integral equation satisfied by the vertex function Λ^μ. The three-point propagator $G_\mu[x, y, z|a]$ satisfies:

$$\left(-i\gamma^\nu \frac{\partial}{\partial x^\nu} + m\right) G_\mu[x, y, z|a] =$$

$$= -e\gamma_x^\nu \int d^4w \lambda_{\nu\sigma}(w-x) G_\mu{}^\sigma[x, y, z, w|a], \tag{B1}$$

and therefore:

$$\left(-i\gamma^\nu \frac{\partial}{\partial x^\nu} + m\right) \mathscr{G}_\mu[x, y, z|a] = -e\gamma_x^\nu \mathscr{G}_{\mu\nu}^{(B)}[x, y, z, x|a], \tag{B2}$$

where the gauge-invariant four-point function $\mathscr{G}_{\mu\nu}^{(B)}$ is defined as:

$$\mathscr{G}_{\mu\nu}^{(B)}[x, y, z_1, z_2|a] = i\langle 0|T(\psi(x)\Phi[a]\bar{\psi}(y)B_\mu(z_1)B_\nu(z_2))\,|0\rangle. \tag{B3}$$

We will use the following notation:

$$\mathscr{U}_\mu[x, y, z|a] = \gamma_x^\nu \mathscr{G}_{\mu\nu}^{(B)}[x, y, z, x|a]. \tag{B4}$$

It follows then from (B2) that:

$$\tilde{\mathscr{G}}_\mu[p, q|a] = -e\tilde{S}_F(p+q)\tilde{\mathscr{U}}_\mu[p, q|a], \tag{B5}$$

where we choose x–y and x–z in $\mathscr{U}_\mu$ as the independent space-time variables. It follows from (A2) that the propagator $\mathscr{U}_\mu$ should be defined in the following way:

$$\tilde{\mathscr{U}}_\mu[p, q|a] = \lim_{\xi\to 0} \left\{\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}(e^{ik\xi}+e^{-ik\xi})\gamma^\nu \tilde{\mathscr{G}}_{\mu\nu}^{(B)}[p, q, k|a] - \right.$$

$$\left. - D_1(\xi)\tilde{\mathscr{G}}_\mu[p, q|a] + iD_{2\nu}(\xi)\,(p+q)^\nu \tilde{\mathscr{G}}_\mu[p, q|a] - D_3(\xi)\tilde{\mathscr{U}}_\mu[p, q|a]\right\}. \tag{B6}$$

As the independent variables in $\tilde{\mathscr{G}}_{\mu\nu}^{(B)}[x, y, z_1, z_2|a]$, x–y, x–z_1 and x–z_2 have been chosen. Splitting now the four-point propagator $\mathscr{G}_{\mu\nu}^{(B)}$ into its strongly connected, weakly connected and disconnected parts and them proceeding in exactly the same way as in the Section 4, we find the following equation for the vertex function:

$$\Lambda^\mu[p, q|a] = -\lim_{\xi\to 0}\left\{\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}(e^{ik\xi}+e^{-ik\xi})\times\right.$$

$$\times (i\gamma^\nu \tilde{G}[p+q+k|a]\tilde{\Gamma}^{\mu\lambda}[p, q, k|a]\tilde{\mathscr{G}}_{\nu\lambda}^{(B)}[k|a] +$$

$$+ ie^2\gamma^\nu \tilde{G}[p+q+k|a]\tilde{\Gamma}^\mu[p+k, q|a]\tilde{G}[p+k|a]\tilde{\Gamma}^\lambda[p, k|a]\tilde{\mathscr{G}}_{\nu\lambda}^{(B)}[k|a] +$$

$$\left. + D_3(\xi)\tilde{\Gamma}^\mu[p, q|a]\right\}, \tag{B7}$$

where the tensor $\tilde{\Gamma}^{\mu\lambda}$ describes the strongly connected part of $\mathscr{G}^{(B)}_{\mu\nu}[x, y, z_1, z_2|a]$. It follows from (4.10) that:

$$\gamma^\mu D_3(\xi) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}(e^{ik\xi}+e^{-ik\xi})\times$$
$$\times\{i\gamma^\nu\tilde{G}[p+k|a]\tilde{\Gamma}^{\mu\lambda}[p, 0, k|a]\tilde{\mathscr{G}}^{(B)}_{\nu\lambda}[k|a]+$$
$$+ie^2\gamma^\nu\tilde{G}[p+k|a]\tilde{\Gamma}^{\mu}[p+k, 0|a]\tilde{G}(p+k|a]\tilde{\Gamma}^{\lambda}[p, k|a]\tilde{\mathscr{G}}^{(B)}_{\nu\lambda}[k|a]\}|_{\gamma p=m}. \tag{B8}$$

In the second order of perturbation, formula (B8) gives:

$$\gamma^\mu D_3^{(2)}(\xi) = -\frac{ie^2}{2}\int \frac{d^4k}{(2\pi)^4}(e^{ik\xi}+e^{-ik\xi})\times$$
$$\times\gamma^\nu\tilde{S}_F(p+k)\gamma^\mu\tilde{S}_F(p+k)\gamma^\lambda\tilde{\mathscr{D}}^F_{\nu\lambda}(k)|_{\gamma p=m}. \tag{B9}$$

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