THE MULTIPLICATION LAW IN QUANTUM PHYSICS. PART II. THE RELATIVISTIC QUANTUM FIELD THEORY

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A new construction of the quantum field theory is proposed. The general multiplication law for relativistic quantum field theory is discussed. The method is illustrated on the examples of the free field theory and the interacting fields in the so-called one-particle approximation.

1. Introduction

Recently, we have shown [1] that the Heisenberg quantum theoretical multiplication law is only a particular case of some more general multiplication rule. In the case of the usual non-relativistic quantum mechanics the new generalized multiplication law leads only to some numerical changes in the final results. In particular, our formulation throws some new light on the zero-point energy problem. The situation is, however, completely different when we pass to the cases of physical systems for which an infinite number of transition amplitudes are equal as a manifestation of some internal symmetries. In such a case the conventional Heisenberg multiplication law is meaningless while the generalized law evidently works.

The aim of the present paper is to extend the idea of the generalized multiplication law to the case of relativistic field theory. First of all, we show that the conditions of Lorentz invariance and local commutativity require an equality of an infinite number of representatives of the field quantity. We restrict our discussion to the case of field theories with one scalar field describing one type of particle. Such a restriction is made mainly for the sake of simplicity but it is not easy to generalize our results.

After pointing out the much more complicated nature of the multiplication problem in the case of field theory, we limit our attention to the case of the free field theory, leaving the other cases for a separate paper. Our construction of field theory fully avoids the introduction of creation and annihilation operators for particles with sharp values of the momenta nor do we use any element of the Hamiltonian approach to quantum field

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theory. This is not accidental, since already in the domain of classical field theories the Hamiltonian approach does not have a unique meaning. Of course, our final results are completely equivalent to the usual results of quantum field theory but the way of obtaining them is quite different and in contradistinction to the usual formalism is always meaningful. Therefore, we may avoid the infinite renormalization procedure which is highly questionable from the logical point of view.

2. The passage from classical to quantum field theory

We start our consideration with the description of the way in which we pass from the classical to quantum field theory. By classical field theory we mean here the wave mechanics of the first quantized theory. Consequently, our basic object is the wave function $\psi(x)$ describing some collection of elementary particles. This wave function obeys some non-linear wave equation of the type

$$(\Box + m^2)\psi(x) = F(\psi(x)), \tag{2.1}$$

where the non-linear term on the right-hand side describes the interaction between the particles which are present in the considered collection of them. In general we may write the solution of (2.1) in the form

$$\psi_P(x) = e^{iPx}\varphi_P(x), \tag{2.2}$$

where P is the vector of the total four-momentum of the system under consideration, $\varphi_P(x)$ describes the relative motion of the particles constituting the system and, as usual in the exponent, we have the Lorentz invariant scalar product of P and x. For all non-linear wave equations (2.1) the explicit form of $\varphi_P(x)$ is unknown but, assuming some reasonable restriction on its form, we may write $\varphi_P(x)$ as a Fourier integral

$$\varphi_P(x) = \int \varphi_P(Q)e^{-iQx}d^4Q. \tag{2.3}$$

In the case when the system under consideration consists only of a single particle the corresponding $\varphi_P(x)$ is a constant (depending on P) and the below presented analysis must appropriately changed [2]. This case is described in the Appendix.

We shall have an equivalent amount of information considering instead of (2.2) and (2.3), directly the set

$$\{\varphi_P(Q)e^{i(P-Q)x}\}, \tag{2.4}$$

where the role of the two four-momenta P and Q is completely unsymmetric. Following the Heisenberg reinterpretation of all mechanical quantities [3], we now make the same reinterpretation of the field quantities (2.4) and from now on we shall replace the classical field $\psi(x)$ by a set of representatives

$$\{\varphi(P, Q)e^{i(P-Q)x}\}, \tag{2.5}$$

where both P and Q are now possible experimental values of the total four-momentum for the considered system and $\varphi(P, Q)$ is connected with the amplitude for the transition

probability between the states of the system with these four-momenta, respectively. In order to write down the precise form of the connection between $\varphi(P, Q)$ and the corresponding transition amplitudes we must further specify some details. First of all, for the sake of simplicity, we shall assume that our field theory describes the collections of only one type of scalar particles with mass M. Then, since both P and Q are true experimental values of the total four-momentum and the particles may be observed as individuals only when they asymptotically become free, we must assume that both P and Q are elements of the set

$$\Pi_N = \{0, \sum_{j=1}^{n} p_j | p_j^2 = M^2, p_{j,0} > 0\}_{n=1}^{N}. \tag{2.6}$$

This assumption plays the role of the spectrum condition in our field theory. It is easy to see the difference between (2.6) and the spectrum condition usually adopted in field theories, which is simply a particular case of (2.6) with infinite N. The choice of N — the maximal number of particles which are present in the considered physical system — is governed by the maximal value of energy concentrated in the system and for really existing systems is obviously finite. The choice of infinite value for N is, however, a very convenient idealization.

Our next step consists in the introduction of wave functions for the considered collections of particles. These wave functions of n-particle system, as usual, form the n-particle Hilbert space H_n given by the complex valued symmetric functions of n arguments (being four-momenta on the mass hyperboloid) and subjected to the condition of square integrability with respect to the Lorentz invariant measure defined on the mass hyperboloid. For $n = 0$, H_0 is simply the set of all complex numbers. Then, for the transition from the n-particle state of the system to the m-particle state, we could try to define the transition amplitude by

$$T(u_m, v_n; x) = \int \prod_{j=1}^{m} \frac{d^3 p_j}{p_{j,0}} \prod_{l=1}^{n} \frac{d^3 q_l}{q_{l,0}} u_m(p_1, \dots p_m) \varphi(P, Q) e^{i(P-Q)x} v_n(q_1 \dots q_n), \tag{2.7}$$

where

$$P = \sum_{j=1}^{m} p_j, \qquad Q = \sum_{l=1}^{n} q_l,$$

and $u_m \in H_m$, $v_n \in H_n$, with an obvious modification when P or/and Q are the zero element of the set (2.6). This definition cannot, however, be admitted because the expression (2.7) is not bounded, this contradicting the bounded character of the corresponding transition probabilities. In order to get bounded transition amplitudes we may use the Bohr and Rosenfeld argument [4] concerning the measurability of field quantities. Consequently, we pass to the smeared field representatives given by

$$\varphi_f(P, Q) = \int d^4x f(x) \varphi(P, Q) e^{i(P-Q)x} = \varphi(P, Q) \tilde{f}(P-Q), \tag{2.8}$$

with a suitable choice of the test function $f(x)$. The test functions may be chosen different for different field representatives and we shall interpret them as wave functions of the

emitted or absorbed matter associated with the transition from the initial to final states of the field. The physical transition amplitudes are then defined as

$$T_f(u_m, v_n) = \int \prod_{j=1}^{m} \frac{d^3 p_j}{p_{j0}} \prod_{l=1}^{n} \frac{d^3 q_l}{q_{l,0}}\, u_m^*(p_1, \ldots p_m)\varphi_f(P, Q)v_n(q_1, \ldots q_n), \tag{2.9}$$

with an obvious modification in the cases when the field representatives are different from zero only for some special correlations between the momenta P and Q from the set (2.6). We shall introduce such changes below on the example of the free field theory.

Each transition amplitude (2.9) defines a bounded sesquilinear form on the Cartesian product of the Hilbert spaces H_n and H_m. According to the well-known reconstruction theorem [5], such sesquilinear form defines a bounded operator with the domain H_n and range H_m, respectively. The set of all such bounded operators is then unified by the introduction of a single, in general, unbounded operator $\varphi(\{f\})$ defined on some dense domain in the Fock space

$$\mathscr{F}_N = \bigoplus_{n=0}^{N} H_n, \tag{2.10}$$

in such a way that the restriction of $\varphi(\{f\})$ to H_n with the range H_m coincides with the operator defined by (2.9). The operator $\varphi(\{f\})$ we shall call the field operator generated by the given set of field representatives. Our construction of the field operator is slightly different from the conventional construction performed up to the present time. The first difference consists in the fact that the Fock space (2.10) contains only wave functions for maximally N-particle states in accordance with the form of the spectrum condition (2.6). The case of infinite N is only a particular case of our construction. The second difference is connected with the use different smearing functions for different field representatives. In this way the field operator depends on a set of test functions which may be reduced to a single test function in some particular cases. Consequently, our field operator has a much more general character than the usual smeared operators of the conventional quantum field theory.

The field representatives $\varphi(P, Q)$ should be calculated from the field equation

$$[-(P-Q)^2 + m^2]\varphi(P, Q) = F(\varphi)(P, Q), \tag{2.11}$$

where the meaning of the right-hand side will be elaborated in the next section. Since this basic equation is non-linear with respect to the field representatives it cannot generally be solved exactly and we must be satisfied with some approximate solutions that furnish adequate information about the system. We shall always assume that the approximate solutions can be obtained by means of the asymptotic expansions of $\varphi(P, Q; \lambda)$ with respect to the coupling constant determining the magnitude of the interaction. Consequently, we may choose an asymptotic sequence of functions $\{\alpha_n(\lambda)\}$ such that the field representatives $\varphi(P, Q; \lambda)$ may be written as

$$\varphi(P, Q; \lambda) \approx \sum_{n=0}^{L} \varphi_n(P, Q; \lambda) + O(\alpha_L), \tag{2.12}$$

where we use the terminology of Ref. [6]. We shall later on discuss the technical details concerning the asymptotic method and in particular the choice of the asymptotic sequence $\{\alpha_n(\lambda)\}$. Here we mention only that this choice will crucially differentiate our method from the usual perturbation approach to the non-linear field equations (2.11). But this is not the only difference we propose which will be especially clear from the content of the next section where we discuss the multiplication problem for quantum field theories.

As has been shown in Part I, before formulating the multiplication law we have to exploit the symmetry properties of the considered system. As a first symmetry we have here the identity of particles which is automatically taken into account in the structure of the set (2.6). The next symmetry is the Lorentz invariance of the considered field theory. It is easy to see that it gives the condition

$$\varphi(\Lambda P, \Lambda Q) = \varphi(P, Q) \tag{2.13}$$

for all Lorentz transformations Λ. The relativistic invariance condition dictates, therefore, that an infinite number of the field representatives must be equal and this locates the field theory in a subclass of quantum theories discussed in Sec. 4 of Part I.

A further symmetry follows from the local commutativity condition. Since we do not work in general with an infinite number of particles we must suitably change the formulation of this condition. For finite N we shall always require that

$$[\varphi(\{f\}), \varphi(\{g\})]\, \psi = 0 \tag{2.14}$$

for all $\psi \in \mathscr{F}_{N-1}$ and for space-like separations of the supports of the test functions in the set $\{f\}$ and $\{g\}$, correspondingly. The condition (2.14) gives further relations between the field representatives which are, however, very difficult to extract in a general case. The situation improves considerably if we use the asymptotic expansions (2.12) for the field representatives. Since these representatives are given by (2.12) up to the order α_L only, we may require that the right-hand side of (2.14) is, instead of zero, a quantity of order α_{L+1}. This gives us a recursion procedure for obtaining the relations for the coefficients in (2.12) from the condition (2.14). Instead of elaborating the details of such procedure for the general case we prefer to do this separately for the considered examples. This is a much more economical way in view of the complexity of the condition (2.14).

3. The multiplication problem for quantum field theory

Following the procedure of Part I, we start with the general expression for the representatives of the product of two fields φ and ψ. This is given by

$$(\varphi\psi)(P, Q) = \sum_{k,l,s,u \in Z} \int a(P, Q; p'_1 \dots p'_k; q'_1 \dots q'_l; p''_1 \dots p''_s; q''_1 \dots q''_u) \times$$

$$\times \varphi(P', Q')\psi(P'', Q'') \prod_{\alpha=1}^{k} \frac{d^3 p'_\alpha}{p'_{\alpha 0}} \prod_{\beta=1}^{l} \frac{d^3 q'_\beta}{q'_{\beta 0}} \prod_{\gamma=1}^{s} \frac{d^3 p''_\gamma}{p''_{\gamma 0}} \prod_{\delta=1}^{u} \frac{d^3 q''_\delta}{q''_{\delta 0}}, \tag{3.1}$$

where

$$P' = \sum_{\alpha=1}^{k} p'_\alpha, \quad P'' = \sum_{\gamma=1}^{s} p''_\gamma,$$

$$Q' = \sum_{\beta=1}^{l} q'_\beta, \quad Q'' = \sum_{\delta=1}^{u} q''_\delta. \tag{3.2}$$

Already at this point we see the much more complicated character of the multiplication problem in the case of field theory in comparison with the analogous problem in the case of non-relativistic quantum mechanics. Each coefficient a_{mnklsu} of Part I becomes now a function of $8+3$ $(k+l+s+u)$ arguments.

From the analogy of the Rydberg-Ritz combination principle we get the condition that the unknown functions in (3.1) may be different from zero only on the manifold determined by the equality

$$Q+P'+P'' = P+Q'+Q'', \tag{3.3}$$

which must hold identically for all values of the one-particle momenta present (3.2). This is possible only when

$$n+k+s = m+l+u \tag{3.4}$$

where m and n are the numbers of one-particle momenta constituting P and Q, respectively. We see therefore that the analogue of the Rydberg-Ritz combination principle gives in the present case rather a weak restriction on the general form of the multiplication law. In order to proceed further we must know the type of the functional dependence of the field representatives on their arguments. But even knowing this we introduce a new function each time we perform the multiplication step. These new functions play an analogous role to that which the form factors play in the conventional quantum field theory. Consequently, we have here the problem of how to remove the form factors (or rather how to fix them) instead of introducing them as happens in the usual approach to field theory. This fact permits the hope that in our approach it should be easier to get an agreement between theoretical results and the experimental data.

To obtain the field representatives we must solve the field equation (2.11). In order to do this, however, we must specify the form factors in (3.1) and at this point a specific closed circle arises. There are many ways of cutting this circle. The first possibility is to assume a priori the functional shape of the field representatives and of the form factors and with this to try to solve the field equation. This way, however, is very ineffective and in practice can be followed only in the cases of the free field theory and the interacting fields in the so-called one-particle approximation. The details of these cases are discussed in the rest of this paper. Other models based on such a procedure are practically not solvable.

Another way of cutting the closed circle is to use the asymptotic expansions mentioned in Sec. 2. Owing to the fact that the zero order of approximation coincides with the free field theory where the field representatives are completely known, we may calculate

the field representatives up to an arbitrary required order of approximation and discuss the corresponding multiplication problem. We shall describe the details of such a procedure in the last part of the present series of papers.

4. *The free field theory*

As a first illustration of our construction of quantum field theory we discuss here the free field theory. From the field equation

$$[-(P-Q)^2+m^2]\,\varphi(P, Q) = 0. \tag{4.1}$$

we conclude that the field representatives of the free field are different from zero only when

$$(P-Q)^2 = m^2. \tag{4.2}$$

This means, however, that if we do not wish to have any restrictions on the values of the one--particle momenta in the set (2.6) we must conclude that the mass of the particles is equal to m and that the field representatives are different from zero only in two cases. In the first case, if P is an element of (2.6) with n momenta, then Q is an element of (2.6) with $n+1$ momenta but among them n must be equal to those present in P. The second case consists in reversing the role of P and Q. The physical transition amplitudes are then given by

$$T_f(u_{n+1}, v_n) = \int \prod_{j=1}^{n+1} \frac{d^3p_j}{p_0}\, u^*_{n+1}(p_1, \ldots, p_{n+1})\varphi_f(P, Q)v_n(p_1, \ldots p_n) \tag{4.3}$$

and

$$T_f(u_n, v_{n+1}) = \int \prod_{j=1}^{n+1} \frac{d^3p_j}{p_0}\, u^*_n(p_1, \ldots p_n)\varphi_f(P, Q)v_{n+1}(p_1, \ldots, p_{n+1}) \tag{4.4}$$

and define two bounded sesquilinear forms on the Cartesian products $H_{n+1} \times H_n$ and $H_n \times H_{n+1}$, respectively. These two forms are mutually adjoint if

$$\varphi(\sum_{j=1}^{n+1} p_j, \sum_{j=1}^{n} p_j) = \varphi^*(\sum_{j=1}^{n} p_j, \sum_{j=1}^{n+1} p_j). \tag{4.5}$$

In the following we shall assume that the spectrum condition (2.6) is taken with infinite N. The symmetric field operator generated by the non-zero field representatives is then given in the Fock space $\mathscr{F}_\infty$ by

$$(\varphi(\{f\})u)_n(p_1, \ldots p_n) =$$

$$= \frac{1}{n}\sum_{j=1}^{n} \varphi(p_1+\ldots+p_n, p_1+\ldots+\hat{p}_j+\ldots+p_n)\tilde{f}_n(p_j)u_{n-1}(p_1, \ldots \hat{p}_j \ldots p_n) +$$

$$+ \int \frac{d^3p}{p_0}\, \varphi(p_1+\ldots+p_n, p_1+\ldots+p_n+p)\tilde{f}^*_{n+1}(p)u_{n+1}(p_1, \ldots p_n, p), \tag{4.6}$$

where the hat over the momentum p_j means that this particular momentum has to be omitted in the corresponding place. As usual the field operator (4.6) is defined on the dense domain of all finite vectors in the Fock space.

Having constructed the field operator we may elaborate the restrictions on the field representatives which follow from the local commutativity condition (2.14). After a standard calculation we then get

$$\varphi(\sum_{j=1}^{n+1} p_j, \sum_{j=1}^{n} p_j) = \varphi_{10}\sqrt{n+1}$$

and

$$\varphi(\sum_{j=1}^{n} p_j, \sum_{j=1}^{n+1} p_j) = \varphi_{01}\sqrt{n+1}\,, \tag{4.7}$$

where φ_{10} and φ_{01} are the field representatives describing the transition between the vacuum and one-particle states. According to (2.13), these representatives are constants independent of the particular momentum of the one-particle state. We see therefore that, in addition to the Lorentz invariance condition (2.13), the local commutativity condition (2.14) equates an infinite number of field representatives in the set (2.5) and this further confirms the statement that the relativistic local quantum field theories belong to the class of physical systems discussed in Sec. 4 of Part I.

From (4.7) it follows that the set of equivalence classes of the free field representatives coincides with the quantum mechanical oscillator problem. Since this is a result of the symmetries for the considered field, we may assume that the form factors in (3.1) do not depend on P and Q. Then the multiplication problem for the free field theory may be reduced to the general multiplication problem for the harmonic oscillator dynamics. In Part I we have shown that the general multiplication law for the oscillator is parametrized by the value of the ground state energy. Since we have here to do with an infinite number of oscillators, the only consistent choice of this ground state energy is the zero value. Then, calculating the powers of the oscillator matrix X, we get the reduced representatives of the powers of the free field theory. The mn-th matrix element of $:X^k:$ corresponds to the representatives of the k-th power of the free field attached to the momenta P and Q such that P is the sum of m one-particle momenta and Q is the sum of n one-particle momenta. Owing to the non-vanishing of the field representatives of the free field only for some special correlation between the momenta constituting P and Q, the same is true for the non-vanishing representatives of the powers of the free field. We do not write down these correlations since they are easy to obtain in each case. In exactly the same way as we have constructed the field operator from the field representatives we may proceed with the representatives of the powers of the field. The operators in the Fock space $\mathscr{F}_\infty$ generated by these representatives always coincide with the Wick ordered powers of the free field operator. This confirms the correctness of our method in the case of the free field theory.

It is obvious that we could proceed in the reverse way without any reference to the results of Part I. Having established the fact that the field representatives for the free field

are constant, we may reduce the multiplication problem (3.1) to the form

$$\varphi^2(P, Q) = \sum_{k,l,s,u \in Z} a_{klsu}(P, Q)\varphi_{kl}\varphi_{su}, \tag{4.8}$$

where

$$\varphi_{mn} = \varphi_{01}\sqrt{m+1}\,\delta_{m+1,n} + \varphi_{10}\sqrt{n+1}\,\delta_{m,n+1},$$

and ask how to choose $a_{klsu}(P, Q)$ in order to get a local operator generated by (4.8). It is easy to find that the answer is exactly the same as above. However, if the generalized multiplication law in the case of non-relativistic physics did not exist, this would show that non-relativistic quantum mechanics and relativistic quantum field theory are two different theories — which is an additional justification for writting Part I of this series.

We have already mentioned in Part I that the procedure proposed there in Sec. 4 is not quite unique. We may use this non-uniqueness in order to construct the constants of motion for the free field theory such as, for example, the operator of the number of particles. It is clear that the results are quite the same as in the usual approach to quantum field theory.

5. The interacting fields in one-particle approximation

In this section we give some application of our scheme to the case of interacting fields in the so-called one-particle approximation. We have already stressed the fact that the set of all possible four-momenta P and Q for the considered system must coincide with the real experimental situation. If we know, therefore, that the total amount of energy concentrated in the system does not allow the production of a large number of particles we may assume from the very beginning that the set (2.6) does not contain terms with a large number of one-particle momenta. It is also one of the advantages of our approach that it allows such approximations to be made from the start.

The simplest case is, of course, the system containing at most one particle. Intuitively speaking, this problem should be trivial in any theory of elementary particles. On the examples to the $\lambda\varphi^3$ and $\lambda\varphi^4$ theories we shall show that our approach fulfills this requirement and it is superfluous to display again the difference between our approach and the usual one.

In the considered example the set (2.6) is replaced by the set

$$\{0, p | p^2 = M^2; p_0 > 0\}. \tag{5.1}$$

The set of field representatives (2.5) reduces now to a simple set containing four types of elements only; the representative φ_{00} describing the vacuum mean value of the field, the representatives $\varphi_{01}\,e^{-ipx}$ and $\varphi_{10}\,e^{ipx}$ describing the vacuum one-particle state transitions, and the representative φ_{11} describing the mean value of the field in the one-particle states. The set of equivalence classes of the field representatives, or more simply the reduced set of field representatives, is therefore given by a 2×2 matrix

$$\begin{pmatrix} \varphi_{00}, & \varphi_{01} \\ \varphi_{10}, & \varphi_{11} \end{pmatrix}. \tag{5.2}$$

The Fock space $\mathscr{F}_1$ is the direct sum

$$\mathscr{F}_1 = C \oplus H_1 \tag{5.3}$$

and the smeared values of the field representatives define in it a field operator given by

$$(\varphi(f)u)_0 = \varphi_{00}u_0 + \varphi_{01}\int \frac{d^3p}{p_0}\tilde{f}^*(p)u_1(p)$$

$$(\varphi(f)u)_1(p) = \varphi_{10}\tilde{f}(p)u_0 + \varphi_{11}u_1(p). \tag{5.4}$$

This field operator is a local operator on the vacuum if

$$\varphi_{00} = \varphi_{11}. \tag{5.5}$$

Using the principle formulated in Part I, we may apply to the reduced set of field representatives (5.2) the usual matrix multiplication law and obtain in this way the reduced representatives for the powers of the field (5.4). Although other choices are not excluded we do so for the sake of simplicity. With an arbitrary choice we may obtain the explicit form of the solution of field equations. For our choice the solutions are as follows:

The $\lambda\varphi^3$ theory:

$$\varphi_{00} = \varphi_{11} = \frac{-m^2 \pm \sqrt{m^4 - 4\lambda^2|\varphi_{01}|^2}}{2\lambda},$$

$$M^2 = \pm\sqrt{m^4 - 4\lambda^2|\varphi_{01}|^2},$$

and only the upper sign gives a physical solution. This solution exists only for

$$4\lambda^2|\varphi_{01}|^2 \leqslant m^4,$$

which can be considered either as a restriction on λ^2 or on $|\varphi_{01}|^2$. It is interesting to note that for very large absolute value of the coupling constant the transition from one-particle state to the vacuum state must be very weak.

The $\lambda\varphi^4$ theory:

In this case there exist two solutions. For the first

$$\varphi_{00} = \varphi_{11} = 0,$$

$$M^2 = m^2 + \lambda|\varphi_{01}|^2,$$

which is always meaningful for positive values of the coupling constant. For negative values of this constant we must have

$$\lambda|\varphi_{01}|^2 \geqslant -m^2,$$

which again may be considered as a restriction either on $|\varphi_{01}|^2$ or on λ. Here for very small negative values of the coupling constant the transition from vacuum to one-particle states are not restricted but they should be very weak for large negative values of the coupling constant. It is interesting to note that the form of the field operator coincides with the form of the free field operator of mass M.

The second solution exists only for negative values of the coupling constant and is given by

$$\varphi_{00} = \varphi_{11} = \pm \sqrt{-\frac{m^2}{\lambda} - 3|\varphi_{01}|^2};$$

$$M^2 = -2(m^2 + 4\lambda|\varphi_{01}|^2).$$

It exists for

$$-\frac{m^2}{3} \leqslant \lambda|\varphi_{01}|^2 \leqslant -\frac{m^2}{4},$$

which once more gives some correlation between λ and $|\varphi_{01}|^2$. The maximum value of M^2 is here bounded by

$$M^2_{\max} = \tfrac{2}{3}\, m^2.$$

Finally, it should be noted that, apart from the solutions quoted above, there exist in both cases also nonlocal solutions which do not satisfy (5.5). Most of them correspond to the zero value of the mass M.

APPENDIX

In this Appendix we describe the strictly one-particle problem in the framework of our reinterpretation of classical field theory. In the first quantized wave mechanics the wave function of a single particle with mass M is usually written in the form of a Fourier integral

$$\psi(x) = \int \frac{d^3p}{p_0} (\psi(p)e^{-ipx} + \psi^*(p)e^{ipx}), \tag{A1}$$

where

$$p_0 = +\sqrt{\vec{p}^2 + M^2}, \tag{A2}$$

and the squares of the Fourier coefficients $|\psi(p)|^2$ are interpreted as being proportional to the probability that the particle moves with the momentum $\vec{p}$. We have an equivalent amount of information working directly with the set

$$\left\{\psi(p)e^{-ipx}|p^2 = M^2;\, p_0 > 0;\, \int \frac{d^3p}{p_0} |\psi(p)|^2 = N_\psi^2 < \infty\right\}, \tag{A3}$$

and strictly speaking only the scheme (A3) follows from the original de Broglie idea [7] while (A1) has only a limited range of application. The assumption that the first quantized wave function of a single particle is always represented by a set of type (A3) is in fact the content of our reinterpretation.

Having two sets of the form (A3) (with the same N_ψ), we define their superposition by

$$(\varphi \oplus \psi)(p) = \frac{\varphi(p) + \psi(p)}{\sqrt{2\left(1 + \frac{\mathrm{Re}\,(\varphi, \psi)}{N^2}\right)}}, \tag{A4}$$

where

$$(\varphi, \psi) = \int \frac{d^3p}{p_0} \varphi^*(p)\psi(p). \tag{A5}$$

The reason for the fixed value of N will be clear from (A9) below. The superposition principle (A4) is valid independently of the wave equation obeyed by φ and ψ and has three important features: First, the representatives of a superposition have the same value of N; second, if $\varphi(p)$ and $\psi(p)$ do not overlap in the sense of (A5) then

$$(\varphi \oplus \psi) = \frac{\varphi(p) + \psi(p)}{\sqrt{2}} \tag{A6}$$

which coincides with the usual linear superposition principle; third, the superposition of $\psi(p)$ with itself is equal to $\psi(p)$.

Now, we shall define the multiplication law for the sets of type (A3). Following the procedure of Part I, we start with the general expressions for the product and require that if we substitute into them the representatives together with the exponential factors, the representatives of the products attached to the momentum p should automatically be accompanied by the right exponential factors. In this we see an important difference between the present case and the cases considered in Part I and in this paper. This difference is caused by the fact that, while we always had the situation that the linear combinations of momenta appearing in the corresponding exponential factors are possible momenta for the considered system, this property cannot be met in the present case owing simply to the fact that the sum of two one-particle four-momenta is not a one-particle four-momentum. This fact has serious consequences. In particular, in the present case it is impossible to represent in the class of the sets (A3) any even function of the set (A3) while the odd powers are almost uniquely represented by

$$\psi^{2n+1}(p) = N_\psi^{2n}\psi(p). \tag{A7}$$

This implies that any non-linear one-particle wave equation

$$(\Box + m^2)\psi(x) = F(\psi(x)), \tag{A8}$$

with

$$F(-z) = -F(z)$$

is trivially soluble. With the definition (A7) it follows from (A8) that the non-zero solution exists only for

$$M^2 = m^2 - \frac{F(N)}{N} \tag{A9}$$

and equation (A8) does not restrict the values of $\psi(p)$. It is clear that this is indeed a correct result, since with the usual statistical interpretation of $|\psi(p)|^2$ this quantity should be determined by the initial conditions and not by the equation of motion. It follows also from (A9) that any non-linear wave equation (A8) is equivalent in the present scheme to the Klein-Gordon wave equation with mass M given by (A9). This is in excellent agreement with the one-particle character of the considered system.

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