# COUPLING BETWEEN QUADRUPOLE AND HEXADECAPOLE DEGREES OF FREEDOM

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The energies of the quadrupole and hexadecapole vibrational states for some nuclei are calculated in the Actinide region. The influence of the coupling between quadrupole and hexadecapole modes on the vibrational energies is studied.

Vibrational excitations in nuclei are caused mainly by the long-range residual forces acting between the nucleons. It is customary to apply the multipole-multipole expansion in the description of these excitations. Using such a description we are facing a question how strong is the influence of the coupling between various multipolarities on the energy of vibrational states.

In the present paper we shall calculate the lowest vibrational energies, introducing quadrupole and hexadecapole degrees of freedom. Our calculations are based on the microscopic approach to the vibrational motion (see for example Ref. [1]; the details of the calculations have been described in Ref. [2]). We consider the total energy of the motion in the harmonic approximation. It can be expressed taking into account the vibrations of the nucleus around the equilibrium point. Thus we obtain two equivalent formulae for the energy

$$E = E_0 + \tfrac{1}{2} \sum_{i,j=2,4} C_{ij} \Delta Q_i \Delta Q_j + \tfrac{1}{2} \sum_{i,j=2,4} B_{ij} \dot{Q}_i \dot{Q}_j \tag{1}$$

and

$$E = E_0 + \tfrac{1}{2} \sum_{i,j=2,4} C_{\varepsilon_i \varepsilon_j} \Delta \varepsilon_i \Delta \varepsilon_j + \tfrac{1}{2} \sum_{i,j=2,4} B_{\varepsilon_i \varepsilon_j} \dot{\varepsilon}_i \dot{\varepsilon}_j, \tag{2}$$

described by the collective coordinates considered either as expectation values of multipole moments (Eq. (1)) or as parameters describing nuclear shape (Eq. (2)). The descrip-

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tion of the shape is defined by Nilsson [3] parametrization (taking into account the matrix elements of Nilsson Hamiltonian

$$H_0 = \frac{1}{2}\hbar\omega_0(\varepsilon_2, \varepsilon_4)\Big[-\Delta_\varrho + \frac{2}{3}\varepsilon_2 \cdot \frac{1}{2}(2\partial^2/\partial\zeta^2 - \partial^2/\partial\xi^2 - \partial^2/\partial\eta^2) +$$

$$+\varrho^2\left(1 - \frac{2}{3}\varepsilon_2 P_2(\cos\theta_t) + 2\varepsilon_4 P_4(\cos\theta_t)\right) - 4\kappa\frac{\mathring{\omega}_0}{\omega_0}\vec{l}_t\cdot\vec{s} - 2\mu\kappa\frac{\mathring{\omega}_0}{\omega_0}(\vec{l}_t^2 - \langle\vec{l}_t^2\rangle)\Big] \tag{3}$$

only between the states with the same oscillator number $N$ ($\Delta N = 0$)).

In both above equations the energy $E_0$ denotes the energy of the equilibrium state of the nucleus. The stiffness ($C_{ij}$) and mass ($B_{ij}$) parameters obtained by aid of the microscopic calculations are given by

$$C_{ij} = \tfrac{1}{2}\sigma_1^{ij} - \kappa_i\delta_{ij}, \tag{4}$$

$$B_{ij} = \tfrac{1}{2}\hbar^2 \sum_{k,l=2,4} \sigma_1^{ik}\Sigma_3^{kl}\sigma_1^{lj}, \tag{5}$$

where

$$\Sigma_n^{ij} = \sum_{\nu\omega} \frac{\langle\nu|\hat{q}_i|\omega\rangle\langle\omega|\hat{q}_j|\nu\rangle(u_\nu v_\omega + u_\omega v_\nu)^2}{(E_\nu + E_\omega)^n}, \tag{6}$$

for $n = 1$ or 3, and $i, j = 2$ or 4, $\sigma_1^{ij}$ denotes a matrix inverse with respect to $\Sigma_1^{ij}$, $\kappa_i$ are the strength parameters of the quadrupole-quadrupole ($i = 2$) and hexadecapole-hexadecapole ($i = 4$) forces. Quantities $E_\nu$ in Eq. (6) denote quasiparticle energies, $\langle\nu|\hat{q}_i|\omega\rangle$ are matrix elements of the single-particle quadrupole $\hat{q}_2 = 2r^2P_2(\cos\theta)$ and hexadecapole $\hat{q}_4 = \varrho^2P_4(\cos\theta_t)$ (notation as in Ref. [3]) operators between Nilsson states (states with $\Delta N = 2$ have also been taken into account) and $u_\nu$, $v_\nu$ are the coefficients in the BCS wave function.

Finally, $Q_2$ and $Q_4$ (see Eq. (1)) are the total (mass) quadrupole and hexadecapole moments

$$Q_i = 2\sum_\nu v_\nu^2\langle\nu|\hat{q}_i|\nu\rangle, \tag{7}$$

$\Delta Q_i$ are their deviations from the equilibrium points. Parameters $\varepsilon_2$ and $\varepsilon_4$ are Nilsson deformation parameters; $\Delta\varepsilon_i$ describe the change of the nuclear shape in comparison with the equilibrium one.

It follows from the equivalence of formulae (1) and (2) that the stiffness ($C_{\varepsilon_i\varepsilon_j}$) and mass ($B_{\varepsilon_i\varepsilon_j}$) parameters are connected with the corresponding parameters $C_{ij}$ and $B_{ij}$ (Eqs (4), (5)).

Their connection is given by the equations

$$B_{\varepsilon_i\varepsilon_j} = \sum_{k,l=2,4} B_{kl}\frac{\partial Q_k}{\partial\varepsilon_i}\frac{\partial Q_l}{\partial\varepsilon_j}. \tag{8}$$

Similar relations are valid for stiffness parameters $C_{\varepsilon_i\varepsilon_j}$. Further, we shall consider the oscillations of the nuclear shape. Therefore we shall work with $B_{\varepsilon_i\varepsilon_j}$ and $C_{\varepsilon_i\varepsilon_j}$ parameters. Let us study the coupling between quadrupole and hexadecapole degrees of freedom. Here we distinguish the following two possibilities:

(*i*) We consider only one-dimensional vibrations of the nucleus along $\varepsilon_2$ or $\varepsilon_4$ axes for fixed values of $\varepsilon_4$ and $\varepsilon_2$ parameters, respectively, corresponding to their equilibrium values.

The appropriate one-dimensional stiffness $(C_{\varepsilon_2}, C_{\varepsilon_4})$ and mass $(B_{\varepsilon_2}, B_{\varepsilon_4})$ parameters can be obtained from the expressions for $C_{\varepsilon_i\varepsilon_j}$ and $B_{\varepsilon_i\varepsilon_j}$, taking into account only one, considered degree of freedom. In this case the energies of the lowest quadrupole $E_2$ and hexadecapole $E_4$ vibrational excitations are described in the following way:

$$E_2 = \hbar \sqrt{\frac{C_{\varepsilon_2}}{B_{\varepsilon_2}}}, \tag{9}$$

$$E_4 = \hbar \sqrt{\frac{C_{\varepsilon_4}}{B_{\varepsilon_4}}}. \tag{10}$$

(*ii*) In contrast to (*i*) we consider the full dynamical influence of both degress of freedom on the vibrational motion. We investigate the normal modes of the motion in the plane $(\varepsilon_2, \varepsilon_4)$. In this case we transform equation (2) to the canonical form, obtaining the following expressions for the quadrupole and hexadecapole vibrational energies, expressed by the eigen-oscillations of the system

$$E_2^t = \hbar \sqrt{\frac{-B - \sqrt{B^2 - 4AC}}{2A}}, \tag{11}$$

$$E_4^t = \hbar \sqrt{\frac{-B + \sqrt{B^2 - 4AC}}{2A}}, \tag{12}$$

where

$$A = B_{\varepsilon_2\varepsilon_2} B_{\varepsilon_4\varepsilon_4} - B_{\varepsilon_2\varepsilon_4}^2 \tag{13}$$

$$B = 2C_{\varepsilon_2\varepsilon_4} B_{\varepsilon_2\varepsilon_4} - C_{\varepsilon_2\varepsilon_2} B_{\varepsilon_4\varepsilon_4} - C_{\varepsilon_4\varepsilon_4} B_{\varepsilon_2\varepsilon_2} \tag{14}$$

$$C = C_{\varepsilon_2\varepsilon_2} C_{\varepsilon_4\varepsilon_4} - C_{\varepsilon_2\varepsilon_4}^2. \tag{15}$$

Microscopic method requires the knowledge of strengths of the multipole forces, $\kappa_2$ and $\kappa_4$, (see Eq. (4)). Parameters $\kappa_2$ have been obtained from the fit to the experimental values of the energies of the lowest quadrupole states (substituted into Eq. (11) for $E_2^t$). Parameters $\kappa_4$ have been obtained from the comparison of the appropriate parts of the two Hamiltonians, describing the deviation from the hexadecapole equilibrium point, namely:

a) the Nilsson Hamiltonian, and

b) the one-body Hamiltonian obtained from collective Hamiltonian with multipole forces by making use of Hartree method

$$H = H_0 - \kappa_2 Q_2 \hat{q}_2 - \kappa_4 Q_4 \hat{q}_4. \tag{16}$$

We get [4]

$$\kappa_4 = -\hbar\omega_0 \left(\frac{\partial Q_4}{\partial \varepsilon_4}\right)^{-1}. \tag{17}$$

The results on $\kappa_2$ and $\kappa_4$ are listed in Table I.

TABLE I

The strengths of quadrupole ($\kappa_2$) and hexadecapole ($\kappa_4$) forces

| Z | N | $\kappa_2\left[\left(\frac{M\omega_0}{\hbar}\right)^2 \text{MeV}\right]$ | $\kappa_4$[MeV] |
|---|---|---|---|
| 92 | 146 | $8.144 \quad 10^{-4}$ | $-4.376 \quad 10^{-2}$ |
| 94 | 144 | $8.079 \quad 10^{-4}$ | $-4.646 \quad 10^{-2}$ |
| 94 | 146 | $8.388 \quad 10^{-4}$ | $-4.450 \quad 10^{-2}$ |

Table II presents the values of stiffness $C_{\varepsilon_2\varepsilon_2}$, $C_{\varepsilon_4\varepsilon_4}$ and mass $B_{\varepsilon_2\varepsilon_2}$, $B_{\varepsilon_4\varepsilon_4}$ parameters obtained in our calculations. The presented values correspond to the equilibrium points of quadrupole and hexadecapole deformations.

For the wider discussion of mass and especially stiffness parameters (their comparison with values obtained from Strutinsky method) see Ref. [4].

TABLE II

Diagonal stiffness and mass parameters calculated according to microscopic formulae in the points corresponding to the equilibrium

| Z | N | $\varepsilon_2$ | $\varepsilon_4$ | $C_{\varepsilon_2\varepsilon_2}$[MeV] | $C_{\varepsilon_4\varepsilon_4}$[MeV] | $B_{\varepsilon_2\varepsilon_2}\left[\frac{\hbar^2}{\text{MeV}}\right]$ | $B_{\varepsilon_4\varepsilon_4}\left[\frac{\hbar^2}{\text{MeV}}\right]$ |
|---|---|---|---|---|---|---|---|
| 92 | 146 | −0.21 | −0.04 | 496.7 | 1295 | 388 | 18.10 |
| 94 | 144 | −0.21 | −0.04 | 521.3 | 1207 | 412.7 | 13.89 |
| 94 | 146 | −0.23 | −0.03 | 365 | 1248 | 394.5 | 19.20 |

The comparison between corresponding energies calculated in cases (*i*) and (*ii*) shows the influence of the coupling between multipolarities on the vibrational energy. The numerical results are presented in Table III. Columns Nos 3 to 6 present the quadrupole and hexadecapole excitation energies obtained form formulae (9,, (11) and (10), (12), respectively. The details of the numerical calculations are presented in Ref. [4].

The examination of the results indicates that the dynamical influence of the hexadecapole modes on the quadrupole energy is small. It is of the order of (1–3)%, while the influence of the quadrupole modes on the hexadecapole vibrational energy is much stronger (of the order of (10–20)%).

TABLE III

Comparison between quadrupole ($E_2$ and $E_2^t$) and hexadecapole ($E_4$ and $E_4^t$) vibrational energies, calculated without and with the coupling between these two modes

| Z | N | $E_2$[MeV] | $E_2^t$[MeV] | $E_4$[MeV] | $E_4^t$[MeV] |
|---|---|---|---|---|---|
| 92 | 146 | 0.97 | 0.96 | 8.18 | 9.16 |
| 94 | 144 | 1.00 | 0.97 | 8.66 | 10.41 |
| 94 | 146 | 0.88 | 0.86 | 7.75 | 9.75 |

The other problem which can be examined is to consider the coupling between diagonal and nondiagonal parameters. The quantities $B^2_{\varepsilon_2\varepsilon_4}$ ($C^2_{\varepsilon_2\varepsilon_4}$) are of the same order of magnitude as the product of the diagonal $B_{\varepsilon_2\varepsilon_2}$ ($C_{\varepsilon_2\varepsilon_2}$) and $B_{\varepsilon_4\varepsilon_4}$ ($C_{\varepsilon_4\varepsilon_4}$) parameters; the expressions

$$B_{\varepsilon_2\varepsilon_2}B_{\varepsilon_4\varepsilon_4}(B^2_{\varepsilon_2\varepsilon_4})^{-1} \quad \text{and} \quad C_{\varepsilon_2\varepsilon_2}C_{\varepsilon_4\varepsilon_4}(C^2_{\varepsilon_2\varepsilon_4})^{-1}$$

are of the order of few units. Thus, it follows from our calculations that these two degrees of freedom are coupled strongly through non-diagonal terms and that this coupling influences much more the hexadecapole than the quadrupole vibrational energies.

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