INTERACTION OF GRAVITATIONAL RADIATION WITH A UNIFORMLY MAGNETIZED SPHERE*

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Maxwell equations in the field of a gravitational wave are linearized by means of the weak field approximation. Then the equations are solved in the case of a uniformly magnetized sphere and the dipole electromagnetic radiation power is calculated. These results are applied to compute the electromagnetic radiation emitted by the Earth and magnetic neutron stars when hit by gravitational radiation.

1. Maxwell equations in weak gravitational fields

The production of electromagnetic radiation by gravitational waves falling on a static magnetic field with plane symmetry was studied by Boccaletti et al. [1].

The case of a spherically symmetric magnetic field is, however, more interesting in astrophysics; in the present work we will therefore calculate the emission of electromagnetic waves by a uniformly magnetized sphere whe.. gravitational radiation falls on it.

Maxwell equations in a gravitational field in vacuum are:

$$\frac{\partial(\sqrt{-g}\, g^{\alpha\beta} g^{\gamma\delta} F_{\beta\delta})}{\partial x^{\gamma}} = 0, \tag{1}$$

$$F_{[\alpha\beta,\gamma]} = 0, \tag{2}$$

where $g^{\alpha\beta}$ is the metric tensor with signature -2, the Greek indices run from 0 to 3 and the Latin ones from 1 to 3.

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In the weak field approximation the metrics is written as: [2]

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$
$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$$

where $\eta_{\alpha\beta}$ is the Minkowski tensor with signature -2 and $h_{\alpha\beta}$ are small corrections due to the gravitational field; we shall neglect powers of $h_{\alpha\beta}$ greater than 1.

In the same approximation the electromagnetic tensor is to be thought of as the sum of two terms: one, $f_{\alpha\beta}^{(0)}$, represents the unperturbed field and the other, $f_{\alpha\beta}$ represents the perturbation due to the gravitational field. The latter is to be considered of the same order of $h_{\alpha\beta}$; therefore second and higher order terms in $h_{\alpha\beta}$ and $f_{\alpha\beta}$ are to be neglected.

We shall consider only the case where the unperturbed term $f_{\alpha\beta}^{(0)}$ is a static magnetic field, that is:

$$H_1^{(0)} = f_{23}^{(0)}, \quad H_3^{(0)} = f_{12}^{(0)},$$
$$H_2^{(0)} = f_{31}^{(0)}, \quad f_{0k}^{(0)} = 0.$$

In this approximation and with the splitting of $f_{\alpha\beta}$ into space and time components, Equations (1) become:

$$-\frac{\partial f_{0k}}{\partial x^k} = \frac{\partial(g^{03}H_2^{(0)} - g^{02}H_3^{(0)})}{\partial x^1} + \frac{\partial(g^{01}H_3^{(0)} - g^{03}H_1^{(0)})}{\partial x^2} + \frac{\partial(g^{02}H_1^{(0)} - g^{01}H_2^{(0)})}{\partial x^3}, \quad \text{(sum over } k) \tag{3}$$

$$\frac{\partial f_{ik}}{\partial x^k} - \frac{\partial f_{i0}}{\partial x^0} = -\frac{\partial f_{ik}^{(0)}[\sqrt{-g}\, g^{ii} g^{kk}]_h}{\partial x^k} + f_{ik}^{(0)} \frac{\partial g^{0k}}{\partial x^0} + \frac{\partial(g^{ij} f_{jk}^{(0)} + g^{kl} f_{il}^{(0)})}{\partial x^k}, \quad \text{(sum over } j, k, l) \tag{4}$$

where $[\ \ ]_h$ means that in the expression in brackets only terms of the first order in $h_{\alpha\beta}$ are to be considered and not zero order terms i.e. flat space terms. The right-hand side terms in (3) and (4), which arise from the perturbation of the gravitational field can be interpreted as a current S_α with components:

$$S_0 = \frac{\partial(g^{03}H_2^{(0)} - g^{02}H_3^{(0)})}{\partial x^1} + \frac{\partial(g^{01}H_3^{(0)} - g^{03}H_1^{(0)})}{\partial x^2} + \frac{\partial(g^{02}H_1^{(0)} - g^{01}H_2^{(0)})}{\partial x^3},$$

$$S_i = \frac{-\partial f_{ik}^{(0)}[\sqrt{-g}\, g^{ii} g^{kk}]_h}{\partial x^k} + f_{ik}^{(0)} \frac{\partial g^{0k}}{\partial x^0} + \frac{\partial(g^{ij} f_{jk}^{(0)} + g^{kl} f_{il}^{(0)})}{\partial x^k}.$$

One can easily check that such a perturbation current satisfies the continuity equation:

$$\frac{\partial S^\alpha}{\partial x^\alpha} = 0$$

2. Dipole solutions of Maxwell equations for a uniformly magnetized sphere at great distances

Let us consider a plane gravitational wave produced by a mass quadrupole oscillator. In the linear approximation the metrics is given by [3]:

$$h_{11} = \frac{\varphi}{2}(2\cos^2\theta - \sin^2\theta), \quad h_{13} = \varphi \sin\theta\cos\theta,$$

$$h_{22} = -\frac{\varphi}{2}\sin^2\theta, \quad h_{03} = -\varphi\sin\theta\cos\theta,$$

$$h_{33} = \frac{\varphi}{2}\sin^2\theta, \quad h_{01} = -\varphi\cos^2\theta,$$

$$h_{00} = \frac{\varphi}{2}(1+\cos^2\theta), \quad h_{02} = h_{12} = h_{23} = 0,$$

$$g = -1+\varphi\sin^2\theta, \quad \varphi = ae^{ik(x_1-x_0)},$$

where θ is the angle between the oscillation direction of the quadrupole and the x-axis taken as the direction of propagation of the gravitational wave.

Let us assume that this wave falls on a uniformly magnetized sphere with its center in the origin of the coordinates. For the sake of simplicity we assume that the constant magnetic field $H^{(0)}$ with components $H_1^{(0)}$, $H_2^{(0)}$, $H_3^{(0)}$ is zero outside the sphere. With the choice of this metrics, (3) and (4) inside the sphere become:

$$\frac{\partial f_{12}}{\partial x^2} + \frac{\partial f_{13}}{\partial x^3} - \frac{\partial f_{10}}{\partial x^0} = -H_2^{(0)} ik\varphi \sin\theta\cos\theta,$$

$$\frac{\partial f_{21}}{\partial x^1} + \frac{\partial f_{23}}{\partial x^3} - \frac{\partial f_{20}}{\partial x^0} = -H_3^{(0)} ik\varphi \frac{\sin^2\theta}{2},$$

$$\frac{\partial f_{31}}{\partial x^1} + \frac{\partial f_{32}}{\partial x^2} - \frac{\partial f_{30}}{\partial x^0} = -H_2^{(0)} ik\varphi \frac{\sin^2\theta}{2},$$

$$\frac{\partial f_{01}}{\partial x^1} + \frac{\partial f_{02}}{\partial x^2} + \frac{\partial f_{03}}{\partial x^3} = H_2^{(0)} ik\varphi \sin\theta\cos\theta.$$

Together with these we consider the remaining four Maxwell Equations (2) which allow us to define:

$$f_{12} = H_3, \quad f_{23} = H_1,$$

$$f_{31} = H_2, \quad f_{i0} = E_i.$$

The above identification is possible as we look for solutions of the Maxwell equations at infinity where electric and magnetic fields are of the same order of $h_{\alpha\beta}$, in fact in this case the tetrad components of the electromagnetic tensor, which are the true $\vec{H}$ and $\vec{E}$

[4] are the same as the coordinate components. This amonnuts to considering the equations written in the usual flat space, that is:

$$\operatorname{div} \vec{E} = 4\pi \varrho, \ \operatorname{div} \vec{H} = 0,$$

$$\operatorname{rot} \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}, \quad \operatorname{rot} \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J},$$

where

$$\vec{J} \equiv \left(-\frac{i\omega}{4\pi}\varphi H_2^{(0)} \sin\theta\cos\theta; -\frac{i\omega}{4\pi}\varphi H_3^{(0)}\frac{\sin^2\theta}{2}; -\frac{i\omega}{4\pi}\varphi H_2^{(0)}\frac{\sin^2\theta}{2}\right)$$

and

$$\varrho = -\frac{1}{4\pi} ik\varphi H_2^{(0)} \sin\theta\cos\theta. \quad (\omega = ck)$$

Using standard electrodynamics, in the dipole approximation, we obtain:

$$H_x = \frac{e^{ikR-i\omega t}}{R^3}(k^2R - ik)\frac{aR_1^3}{3}\left[\frac{\sin^2\theta}{2}H_3^{(0)}z - \frac{\sin^2\theta}{2}H_2^{(0)}y\right],$$

$$H_y = \frac{e^{ikR-i\omega t}}{R^3}(k^2R - ik)\frac{aR_1^3}{3}\left[\sin\theta\cos\theta H_2^{(0)}z - \frac{\sin^2\theta}{2}H_3^{(0)}x\right],$$

$$H_z = \frac{e^{ikR-i\omega t}}{R^3}(k^2R - ik)\frac{aR_1^3}{3}\left[\sin\theta\cos\theta H_2^{(0)}y - \frac{\sin^2\theta}{2}H_3^{(0)}x.\right],$$

$$E_x = \frac{e^{ikR-i\omega t}}{R^3}\frac{aR_1^3}{3}(1-ikR-k^2R^2)\sin\theta\cos\theta H_2^{(0)} + \frac{e^{ikR-i\omega t}}{R^5}\times$$
$$\times\frac{aR_1^3}{3}(k^2R^2+3ikR-3)x\left[\sin\theta\cos\theta H_2^{(0)}x + \frac{\sin^2\theta}{2}H_3^{(0)}y + \frac{\sin^2\theta}{2}H_2^{(0)}z\right],$$

$$E_y = \frac{e^{ikR-i\omega t}}{R^3}\frac{aR_1^3}{3}(1-ikR-k^2R^2)\frac{\sin^2\theta}{2}H_3^{(0)} + \frac{e^{ikR-i\omega t}}{R^5}\frac{aR_1^3}{3}\times$$
$$\times(k^2R^2+3ikR-3)y\left[\sin\theta\cos\theta H_2^{(0)}x + \frac{\sin^2\theta}{2}H_3^{(0)}y + \frac{\sin^2\theta}{2}H_2^{(0)}z\right],$$

$$E_z = \frac{e^{ikR-i\omega t}}{R^3}\frac{aR_1^3}{3}(1-ikR-k^2R^2)\frac{\sin^2\theta}{2}H_2^{(0)} + \frac{e^{ikR-i\omega t}}{R^5}\frac{aR_1^3}{3}\times$$
$$\times(k^2R^2+3ikR-3)z\left[\sin\theta\cos\theta H_2^{(0)}x + \frac{\sin^2\theta}{2}H_3^{(0)}y + \frac{\sin^2\theta}{2}H_2^{(0)}z\right],$$

(R is the radius of the sphere).

As we are interested only in wave propagation, we take into account only the terms which behave as $1/R$ at infinity, which amounts to considering $R \gg R_1$. We have thus:

$$H_x = \frac{e^{ikR-i\omega t}}{R^2}\frac{k^2 aR_1^3}{3}\left(\frac{\sin^2\theta}{2}H_z^{(0)}z - \frac{\sin^2\theta}{2}H_y^{(0)}y\right),$$

$$H_y = \frac{e^{ikR-i\omega t}}{R^2}\frac{k^2 aR_1^3}{3}\left(\sin\theta\cos\theta H_y^{(0)}z - \frac{\sin^2\theta}{2}H_z^{(0)}x\right),$$

$$H_z = \frac{e^{ikR-i\omega t}}{R^2}\frac{k^2 aR_1^3}{3}\left(\sin\theta\cos\theta H_y^{(0)}y - \frac{\sin^2\theta}{2}H_z^{(0)}x\right),$$

$$E_x = \frac{e^{ikR-i\omega t}}{R^3}\frac{k^2 aR_1^3}{3}\left[\sin\theta\cos\theta H_y^{(0)}x^2 + \frac{\sin^2\theta}{2}H_z^{(0)}xy + \right.$$
$$\left. + \frac{\sin^2\theta}{2}H_y^{(0)}xz - \sin\theta\cos\theta H_y^{(0)}R^2\right],$$

$$E_y = \frac{e^{ikR-i\omega t}}{R^3}\frac{k^2 aR_1^3}{3}\left[\sin\theta\cos\theta H_y^{(0)}xy + \frac{\sin^2\theta}{2}H_z^{(0)}y^2 + \right.$$
$$\left. + \frac{\sin^2\theta}{2}H_y^{(0)}yz - \frac{\sin^2\theta}{2}H_z^{(0)}R^2\right],$$

$$E_z = \frac{e^{ikR-i\omega t}}{R^3}\frac{k^2 aR_1^3}{3}\left[\sin\theta\cos\theta H_y^{(0)}xz + \frac{\sin^2\theta}{2}H_z^{(0)}yz + \right.$$
$$\left. + \frac{\sin^2\theta}{2}H_y^{(0)}z^2 - \frac{\sin^2\theta}{2}H_y^{(0)}R^2\right].$$

The Poynting vector is:

$$\vec{S} = \frac{c}{8\pi}\vec{E}\times\vec{H}^*, \qquad * = \text{complex conjugation}$$

and therefore the energy flux is:

$$\frac{c}{8\pi}|H|^2 \ \text{erg/cm}^2\cdot\text{sec}.$$

Integrating on a spherical surface of radius R we get:

$$W_{\text{em}} = \frac{ck^4a^2R_1^6}{27}\left[\frac{\sin^4\theta}{4}H_y^{(0)2} + \sin^2\theta\cos^2\theta H_y^{(0)2} + \frac{\sin^4\theta}{4}H_z^{(0)2}\right].$$

Averaging over all possible directions of the mass quadrupole oscillations we get:

$$W_{\text{em}} = \frac{\omega^4a^2R_1^6}{27c^3}\left(\frac{4}{15}H_y^{(0)2} + \frac{2}{15}H_z^{(0)2}\right).$$

After averaging over all possible orientations of the magnetic field we find:

$$W_{em} = \frac{2}{405} \frac{\omega^4 a^2 R_1^6}{c^3} H^{(0)^2},$$

where $H^{(0)}$ is the magnetic field strength.

Expressing the gravitational wave amplitude a by means of the incident gravitational wave energy flux F_g[6], i. e.:

$$a^2 = \frac{128\pi G F_g}{\omega^2 c^3}, \qquad G = \text{gravitational constant},$$

we get:

$$W_{em} = 1.8 \cdot 10^{-70} \omega^2 R_1^6 F_g H^{(0)^2}. \tag{5}$$

3. *Astrophysical applications*

In the preceding calculations we made the approximation that the field outside the sphere was zero. Actually the external field is proportional to μ/R^3 where μ is the magnetic dipole moment of the sphere. We notice that in Formula (5) a factor $H^2 R_1^6 \sim \mu^2$ appears. We are therefore allowed to expect the order of magnitude of the interaction between gravitational radiation and a true magnetic dipole not to be very different from the one given by (5).

In fact we notice that the greatest part of the interaction takes place essentially in the region of high magnetic field, i. e. inside the sphere, while outside it the strength of the magnetic field decreases very fast according to the law $1/R^3$.

Formula (5) can be rewritten as:

$$W_{em} \cong 1.8 \cdot 10^{-70} \omega^2 F_g \mu^2 \tag{6}$$

In this way we can apply our results to the Earth and to magnetic neutron stars.

Magnetic neutron stars behave essentially as magnetic dipoles and their sizes are smaller than the wavelength of the gravitational wave. In fact Weber's experiment seems to show the existence of gravitational waves with a frequency $\nu \cong 1.6 \cdot 10^3$ Hz caused by the gravitational collapse of massive objects in the Galactic center. Presumably the flux on the Earth per event is $10^6 \div 10^7$ erg/cm$^2 \cdot$ sec [7, 8].

In a preceding paper we proposed a model for the galactic center which besides explaining the emission of gravitational waves, gave also an account for the emission of cosmic and γ rays as well as for the emission of infrared and radio waves [9].

In [9] the possible effects of the interaction of gravitational waves produced in the cluster itself with the magnetic dipoles of the neutron stars were not taken into account. In fact we made the assumption that in the center of the Galaxy is a cluster of neutron stars with radius $\sim 10^{17}$ cm and with 10^{11} stars among which 10^{10} have a magnetic moment $\mu \cong 10^{33}$ erg G^{-1} [10].

Gravitational waves are emitted in the pulsating mode during the collision of two neutron stars [11]. These events occur with a frequency 10 yr^{-1}. The flux of gravitational energy per event is in the cluster $\sim 10^{17} \div 10^{18}$ erg/cm^2 · sec, therefore every magnetic star emits, because of (6), an energy which multiplied by the number of magnetic stars gives for the whole cluster $\sim 10^{31} \div 10^{32}$ erg/sec in electromagnetic waves with a frequency $1.6 \cdot 10^3$ Hz. Such amount of energy, absorbed by the interstellar plasma, is, however, negligible compared with the energy absorbed in the radio frequency and therefore does not disturb the condition assumed for the model.

As to the Earth the emission of electromagnetic waves of frequency $\sim 1.6 \cdot 10^3$ Hz, taking into account a flux of gravitational radiation $10^6 \div 10^7$ erg/cm^2 sec and a $\mu = 8 \cdot 10^{25}$ erg G^{-1} is $\sim 10^{-4} \div 10^{-3}$ erg/sec per pulse. We recall that these figures are meaningless if one considers a situation too far from the dipole approximation ($kR_1 < 1$) i. e. if the uniformly magnetized core of the Earth has a radius greater than the wavelength of the incident gravitational wave.

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