MASSIVE ABELIAN GAUGE BOSONS IN FRONT-FORM HAMILTONIANS*

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It is shown how gauge bosons can be supplied with a mass term using the Higgs mechanism for the purpose of regulating Hamiltonians of Abelian gauge theories in the front form of quantum dynamics.

1. Introduction

Introduction of a mass parameter for gauge bosons that is discussed in this article is motivated by appearance of infrared (IR) divergences in quantum Hamiltonians of the gauge theories, in which gauge bosons are massless. The canonical Hamiltonian of QCD in the front form (FF) of dynamics [1] provides an important example of such divergences [2]. Using the parton model [3] language of the infinite momentum frame (IMF) [4], one can say that each of the quarks and gluons in a hadron carries some fraction x of the hadron momentum and some momentum $k^\perp$ that is spatially orthogonal to the hadron momentum. The Hamiltonian contains functions of x that diverge when $x \to 0$. In the evolution of quark and gluon states, one encounters the sums over intermediate states that involve integrals of the type of $\int \mathrm{d}x/x$ and $\int \mathrm{d}^2 k^\perp/k^{\perp 2}$. These integrals produce IR infinities due to small x and $k^\perp$. Literature on the FF of Hamiltonian dynamics in quantum field theory (QFT) is reviewed in Ref. [5]. Examples of application to the Standard Model are offered in Refs. [6, 7].

The issue is that separate regularizations of small-x and $k^\perp$ regions in phase space introduce the frame dependence that is difficult to remove from the theory. Ultraviolet (UV) divergences in $k^\perp$ complicate the situation because the unknown finite parts of the UV counter-terms interfere with the

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small-x dynamics; they introduce unknown functions of x [2]. Examples of functions of x one may expect in the case of asymptotically free theories are described in Ref. [8]. Reference [8] employs the renormalization group procedure for effective particles (RGPEP) [9]. The regularization of IR singularities due to massless gauge bosons by introduction of a mass parameter for them is discussed here for the purpose of application in the RGPEP.

The complex frame dependence in regularization of FF Hamiltonians may be mitigated by introducing a mass parameter for gauge bosons because a mass does not depend on a frame of reference. However, the gauge boson mass parameter introduced arbitrarily leads to severe complications in the RGPEP calculation of counterterms. Knowing that the Higgs mechanism [10–13] leads to a renormalizable theory in the path-integral formulation of QFT with a finite number of counter-terms, one may take advantage of this mechanism in the RGPEP for FF Hamiltonians. This article concerns the issue of how to introduce the mass as a regularization parameter in the case of Abelian theory.

A comment is in order regarding small-x singularities due to fermions. Since fermions are considered to have non-zero masses, their mass parameters can be directly used for the purpose of regularization.

An additional reason of interest in introducing a mass parameter for gauge bosons is that the limit $x \to 0$ in the FF of Hamiltonian dynamics is known to be related to the vacuum issue in QFT. In the context of renormalization group procedure in QCD, the issue is discussed in Ref. [2]. But it has a long history [15], including the possibility that the so-called vacuum condensates are reducible to universal small-x components of hadrons. This possibility leads to the idea that the so-called vacuum condensates are associated with the hadron states rather than the vacuum state [16–19]. A free massless particle has the FF "energy" $k^- = k^{\perp\,2}/k^+$, where $k^\pm = k^0 \pm k^3$. The limit of small $k^\perp$ and $x = k^+/p^+$, with p^+ denoting the hadron momentum, does not imply any definite value of k^- and all scales of the "energy" k^- are mixed up in the IR dynamics. Once a mass scale κ is introduced, the FF "energy" becomes $k^- = (k^{\perp\,2} + \kappa^2)/k^+$ and the small parton fraction x for fixed hadron p^+ necessarily implies a large k^-. Since it is known that renormalization of singularities at small k^+, related to small x, may shed new light on the otherwise perplexing vacuum issue [2], even the slightest possibility of introducing the mass scale κ as a regulating parameter for gauge bosons, resolving the intricate FF scale mixing and turning the limit of small x into an UV one, deserves to be noted. Serving this purpose, this article is of technical nature.

Section 2 introduces the theory we consider. In addition to fermions and gauge bosons, a complex scalar field is introduced. Its potential has a minimum at a definite value of the field modulus, while the field phase is

arbitrary. A limiting theory, for brevity called *massive*, is identified. Later on, this theory is used to derive the quantum Hamiltonian for massive gauge bosons interacting with fermions. In Section 3, following Soper's work [20], two different choices of gauge are described. One choice shows that the Lagrangian density we consider corresponds to a theory of massive vector bosons coupled to fermions. Another one is suitable for constructing an FF quantum Hamiltonian for vector bosons coupled to fermions. Section 3 provides details of the two gauge choices. The FF Hamiltonian density in gauge $A^+ = 0$ is calculated in Section 4, prior to taking the massive limit. The calculation involves solving constraint equations for fermion and gauge-boson fields. The solution yields dynamics that involves the modulus field. Section 5 employs the massive limit to derive the Hamiltonian for fermions coupled to gauge bosons only, while the modulus field is decoupled. The corresponding quantum Hamiltonian is obtained using the standard FF field quantization procedure. Section 6 explains how the phase field provides the third polarization state for gauge vector bosons, and Section 7 concludes the paper with comments concerning regularization.

2. Lagrangian density

The gauge theory we discuss is introduced in terms of a local Lagrangian density of the familiar structure

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_A + \mathcal{L}_{A\phi} - \mathcal{V}_\phi \,, \tag{1}$$

where

$$\mathcal{L}_\psi = \bar\psi \left[(i\partial_\mu - g A_\mu) \gamma^\mu - m \right] \psi \,, \tag{2}$$

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \tag{3}$$

$$\mathcal{L}_{A\phi} = \left[\left(i\partial^\mu - g' A^\mu \right) \phi \right]^\dagger \left(i\partial_\mu - g' A_\mu \right) \phi \,, \tag{4}$$

$$\mathcal{V}_\phi = -\mu^2 \, \phi^\dagger \phi + \frac{\lambda^2}{2} \left(\phi^\dagger \phi \right)^2 \,. \tag{5}$$

It involves a fermion field ψ, an Abelian vector field A with field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and a complex scalar field ϕ with a real potential $\mathcal{V}_\phi$ in which the constant μ^2 is meant to be positive. The Lagrangian density exhibits gauge symmetry in the sense that it does not change its form when one makes the replacements

$$\psi \rightarrow e^{igf} \psi \,, \tag{6}$$

$$A^\mu \rightarrow A^\mu - \partial^\mu f \,, \tag{7}$$

$$\phi \rightarrow e^{ig'f} \phi \,, \tag{8}$$

where f is a function of points in space-time.

The complex field ϕ does not interact directly with the fermion field, but it does interact with fermions indirectly, through the gauge field A. When the interaction of field ϕ with gauge field A is turned off, *i.e.*, when g' is set to zero, the field ϕ becomes a real self-interacting field that does not interact with any other field. The Yukawa-like coupling $\bar\psi\phi\psi$ is excluded by the requirement of gauge invariance.

2.1. Phase field θ

The field ϕ can be written using its modulus $|\phi| = \varphi/\sqrt{2}$ and phase $g'\theta$ [14]

$$\phi = \varphi\, e^{ig'\theta}/\sqrt{2}\,. \tag{9}$$

The real field φ can be considered constant and denoted as such by v, as if it represented a ground-state, or vacuum expectation value of a field operator in a quantum theory. In the theory we consider, the constant v is just a free parameter. The field φ can deviate from the constant v. Such deviation is denoted here by h, in analogy to the Higgs field in the Standard Model (SM). The potential $\mathcal{V}(\phi)$ is a function of φ alone, $\mathcal{V}(\phi) = \mathcal{V}(\varphi/\sqrt{2})$; it does not depend on the phase field θ.

When $\varphi = v+h$ and $h = 0$, the potential has its minimal value $-\mu^4/(2\lambda^2)$ for $v = \sqrt{2}\,\mu/\lambda$. Using this special value of v for $h \neq 0$, one has

$$\mathcal{V}(\phi) = -\frac{\mu^4}{2\lambda^2} + \frac{1}{2}\left(\sqrt{2}\,\mu\right)^2 h^2 + \frac{\lambda}{\sqrt{2}}\,\mu\, h^3 + \frac{\lambda^2}{8}\,h^4\,, \tag{10}$$

which means that a term linear in h is absent. The Lagrangian density depends on the gradient $\partial^\mu\theta$

$$\mathcal{L}_{A\phi} = \frac{1}{2}\,(\partial^\mu\varphi)^2 + \frac{1}{2}\,g'^2(A^\mu + \partial^\mu\theta)^2\varphi^2\,. \tag{11}$$

2.2. Massive limit

We will consider the gauge theory in the limit of $g' \to 0$, $v \to \infty$ and $g'v = \kappa$ kept constant. To be short, we call this limit the *massive* limit. For the sake of further discussion, the massive limit is specified by assuming that the positive mass parameter μ is fixed, but arbitrary, and the field φ self-interaction coupling constant $\lambda \to 0$. In this limit, the field h can be neglected in comparison to the constant v for as long as the divergences caused by the field h are regulated and negligible in comparison with the corresponding powers of the constant v. Hence, $v + h \to v$, $g'\phi \to \kappa$, and

$$\mathcal{L}_{A\phi} = \frac{1}{2}\,(\partial^\mu h)^2 + \frac{1}{2}\,\kappa^2(A^\mu + \partial^\mu\theta)^2\,, \tag{12}$$

$$\mathcal{V}_\phi = -\frac{\mu^4}{2\lambda^2} + \frac{1}{2}\left(\sqrt{2}\,\mu\right)^2 h^2 \,. \tag{13}$$

The field h becomes free, with mass $\sqrt{2}\,\mu$, and the field $A^\mu + \partial^\mu\theta$ obtains the mass κ. Since the Yukawa-like coupling $\bar{\psi}\phi\psi$ is excluded, the modulus field cannot generate the mass of fermions.

2.3. Gauge symmetry in terms of ψ, A^μ, φ and θ

In summary, the Lagrangian density written in terms of fields ψ, A^μ, φ and $\partial^\mu\theta$ is $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_A + \mathcal{L}_{A\phi} - \mathcal{V}_\phi$, where

$$\mathcal{L}_\psi = \bar{\psi}\left[(i\partial_\mu - gA_\mu)\gamma^\mu - m\right]\psi \,, \tag{14}$$

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,, \tag{15}$$

$$\mathcal{L}_{A\phi} = \frac{1}{2}(\partial^\mu\varphi)^2 + \frac{1}{2}g'^2(A^\mu + \partial^\mu\theta)^2\varphi^2 \,, \tag{16}$$

$$\mathcal{V}_\phi = \mathcal{V}\left(\varphi/\sqrt{2}\right) \,. \tag{17}$$

The gauge transformation that leaves the Lagrangian density unchanged is

$$\psi \rightarrow e^{igf}\psi \,, \tag{18}$$

$$A^\mu \rightarrow A^\mu - \partial^\mu f \,, \tag{19}$$

$$\varphi \rightarrow \varphi \,, \tag{20}$$

$$\theta \rightarrow \theta + f \,. \tag{21}$$

The transformation is realized by substitutions

$$\psi = e^{-igf}\tilde{\psi} \,, \tag{22}$$

$$A^\mu = \tilde{A}^\mu + \partial^\mu f \,, \tag{23}$$

$$\varphi = \tilde{\varphi} \,, \tag{24}$$

$$\theta = \tilde{\theta} - f \,. \tag{25}$$

The Lagrangian density as a function of fields without tilde and as a function of fields with a tilde is the same function. Two different choices for the function f are used in what follows.

3. Two choices of gauge

Following Soper [20], we use two different choices of gauge in order to obtain two results. In the first gauge, the result is that the Lagrangian density we consider corresponds to a theory of massive vector fields coupled to

fermions. In the second gauge, an explicit derivation of quantum Hamiltonian for the theory becomes possible. The difference from [20] is that the mass parameter, which Soper introduces in order to obtain gauge symmetry and which he denotes by κ, is in our gauge theory example obtained as a result of a different gauge symmetry, which includes an additional field h in a way resembling but not identical to the Higgs mechanism in the SM. The field h is not involved in generating fermion masses. Our parameter κ that corresponds to Soper's, appears from substitution $\kappa = g'v$. The additional field h decouples only in the massive limit.

3.1. Gauge choice $f = -\theta$

For $f = -\theta$, we have

$$\psi \ \rightarrow \ \tilde{\psi} = e^{igf}\psi = e^{-ig\theta}\psi \,, \tag{26}$$

$$A^\mu \ \rightarrow \ \tilde{A} = A^\mu - \partial^\mu f = A^\mu + \partial^\mu\theta \,, \tag{27}$$

$$\varphi \ \rightarrow \ \tilde{\varphi} = \varphi \,, \tag{28}$$

$$\theta \ \rightarrow \ \tilde{\theta} = \theta + f = 0 \,. \tag{29}$$

These substitutions are realized by setting

$$\psi \ = \ e^{ig\theta}\tilde{\psi} \,, \tag{30}$$

$$A^\mu \ = \ \tilde{A} - \partial^\mu\theta \,, \tag{31}$$

$$\varphi \ = \ \tilde{\varphi} \,, \tag{32}$$

$$\theta \ = \ \tilde{\theta} + \theta \,, \tag{33}$$

which leads to $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_A + \mathcal{L}_{A\phi} - \mathcal{V}_\phi$ with

$$\mathcal{L}_\psi \ = \ \bar{\tilde{\psi}} \left[\left(i\partial_\mu - g\tilde{A}_\mu \right) \gamma^\mu - m \right] \tilde{\psi} \,, \tag{34}$$

$$\mathcal{L}_A \ = \ -\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \,, \tag{35}$$

$$\mathcal{L}_{A\phi} \ = \ \frac{1}{2}\left(\partial^\mu\tilde{\varphi}\right)^2 + \frac{1}{2}g'^2\tilde{A}^{\mu\,2}\tilde{\varphi}^2 \,, \tag{36}$$

$$\mathcal{V}_\phi \ = \ \mathcal{V}\left(\tilde{\varphi}/\sqrt{2}\right) \,. \tag{37}$$

In the massive limit, $\mathcal{L}_\psi$, $\mathcal{L}_A$ remain the same, $\mathcal{L}_{A\phi}$ becomes

$$\mathcal{L}_{A\phi} \ = \ \frac{1}{2}\left(\partial^\mu\tilde{h}\right)^2 + \frac{1}{2}\kappa^2\tilde{A}^2 \,, \tag{38}$$

and $\mathcal{V}_\phi$ is dominated by the constant $\mu^2/(2\lambda^2)$. Thus, in the massive limit, the Lagrangian density is $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_A + \mathcal{L}_{A\phi} - \mathcal{V}_\phi$, where

$$\mathcal{L}_\psi = \bar{\tilde{\psi}} \left[\left(i\partial_\mu - g\tilde{A}_\mu \right) \gamma^\mu - m \right] \tilde{\psi} \,, \tag{39}$$

$$\mathcal{L}_A = -\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \,, \tag{40}$$

$$\mathcal{L}_{A\phi} = \frac{1}{2}\left(\partial^\mu \tilde{h}\right)^2 + \frac{1}{2}\kappa^2 \tilde{A}^2 \,, \tag{41}$$

$$\mathcal{V}_\phi = -\frac{\mu^2}{2\lambda^2} + \frac{1}{2}\left(\sqrt{2}\,\mu\right)^2 \tilde{h}^2 \,. \tag{42}$$

This Lagrangian density yields the action of a free field $\tilde{h}$ of mass $\sqrt{2}\,\mu$ and, additively, of a massive vector field $\tilde{A}$ coupled to the fermion field $\tilde{\psi}$. Assuming that the gauge symmetry is realized in nature and that photons coupled to charged fermions have a very small mass, corresponding to a very small κ, there also ought to exist in nature a scalar field h with an unknown mass whose value of $\sqrt{2}\,\mu$ is not in any way limited by the theory. A different form of the potential density $\mathcal{V}_\phi$ than in Eq. (5) would lead to a different mass of the field h in the massive limit that is realized through some analog of $\lambda \to 0$.

Current upper limit on the photon mass [21] is 10^{-18} eV/c^2, which is extremely low. There are no data suggesting that the field h exists, but the astrophysical data stimulate searches for the dark matter and other exotic particles [21].

3.2. Gauge choice $\tilde{A}^+ = 0$

The original Lagrangian density allows for simultaneous alteration of the fermion field ψ and phase field θ so that the vector field A can be replaced by a similar field $\tilde{A}$ whose component $\tilde{A}^+ = 0$. The $+$-component of a four-vector is defined in the same way as for all tensors, $\pm = 0 \pm 3$. For example, a position four-vector x has components (x^-, x^+, x^1, x^2) and the gradient components are $(\partial^-, \partial^+, \partial^\perp)$. Components 1 and 2 are collectively denoted by $\perp$.

The choice of $\tilde{A}^+ = 0$ stems from the form of dynamics that we use to construct the Hamiltonian [1]. We use the FF, instead of the commonly used form that Dirac called the instant form (IF) [1]. Setting $\tilde{A}^+ = 0$ is useful because it leads to simple and soluble constraint equations. Also, it is invariant with respect to seven kinematic Poincaré transformations that form the FF symmetry group in the Minkowsky space-time. The IF has only six kinematic symmetries [5].

Since the couplings of A to ψ and ϕ have different strengths, given by different dimensionless coupling constants g and g', different changes of phase are needed for fields ψ and ϕ to obtain $\tilde{A}^+ = 0$. One can introduce the function f that satisfies the equation

$$\partial^+ f \,=\, A^+ \,. \tag{43}$$

For example,

$$f(x) \,=\, \frac{1}{4} \left(\int\limits_{-\infty}^{x^-} - \int\limits_{x^-}^{\infty} \right) \mathrm{d}y^- A^+ \left(x^+, y^-, x^\perp \right) \,. \tag{44}$$

An arbitrary function that does not depend on x^- could be added to this definition. The gradient of this f is

$$\partial^- f(x) \,=\, \frac{1}{4} \left(\int\limits_{-\infty}^{x^-} - \int\limits_{x^-}^{\infty} \right) \mathrm{d}y^- \partial^- A^+ \left(x^+, y^-, x^\perp \right) \,, \tag{45}$$

$$\partial^+ f(x) \,=\, A^+ \left(x^+, x^-, x^\perp \right) \,, \tag{46}$$

$$\partial^\perp f(x) \,=\, \frac{1}{4} \left(\int\limits_{-\infty}^{x^-} - \int\limits_{x^-}^{\infty} \right) \mathrm{d}y^- \partial^\perp A^+ \left(x^+, y^-, x^\perp \right) \,. \tag{47}$$

The gauge transformation is realized by substitutions

$$\psi \,=\, e^{-igf} \tilde{\psi} \,, \tag{48}$$
$$A^\mu \,=\, \tilde{A}^\mu + \partial^\mu f \,, \tag{49}$$
$$\varphi \,=\, \tilde{\varphi} \,, \tag{50}$$
$$\theta \,=\, \tilde{\theta} - f \,, \tag{51}$$

which produce the same Lagrangian density $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_A + \mathcal{L}_{A\phi} - \mathcal{V}_\phi$ as the one in Eqs. (14), (15), (16) and (17) except that the fields ψ, A, φ and θ are replaced by fields $\tilde{\psi}$, $\tilde{A}$, $\tilde{\varphi}$ and $\tilde{\theta}$, i.e.,

$$\mathcal{L}_\psi \,=\, \bar{\tilde{\psi}} \left[\left(i\partial_\mu - g\tilde{A}_\mu \right) \gamma^\mu - m \right] \tilde{\psi} \,, \tag{52}$$

$$\mathcal{L}_A \,=\, -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \,, \tag{53}$$

$$\mathcal{L}_{A\phi} \,=\, \frac{1}{2} \left(\partial^\mu \tilde{\varphi} \right)^2 + \frac{1}{2} g'^2 \left(\tilde{A}^\mu + \partial^\mu \tilde{\theta} \right)^2 \tilde{\varphi}^2 \,, \tag{54}$$

$$\mathcal{V}_\phi \,=\, \mathcal{V} \left[\tilde{\varphi}/\sqrt{2} \right] \,, \tag{55}$$

and $\tilde{A}^+ = 0$. In the massive limit,

$$\mathcal{L}_\psi = \bar{\tilde{\psi}} \left[\left(i\partial_\mu - g\tilde{A}_\mu \right) \gamma^\mu - m \right] \tilde{\psi} \,, \tag{56}$$

$$\mathcal{L}_A = -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \,, \tag{57}$$

$$\mathcal{L}_{A\phi} = \frac{1}{2} \left(\partial^\mu \tilde{h} \right)^2 + \frac{1}{2} \kappa^2 \left(\tilde{A}^\mu + \partial^\mu \tilde{\theta} \right)^2 \,, \tag{58}$$

$$\mathcal{V}_\phi = -\frac{\mu^2}{2\lambda^2} + \frac{1}{2} \left(\sqrt{2}\,\mu \right)^2 \tilde{h}^2 \,, \tag{59}$$

which look the same as Eqs. (39), (40), (41) and (42). However, the component $\tilde{A}^+$ of the gauge field is zero. Instead, the gradient of field $\tilde{\theta}$ is present.

4. Calculation of the FF Hamiltonian

The Lagrangian density $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_A + \mathcal{L}_{A\phi} - \mathcal{V}_\phi$ of Eqs. (2), (3), (4) and (5) is written in Secs. 3.1 and 3.2 in two formally equivalent versions that differ by the choice of gauge. In the massive limit, the version of Sec. 3.1 coincides with a theory of a fermion field coupled in a mininmal way to a massive vector field, while the version of Sec. 3.2 coincides with a theory of a fermion field coupled to a massive transverse vector field and this vector field is coupled to a gradient of a massive scalar field.

These two formally equivalent massive limit versions of the gauge theory correspond to the theory developed by Soper [20]. In this section, an FF Hamiltonian that corresponds to the Lagrangian $\mathcal{L}$ is constructed using the gauge $\tilde{A}^+ = 0$ of Sec. 3.2 before the massive limit is taken. The resulting FF Hamiltonian formally reduces in the massive limit to the Soper FF Hamiltonian.

4.1. Equations of motion

The Lagrangian density of Eqs. (2) to (5) implies, through the principle of minimal action, the Euler–Lagrange (EL) equations of motion

$$\left[\left(i\partial_\mu - gA_\mu \right) \gamma^\mu - m \right] \psi = 0 \,, \tag{60}$$

$$-\partial_\alpha \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha \right) = -g\bar{\psi}\gamma^\beta\psi + g'^2 \, \varphi^2 \left(A^\beta + \partial^\beta \theta \right) \,, \tag{61}$$

$$\partial_\mu \partial^\mu \varphi = g'^2 \, \varphi \left(A^\beta + \partial^\beta \theta \right)^2 - \frac{\partial \mathcal{V} \left(\varphi/\sqrt{2} \right)}{\partial \varphi} \,, \tag{62}$$

$$\partial_\mu g'^2 (A^\mu + \partial^\mu \theta) \varphi^2 = 0 \,. \tag{63}$$

The last equation is necessarily satisfied if the first two are. In the massive limit, these equations become

$$[(i\partial_\mu - gA_\mu)\gamma^\mu - m]\psi = 0\,, \tag{64}$$

$$-\partial_\alpha \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) = -g\bar\psi\gamma^\beta\psi + \kappa^2 \left(A^\beta + \partial^\beta\theta\right)\,, \tag{65}$$

$$\partial_\mu\partial^\mu h = -\frac{\partial\mathcal{V}\left(h/\sqrt{2}\right)}{\partial h}\,, \tag{66}$$

$$\kappa^2\,\partial_\mu(A^\mu + \partial^\mu\theta) = 0\,. \tag{67}$$

Assuming non-zero mass κ and introducing the field

$$B = -\kappa\,\theta\,, \tag{68}$$

one obtains the massive EL equations in the form of

$$[(i\partial_\mu - gA_\mu)\gamma^\mu - m]\psi = 0\,, \tag{69}$$

$$\Box A^\beta - \partial^\beta\partial_\alpha A^\alpha = g\bar\psi\gamma^\beta\psi - \kappa^2\left(A^\beta - \kappa^{-1}\partial^\beta B\right)\,, \tag{70}$$

$$\kappa^2\,\partial_\mu\left(A^\mu - \kappa^{-1}\partial^\mu B\right) = 0\,, \tag{71}$$

$$\partial_\mu\partial^\mu h = -\frac{\partial\mathcal{V}\left(h/\sqrt{2}\right)}{\partial h}\,. \tag{72}$$

Note that $\Box B = \kappa\partial_\mu A^\mu$ and the field h is decoupled. Equations (69), (70) and (71) for fields ψ, A and B coincide with Eqs. (2), (3) and (4) of Soper [20]. Equation (72) is absent in Soper's theory. The massive limits described in Secs. 3.1 and 3.2 lead to the EL equations that coincide with Soper's equations in gauges $B = 0$ and $A^+ = 0$, respectively. Our construction of the FF Hamiltonian for massive gauge bosons is carried out in the gauge $\tilde{A}^+ = 0$.

Prior to taking the massive limit, the EL equations in terms of the field B read

$$[(i\partial_\mu - gA_\mu)\gamma^\mu - m]\psi = 0\,, \tag{73}$$

$$\Box A^\beta - \partial^\beta\partial_\alpha A^\alpha = g\bar\psi\gamma^\beta\psi - g'^2\,\varphi^2\left(A^\beta - \kappa^{-1}\partial^\beta B\right)\,, \tag{74}$$

$$\Box\varphi = g'^2\,\varphi\left(A^\beta - \kappa^{-1}\partial^\beta B\right)^2 - \frac{\partial\mathcal{V}\left(\varphi/\sqrt{2}\right)}{\partial\varphi}\,, \tag{75}$$

$$\partial_\mu g'^2\varphi^2\left(A^\mu - \kappa^{-1}\partial^\mu B\right) = 0\,. \tag{76}$$

Again, the last equation must be satisfied if the first two are.

<center>*4.2. Constraint equations in gauge $\tilde{A}^+ = 0$*</center>

In the FF of dynamics, the constraint equations are those EL equations that do not involve differentiation with respect to x^+, *i.e.*, $\partial^- = 2\partial/\partial x^+$. We abbreviate our notation from $\tilde{\psi}$, $\tilde{A}$, $\tilde{\varphi}$ and $\tilde{\theta}$ to ψ, A, φ and θ.

4.2.1. Constraint equations for the fermion field

The 4×4 projection matrices $\Lambda_{\pm} = \frac{1}{2}\gamma^0\gamma^{\pm} = \frac{1}{2}(1 \pm \alpha^3)$, which have the properties $\Lambda_{\pm}\alpha^{\perp} = \alpha^{\perp}\Lambda_{\mp}$ and $\Lambda_{\pm}\beta = \beta\Lambda_{\mp}$, are used to write the fermion field as $\psi = \psi_+ + \psi_-$, where $\psi_{\pm} = \Lambda_{\pm}\psi$. The fermion EL Eq. (60) takes the form of

$$\left[\left(i\partial^- - gA^-\right)\Lambda_+ + i\partial^+\Lambda_- - \left(i\partial^\perp - gA^\perp\right)\alpha^\perp - m\beta \right]\psi = 0, \quad (77)$$

which consists of two coupled equations for ψ_+ and ψ_-

$$\left(i\partial^- - gA^-\right)\psi_+ - \left[\left(i\partial^\perp - gA^\perp\right)\alpha^\perp + m\beta\right]\psi_- = 0, \quad (78)$$

$$i\partial^+\psi_- - \left[\left(i\partial^\perp - gA^\perp\right)\alpha^\perp + m\beta\right]\psi_+ = 0. \quad (79)$$

The second equation yields

$$\psi_- = \frac{1}{i\partial^+}\left[\left(i\partial^\perp - gA^\perp\right)\alpha^\perp + m\beta\right]\psi_+, \quad (80)$$

so that the first one describes the evolution of ψ_+ in x^+,

$$i\partial^-\psi_+ = \left\{ \left[\left(i\partial^\perp - gA^\perp\right)\alpha^\perp + m\beta\right] \right.$$
$$\left. \times \frac{1}{i\partial^+}\left[\left(i\partial^\perp - gA^\perp\right)\alpha^\perp + m\beta\right] + gA^- \right\}\psi_+, \quad (81)$$

while ψ_- is given by the constraint Eq. (80).

4.2.2. Constraint equation for the boson field

The EL Eq. (61) for the field A is

$$\Box A^\beta - \partial^\beta\partial_\alpha A^\alpha = g\bar{\psi}\gamma^\beta\psi - g'^2\,\varphi^2\left(A^\beta + \partial^\beta\theta\right), \quad (82)$$

which in terms of the field $B = -\kappa\theta$ reads

$$\Box A^\beta - \partial^\beta\partial_\alpha A^\alpha = g\bar{\psi}\gamma^\beta\psi - g'^2\,\varphi^2\left(A^\beta - \kappa^{-1}\partial^\beta B\right). \quad (83)$$

Setting $\beta = +$, one gets in gauge $A^+ = 0$ that

$$\partial^+\partial_\alpha A^\alpha \;=\; -g\bar\psi\gamma^+\psi - g'^2\,\varphi^2\,\kappa^{-1}\partial^+ B\,. \qquad (84)$$

This equation contains no derivatives with respect to x^+ and it is a constraint. One gets

$$\partial_\alpha A^\alpha \;=\; -\frac{1}{\partial^+}\left(g\bar\psi\gamma^+\psi + g'^2\,\varphi^2\,\kappa^{-1}\partial^+ B\right)\,, \qquad (85)$$

and, as a result of $\partial_\alpha A^\alpha = \partial^+ A^-/2 - \partial^\perp A^\perp$,

$$A^- \;=\; \frac{2}{\partial^+}\partial^\perp A^\perp - \frac{2}{\partial^{+2}}\left(g\bar\psi\gamma^+\psi + g'^2\,\varphi^2\,\kappa^{-1}\partial^+ B\right)\,. \qquad (86)$$

This constraint makes the evolution of ψ_+ in Eq. (81) nonlinear. It also shows that the $(1/\partial^+)B$ couples to fermions as A^- does.

4.3. Lagrangian density in gauge $A^+ = 0$

By inspection, one finds that the Lagrangian density $\mathcal{L}$ in the gauge $A^+ = 0$ is linear in derivatives of fields with respect to x^+. The fermion Lagrange density is

$$\mathcal{L}_\psi \;=\; \psi_+^\dagger(i\partial^- - gA^-)\psi_+ + \psi_-^\dagger i\partial^+\psi_- - \psi_+^\dagger\left[\left(i\partial^\perp - gA^\perp\right)\alpha^\perp + m\beta\right]\psi_-$$
$$-\psi_-^\dagger\left[\left(i\partial^\perp - gA^\perp\right)\alpha^\perp + m\beta\right]\psi_+\,. \qquad (87)$$

The gauge field density is

$$\frac{-1}{4}F_{\mu\nu}F^{\mu\nu} \;=\; \frac{-1}{4}\left[-\frac{1}{2}\left(\partial^+ A^-\right)^2 + 2\left(\partial^k A^- - \partial^- A^k\right)\partial^+ A^k\right.$$
$$\left. + 2\partial^l A^k\partial^l A^k - 2\partial^l A^k\partial^k A^l\right]\,. \qquad (88)$$

The densities for fields φ and $\partial^\mu B$ are

$$\mathcal{L}_{A\phi} \;=\; \frac{1}{2}\left(\partial^+\varphi\,\partial^-\varphi - \partial^\perp\varphi\,\partial^\perp\varphi\right)$$
$$+ \frac{1}{2}g'^2\left[-\kappa^{-1}\partial^+ B\left(A^- - \kappa^{-1}\partial^- B\right) - \left(A^\perp - \kappa^{-1}\partial^\perp B\right)^2\right]\varphi^2\,, \quad (89)$$
$$\mathcal{V}_\phi \;=\; \mathcal{V}\left(\varphi/\sqrt{2}\right)\,. \qquad (90)$$

4.4. Derivation of the FF Hamiltonian density

The energy-momentum tensor, generally given by

$$T^{\mu\nu} = \sum_i \frac{\partial \mathcal{L}}{\partial \partial_\mu f_i} \, \partial^\nu f_i - g^{\mu\nu} \mathcal{L} \,, \tag{91}$$

leads to the Hamiltonian P^-

$$P^- = \frac{1}{2} \int \mathrm{d}^2 x^\perp \mathrm{d} x^- \, T^{+-} \,. \tag{92}$$

Therefore,

$$P^- = \int \mathrm{d}^2 x^\perp \mathrm{d} x^- \left(\sum_i \frac{\partial \mathcal{L}}{\partial \partial^- f_i} \, \partial^- f_i - \mathcal{L} \right) \,. \tag{93}$$

For Lagrangian densities that are linear in partial derivatives ∂^- of fields, the Hamiltonian densities $\mathcal{H}$ are simply given by the formula $\mathcal{H} = -\mathcal{L}(\partial^- \to 0)$. This simplification occurs for all Lagrangian densities that are quadratic in the IF time derivatives and obey the principles of special relativity. In our case,

$$
\begin{aligned}
\mathcal{H} &= g\psi_+^\dagger A^- \psi_+ - \psi_-^\dagger i \partial^+ \psi_- \\
&+ \psi_+^\dagger \left[\left(i\partial^\perp - gA^\perp \right) \alpha^\perp + m\beta \right] \psi_- + \psi_-^\dagger \left[\left(i\partial^\perp - gA^\perp \right) \alpha^\perp + m\beta \right] \psi_+ \\
&+ \frac{1}{4} \left[-\frac{1}{2} (\partial^+ A^-)^2 + 2\partial^k A^- \partial^+ A^k + 2\partial^l A^k \partial^l A^k - 2\partial^l A^k \partial^k A^l \right] \\
&+ \frac{1}{2} \left(\partial^\perp \varphi \right)^2 + \frac{1}{2} g'^2 \varphi^2 \left[\kappa^{-1}\partial^+ B \, A^- + \left(A^\perp - \kappa^{-1}\partial^\perp B \right)^2 \right] + \mathcal{V}\left(\varphi/\sqrt{2} \right) \,,
\end{aligned}
\tag{94}
$$

where for the optimal choice of v in $\varphi = v + h$,

$$\mathcal{V}\left(\varphi/\sqrt{2} \right) = -\frac{\mu^4}{2\lambda^2} + \frac{1}{2} \left(\sqrt{2}\,\mu \right)^2 h^2 + h^2 \left[\frac{\lambda}{\sqrt{2}} \, \mu \, h + \frac{\lambda^2}{8} \, h^2 \right] \,. \tag{95}$$

Constraints to include are

$$\psi_- = \frac{1}{i\partial^+} \left[\left(i\partial^\perp - gA^\perp \right) \alpha^\perp + m\beta \right] \psi_+ \,, \tag{96}$$

$$A^- = \frac{2}{\partial^+}\partial^\perp A^\perp - \frac{2}{\partial^{+2}} \left[g\bar\psi\gamma^+\psi + g'^2 \, \varphi^2 \, \kappa^{-1}\partial^+ B \right] \,. \tag{97}$$

The field ψ_- is most easy to eliminate. Looking at the constraint Eq. (80), one sees that the terms $-\psi_-^\dagger i\partial^+\psi_-$ and $\psi_-^\dagger \left[(i\partial^\perp - gA^\perp)\alpha^\perp + m\beta\right]\psi_+$ cancel each other and one is left with

$$\psi_+^\dagger \left[\left(i\partial^\perp - gA^\perp\right)\alpha^\perp - m\beta\right]\frac{1}{i\partial^+}\left[\left(i\partial^\perp - gA^\perp\right)\alpha^\perp + m\beta\right]\psi_+. \quad (98)$$

Turning to the vector field, the four terms that involve A^-,

$$\frac{1}{2}g\bar\psi\gamma^+\psi A^- + \frac{1}{4}\left[-\frac{1}{2}\left(\partial^+ A^-\right)^2 + 2\partial^k A^- \partial^+ A^k\right] + \frac{1}{2}g'^2\varphi^2 A^- \kappa^{-1}\partial^+ B, \quad (99)$$

can be shown, using partial integration over the front, to be equivalent to

$$\frac{1}{8}\left(\partial^+ A^-\right)^2. \quad (100)$$

The full Hamiltonian is then

$$\begin{aligned}
\mathcal{H} = {} & \frac{1}{2}\left(\partial^+ A^-/2\right)^2 \\
& + \psi_+^\dagger \left[\left(i\partial^\perp - gA^\perp\right)\alpha^\perp + m\beta\right]\frac{1}{i\partial^+}\left[\left(i\partial^\perp - gA^\perp\right)\alpha^\perp + m\beta\right]\psi_+ \\
& + \frac{1}{2}\left(\partial^l A^k \partial^l A^k - \partial^l A^k \partial^k A^l\right) \\
& + \frac{1}{2}\left(\partial^\perp \varphi\right)^2 + \frac{1}{2}g'^2\varphi^2\left(A^\perp - \kappa^{-1}\partial^\perp B\right)^2 + \mathcal{V}\left(\varphi/\sqrt{2}\right), \quad (101)
\end{aligned}$$

where

$$A^- = \frac{2}{\partial^+}\partial^\perp A^\perp - \frac{2}{\partial^{+2}}\left[g\bar\psi\gamma^+\psi + g'^2\varphi^2\kappa^{-1}\partial^+ B\right]. \quad (102)$$

The FF Hamiltonian density of Eq. (101) is used below to derive the Hamiltonian for massive gauge bosons coupled to fermions, taking advantage of the massive limit.

5. Hamiltonian in the massive limit

In the massive limit, which is defined in Sec. 2.2, the coupling constant $g' \to 0$ and the modulus parameter $v \to \infty$ with the product $g'v = \kappa$ kept constant. One has

$$g'\varphi = g'v(1 + h/v) = \kappa(1 + h/v) \to \kappa. \quad (103)$$

Recall that in terms of the Lagrangian density of Eq. (5), the massive limit is set by demanding that the coupling constant $\lambda \to 0$ for an arbitrary but fixed value of the mass parameter μ. Thus, in the massive limit defined by both g' and λ tending to zero, the Hamiltonian density of Eq. (101) becomes

$$
\begin{aligned}
\mathcal{H} = {}& \frac{1}{2} \left(\partial^+ A^-/2 \right)^2 \\
& + \psi_+^\dagger \left[\left(i\partial^\perp - gA^\perp \right) \alpha^\perp + m\beta \right] \frac{1}{i\partial^+} \left[\left(i\partial^\perp - gA^\perp \right) \alpha^\perp + m\beta \right] \psi_+ \\
& + \frac{1}{2} \left(\partial^l A^k \partial^l A^k - \partial^l A^k \partial^k A^l \right) \\
& + \frac{1}{2} \kappa^2 \left(A^\perp - \kappa^{-1} \partial^\perp B \right)^2 + \frac{1}{2} h \left(-\partial^{\perp 2} + 2\mu^2 \right) h - \frac{\mu^4}{2\lambda^2} , \qquad (104)
\end{aligned}
$$

where

$$
A^- = \frac{2}{\partial^+} \partial^\perp A^\perp - \frac{2}{\partial^{+2}} \left[g\bar{\psi}\gamma^+ \psi + \kappa \partial^+ B \right] . \qquad (105)
$$

Using terms equivalent to $\partial^l A^k \partial^l A^k - \partial^l A^k \partial^k A^l$ through integration by parts, one obtains

$$
\begin{aligned}
\mathcal{H} = {}& 2g^2 \psi_+^\dagger \psi_+ \frac{1}{(i\partial^+)^2} \psi_+^\dagger \psi_+ + 2g\psi_+^\dagger \psi_+ \frac{1}{i\partial^+} \left(i\partial^\perp A^\perp - i\kappa B \right) \\
& + 2\psi_+^\dagger \left[\left(i\partial^\perp - gA^\perp \right) \alpha^\perp + m\beta \right] \frac{1}{2i\partial^+} \left[\left(i\partial^\perp - gA^\perp \right) \alpha^\perp + m\beta \right] \psi_+ \\
& + \frac{1}{2} A^\perp \left(-\partial^{\perp 2} + \kappa^2 \right) A^\perp + \frac{1}{2} B \left(-\partial^{\perp 2} + \kappa^2 \right) B \\
& + \frac{1}{2} h \left(-\partial^{\perp 2} + 2\mu^2 \right) h - \frac{\mu^4}{2\lambda^2} . \qquad (106)
\end{aligned}
$$

This Hamiltonian density differs from Soper's in his Eq. (23) by the appearance of the free field h with mass $\sqrt{2}\,\mu$. It is thus demonstrated that the concept of Abelian gauge symmetry understood as invariance under change of phase of the matter fields, which here means fermions and scalar bosons, leads in the massive limit to his result, but with an addition of a free field of arbitrary mass.

5.1. Quantization

The FF Hamiltonian of Eq. (93), $P^- = \int d^2 x^\perp dx^- \, \mathcal{H}$, defined in terms of the density $\mathcal{H}$ of Eq. (106), is turned into a quantum Hamiltonian operator by a nowadays standard quantization procedure [22, 23]. The procedure amounts to replacement of the classical fields by field operators. The field operators are constructed by imposing the commutation relations among their

spatial Fourier components on the front defined by the condition $x^+ = 0$, whereby the Fourier coefficients acquire the properties of creation and annihilation operators. The space of states in which the resulting Hamiltonian acts is constructed by acting with products of the creation operators on the vacuum state. The vacuum state is annihilated by all annihilation operators in the theory. In order to produce formulas for the quantum fields, it is useful to introduce the concept of free fields at $x^+ = 0$ [6]. The option of doing so only for the fields at $x^+ = 0$, *i.e.*, without considering their canonically conjugated momenta, is unique to the FF of Hamiltonian dynamics.

The fermion constraint Eq. (80) can be written as [6]

$$\psi_- = \psi_{f-} - \frac{g}{i\partial^+} A^\perp \alpha^\perp \psi_+ \,, \tag{107}$$

where the "free" part is defined by

$$\psi_{f-} = \frac{1}{i\partial^+} \left(i\partial^\perp \alpha^\perp + m\beta \right) \psi_+ \,. \tag{108}$$

One then introduces the "free" fermion field ψ_f by writing

$$\psi_f = \psi_+ + \psi_{f-} \,. \tag{109}$$

The vector boson field constraint Eq. (86) can be written as

$$A^- = A_f^- - \frac{2}{\partial^{+2}} \left(g\bar{\psi}\gamma^+\psi + g'^2 \, \varphi^2 \, \kappa^{-1}\partial^+ B \right) \,, \tag{110}$$

where the "free" part is

$$A_f^- = \frac{2}{\partial^+}\partial^\perp A^\perp \,. \tag{111}$$

The "free" vector field A_f^μ is hence introduced by writing

$$A_f^\perp = A^\perp \,, \tag{112}$$

$$A_f^+ = 0 \,, \tag{113}$$

$$A_f^- = \frac{2}{\partial^+}\partial^\perp A^\perp \,. \tag{114}$$

The quantum theory is introduced by replacing the fields ψ_f, A_f, B and h by the corresponding quantum field operators

$$\hat{\psi}_f = \sum_{\sigma=1}^{2} \int [p] \left[u_{p\sigma}\hat{b}_{p\sigma}e^{-ipx} + v_{p\sigma}\hat{d}_{p\sigma}^\dagger e^{ipx} \right]_{x^+=0} \,, \tag{115}$$

$$\hat{A}_f^\mu = \sum_{\sigma=1}^{2} \int [p] \left[\varepsilon_{p\sigma}^\mu \hat{a}_{p\sigma} e^{-ipx} + \varepsilon_{p\sigma}^{\mu*} \hat{a}_{p\sigma}^\dagger e^{ipx} \right]_{x^+=0} , \qquad (116)$$

$$\hat{B} = \int [p] \left[-i\hat{a}_{p3} e^{-ipx} + i\hat{a}_{p3}^\dagger e^{ipx} \right]_{x^+=0} , \qquad (117)$$

$$\hat{h} = \int [p] \left[\hat{a}_{ph} e^{-ipx} + \hat{a}_{ph}^\dagger e^{ipx} \right]_{x^+=0} , \qquad (118)$$

where $[p] = \mathrm{d}p^+ \theta(p^+) \mathrm{d}^2 p^\perp / [2p^+ (2\pi)^3]$, $u_{p\sigma}$ and $v_{p\sigma}$ are the Dirac spinors, $\varepsilon_{p\sigma}$ are polarization four-vectors, σ and λ label states of fermions and gauge bosons with spin projections $\pm\frac{1}{2}$ or ± 1 on the third axis, respectively, and the creation and annihilation operators, denoted by b, d and a, obey commutation or, in the case of fermions, anti-commutation rules of the form of

$$\left[\hat{a}_{p\lambda}, \hat{a}_{q\sigma}^\dagger \right] = 2p^+ (2\pi)^3 \delta \left(p^+ - q^+ \right) \delta^2 \left(p^\perp - q^\perp \right) \delta_{\lambda\sigma} , \qquad (119)$$

with other commutators or anti-commutators equal zero. In other words, the Hamiltonian density $\mathcal{H}$ of Eq. (106) is changed into a quantum Hamiltonian density by the substitution

$$\mathcal{H} = \mathcal{H}(\psi_f, A_f, B, h) \rightarrow \hat{\mathcal{H}} = \mathcal{H}\left(\hat{\psi}_f, \hat{A}_f, \hat{B}, \hat{h} \right) , \qquad (120)$$

and the quantum Hamiltonian operator is obtained through

$$P^- = \int \mathrm{d}^2 x^\perp \mathrm{d}x^- \, \mathcal{H} \rightarrow \hat{P}^- = \int \mathrm{d}^2 x^\perp \mathrm{d}x^- \, \hat{\mathcal{H}} . \qquad (121)$$

This substitution is complemented by the normal ordering, whereby all creation operators are moved to the left of all annihilation operators and the terms that result from commuting the operators are dropped. To simplify notation for the quantum theory, the operator symbol $\hat{\ }$ is omitted in further formulas. In the quantum FF Hamiltonian density written in the form of

$$\mathcal{H} = \frac{1}{2} \bar{\psi}_f \gamma^+ \frac{-\partial^{\perp 2} + m^2}{i\partial^+} \psi_f + \frac{1}{2} A^\perp \left(-\partial^{\perp 2} + \kappa^2 \right) A^\perp + \frac{1}{2} B \left(-\partial^{\perp 2} + \kappa^2 \right) B$$

$$+ g\bar{\psi}_f \slashed{A}_f \psi_f + g\bar{\psi}_f \gamma^+ \psi_f \frac{1}{i\partial^+} (-i\kappa B)$$

$$+ \frac{1}{2} g^2 \bar{\psi}_f \gamma^+ \psi_f \frac{1}{(i\partial^+)^2} \bar{\psi}_f \gamma^+ \psi_f + g^2 \bar{\psi}_f \slashed{A}_f \frac{\gamma^+}{2i\partial^+} \slashed{A}_f \psi_f$$

$$+ \frac{1}{2} h \left(-\partial^{\perp 2} + 2\mu^2 \right) h - \frac{\mu^4}{2\lambda^2} , \qquad (122)$$

the first line describes the free FF energies of: fermions through the field ψ_f; transverse gauge bosons of mass κ through the field A_f; and gauge

bosons of mass κ carrying the third state of polarization through the field B. The contribution of field B to polarization of massive gauge bosons is explained in the next section. The second line describes the couplings of fermions to the transverse bosons and longitudinal bosons. The third line provides the interactions that result from solving constraints. The first term is the FF analog of the IF Coulomb potential, with the inverse of $\partial^{+\,2}$ being the analog of inverse of Laplacian in the IF Gauss law. The second term describes the interaction due to the instantaneous fermion propagation down the front third axis. The fourth line describes the FF energy of free quanta of field h, with mass $\sqrt{2}\,\mu$. Formal discussion of the quantum theory with Hamiltonian defined using Eq. (122) can now be pursued along the lines indicated in Refs. [20, 22, 23] without change, since the field h is decoupled. Note, however, that the regularization that involves a mass parameter for gauge bosons alters the Bloch and Nordsieck mean-field approximation of Refs. [24, 25], because the states of modes with infinitesimally small k^+ that could be approximated using the mean field are blocked by the mass regularization parameter from being copiously produced. One has to go to the limit of $\kappa \to 0$ to validate the mean field approximation.

6. Polarization of massive gauge bosons

The role of massive gauge boson field $\tilde{B}$ (we use the tilde notation of Sec. 3) is to provide dynamical effects of the third polarization state that a massive vector field can have besides the two transverse polarization states described by $\tilde{A}^\perp$ in gauge $\tilde{A}^+ = 0$. One can see this by proceeding in a way analogous to Ref. [20], except that in the case of Lagrangian density of Eq. (1), one works in the massive limit. In that limit, the constraint Eq. (86) for $\tilde{A}^-$,

$$\tilde{A}^- \;=\; \frac{2}{\partial^+}\partial^\perp \tilde{A}^\perp - \frac{2}{\partial^{+\,2}}\left(g\bar{\tilde{\psi}}\gamma^+\tilde{\psi} + g'^2\;\tilde{\varphi}^2\;\kappa^{-1}\partial^+\tilde{B} \right),\qquad (123)$$

becomes

$$\tilde{A}^- \;=\; \frac{2}{\partial^+}\partial^\perp \tilde{A}^\perp - \frac{1}{\partial^+}2\kappa\tilde{B} - \frac{2}{\partial^{+\,2}}g\bar{\tilde{\psi}}\gamma^+\tilde{\psi}.\qquad (124)$$

When the interaction with fermions is turned off, $g \to 0$,

$$\tilde{A}^- \;=\; \frac{2}{\partial^+}\partial^\perp \tilde{A}^\perp - \frac{1}{\partial^+}2\kappa\tilde{B}.\qquad (125)$$

To see the third polarization that corresponds to $\tilde{B}$, we change the gauge from $\tilde{A}^+ = 0$ to the one with $B = 0$, described in Sec. 3.1. The change is

accomplished using Eqs. (22), (23), (24), (25), which in terms of $\tilde{B} = -\kappa\tilde{\theta}$ read

$$\tilde{\psi} = e^{igf}\psi\,, \tag{126}$$

$$\tilde{A}^\mu = A^\mu - \partial^\mu f\,, \tag{127}$$

$$\tilde{\varphi} = \varphi\,, \tag{128}$$

$$\tilde{B} = B - \kappa f\,. \tag{129}$$

Demanding $B = 0$, one obtains $f = -\kappa^{-1}\tilde{B}$ and

$$A^\mu = \tilde{A}^\mu - \kappa^{-1}\partial^\mu\tilde{B}\,. \tag{130}$$

This can be written as

$$A^\mu = \tilde{A}^\mu + \kappa^{-1}i\partial^\mu i\tilde{B}\,. \tag{131}$$

Setting $\tilde{A}^\perp = 0$ and using Eqs. (125) and (117) for free $\tilde{B}$, one obtains from Eq. (131) that

$$A^\perp = \int [p]\,\frac{p^\perp}{\kappa}\left[a_{p3}e^{-ipx} + a_{p3}^\dagger e^{ipx}\right]\,, \tag{132}$$

$$A^+ = \int [p]\,\frac{p^+}{\kappa}\left[a_{p3}e^{-ipx} + a_{p3}^\dagger e^{ipx}\right]\,, \tag{133}$$

$$A^- = \int [p]\left(\frac{-2\kappa}{p^+} + \frac{p^{\perp\,2} + \kappa^2}{\kappa p^+}\right)\left[a_{p3}e^{-ipx} + a_{p3}^\dagger e^{ipx}\right]\,. \tag{134}$$

These components together form the field

$$A^\mu = \int [p]\,\varepsilon_{p3}^\mu\left[a_{p3}e^{-ipx} + a_{p3}^\dagger e^{ipx}\right]\,, \tag{135}$$

where the real polarization four-vector has components

$$\varepsilon_{p3} = \left(\varepsilon_{p3}^- = \frac{p^{\perp\,2} - \kappa^2}{\kappa p^+},\ \varepsilon_{p3}^+ = \frac{p^+}{\kappa},\ \varepsilon_{p3}^\perp = \frac{p^\perp}{\kappa}\right)$$

$$= \frac{p}{\kappa} - \eta\frac{\kappa}{p^+}\,, \tag{136}$$

and $\eta^+ = \eta^\perp = 0$, while $\eta^- = 2$. This polarization four-vector has the Minkowski product with the free four-momentum p, corresponding to the mass κ, equal zero and its square equals -1. It complements the two transverse linear polarization four-vectors for the field A that one obtains from the free $A^\perp$ by setting A^+ and B to zero,

$$\varepsilon_{p\sigma} = \left(\varepsilon_{p\sigma}^- = 2p^\perp\varepsilon_\sigma^\perp/p^+,\ \varepsilon_{p\sigma}^+ = 0,\ \varepsilon_{p\sigma}^\perp = \varepsilon_\sigma^\perp\right)\,, \tag{137}$$

where $\varepsilon_\sigma^\perp = (1 + \sigma, 1 - \sigma)/2$ with $\sigma = \pm 1$. Together, the set of three polarization four-vectors corresponds to massive vector quanta in agreement with the classification of representations of the Poincaré group [26].

7. Conclusion

The FF Hamiltonian density of Eq. (101) leads to an interacting quantum theory that is not *a priori* limited to the massive limit. Such a theory involves interactions of the field h with both fields A and B. However, for the purpose of regularization of Abelian gauge theories in the FF of Hamiltonian dynamics such as FF QED, it appears sufficient to consider the massive limit alone. Namely, in the RGPEP, the regularization mass parameter appears in the Hamiltonian interaction terms in the form of regulating factors such as in Ref. [8], which for a particle of mass κ can be replaced by

$$\exp\left(-\frac{k^{\perp 2} + \kappa^2}{x\Delta^2}\right). \tag{138}$$

The argument of the exponent can be written as $-m^2(k^\perp, x)/\Delta^2$, where m^2 denotes the particle contribution to the square of total free invariant mass of all particles created or annihilated by a term. Such functions simultaneously regulate UV and small-x singularities. Small-x region is regulated by limiting x to values greater than about κ^2/Δ^2. For extremely small κ, the UV cutoff parameter Δ provides regularization of small x at extremely small values, the smaller the larger Δ. This feature implies that a large range of small momenta k^+ near zero [15, 27] is included that otherwise would be excluded by using regulating functions such as $\theta(x - \delta)$ with a small parameter δ. In view of the literature on the vacuum in QFT, such as [15, 27], this is a welcome feature.

The fact that the massive limit of an Abelian gauge theory yields also the free field h of arbitrary mass does not produce obstacles because the field is decoupled by sending the independent coupling constant λ to zero. Less clear is the issue of quantum coupling of B-field quanta to fermions. The coupling is of the order of κ/p^+, where p^+ is the momentum of the B boson, equal to the momentum carried by the fermion current $g\bar{\psi}\gamma^+\psi$, which can be arbitrarily small when the momentum transfer to or from the fermion approaches zero. Therefore, it appears that in the limit of $\kappa \to 0$, the B bosons decouple. However, the coupling strength is inversely proportional to x that the boson carries and this factor causes new small-x divergence for $p^+ \ll \kappa$. The net effect requires precise studies that await completion.

As a final remark, we stress that the regularization of FF Hamiltonian dynamics of Abelian gauge theories using mass parameter for gauge quanta requires an extension to the non-Abelian gauge theories in order to become

a candidate for regulating Hamiltonian perturbation theory in the SM. The extension requires that the massive limit is replaced by the condition of constant modulus field, which thus becomes a zero mode of the FF dynamics [5]. The only dynamical quantity is the phase of the zero mode, while its modulus becomes a regularization parameter, whose role is to be eventually eliminated by the RGPEP as a consequence of universality. Since such an extension introduces gauge bosons of third polarization and the latter introduce additional singular interaction terms in the Hamiltonians, the following observation regarding the mass κ as a regularization parameter in the RGPEP is in order.

When one solves equations of the RGPEP [9], the interaction terms in the resulting Hamiltonian are softened by vertex form factors of the form of

$$\exp\left[-t\left(\mathcal{M}_{\mathrm{c}}^2 - \mathcal{M}_{\mathrm{a}}^2\right)^2\right], \tag{139}$$

where $t = s^4$ and s is the size parameter for effective particles that plays the role of a renormalization group scale parameter. Symbols $\mathcal{M}_{\mathrm{c}}$ and $\mathcal{M}_{\mathrm{a}}$ denote total free invariant masses of the particles created and annihilated, respectively, by a term in the Hamiltonian that defines a vertex. Once one introduces the mass parameter κ for quanta of fields A and B, their contribution to the invariant masses in the RGPEP form factors is additive and in the form of

$$\frac{k^{\perp 2} + \kappa^2}{x}. \tag{140}$$

This form causes that the RGPEP vertex form factors regulate both singularities due to large $k^\perp$ and small x for non-zero size parameters s. Therefore, when one introduces the mass κ in a theory based on gauge symmetry and uses the massive limit, there is no need for any separate regularization in the RGPEP besides the factors that automatically result form solving its equations.

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