ON BOGOMOLNY EQUATIONS IN THE SKYRME MODEL

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Using the concept of strong necessary conditions (CSNC), we derive a complete decomposition of the minimal Skyrme model into a sum of three coupled BPS submodels with the same topological bound. The bounds are saturated if corresponding Bogomolny equations, different for each submodel, are obeyed.

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1. Introduction

BPS models are classical field theories admitting reduction of the full second order static equations of motion to a set of first order equations (so-called Bogomolny or BPS equations [1–5]), whose solutions saturate a pertinent topological lower bound on the static energy.

BPS models play an important role in physics. Owing to Bogomolny equations, obtaining of exact solutions is possible. Such solutions significantly enlarge our understanding of considered nonlinear models. In fact, one may treat BPS models as "harmonic oscillators" of nonlinear classical field theories (with a topological charge), where many questions can find analytical and exact answers. Moreover, due to the saturation of a pertinent energy bound, solutions of Bogomolny equations are necessary the lowest energy states in each topological sector, which guarantees the topological stability of solitons carrying a non-trivial value of the corresponding topological charge.

Therefore, models with the BPS property are wanted. There exist several methods of derivation of BPS equations: first of them is based on the original Bogomolny trick *i.e.*, completing to a square [1–6]. Some other approaches are: the first order formalism [7, 8] and on-shell method [9, 10]. However, a completely general method, which allows for a systematic derivation (if possible) of BPS equations, is called *the concept of strong necessary conditions* (CSNC). It was originally introduced and analyzed in [11–24], and it has been very recently further developed by Adam and Santamaria in [25], who proposed a so-called *first order Euler–Lagrange* (FOEL) formalism.

In this paper, we apply the CSNC method to derive a *complete* BPS structure of the generalized Skyrme model [26, 27], which is widely considered as a candidate for a low-energy limit of QCD, which has an ability to describe all baryonic (colorless) states in the nature — from single baryons and atomic nuclei to nuclear matter and neutron stars. In other words, we accomplish the program started recently in [28, 29].

2. The Skyrme model

The generalized SU(2) Skyrme model is defined by the following Lagrange density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 \,, \tag{1}$$

where we have a two-derivative term (kinetic or Dirichlet term)

$$\mathcal{L}_2 = -\frac{1}{2}\mathrm{Tr}(L_\mu L^\mu)\,, \tag{2}$$

a four-derivative term (Skyrme term)

$$\mathcal{L}_4 = \frac{1}{16}\mathrm{Tr}\left([L_\mu, L_\nu]^2\right)\,, \tag{3}$$

a six-derivative term (sometimes referred as the BPS term)

$$\mathcal{L}_6 = \lambda^2\pi^2\mathcal{B}_\mu\mathcal{B}^\mu\,, \tag{4}$$

where

$$\mathcal{B}^\mu = \frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(L_\nu L_\rho L_\sigma)$$

is the topological (baryonic) current. Here, $L_\mu = U^\dagger\partial_\mu U$ is the left invariant current and $U \in \mathrm{SU}(2)$ is the Skyrme matrix field. Finally, we have a non-derivative term, that is a potential $\mathcal{L}_0 = -m^2\mathcal{V}(\mathrm{Tr}(U))$, where m can be related to a mass of small perturbations (pions). Note that for the first terms in the Lagrangian, we omit the usual coupling constants f_π and e, which can be re-introduced by a suitable change of length and energy units.

At the beginning, let us remind that the generalized Skyrme model is not an example of a BPS model. It is a consequence of the fact that even the minimal Skyrme model *i.e.*, $\mathcal{L}_{24} = \mathcal{L}_2 + \mathcal{L}_4$ does not possess nontrivial solutions saturating a corresponding topological bound, a so-called Faddeev bound [30, 31]

$$E \geq 12\pi^2 |B| \,. \tag{5}$$

However, there is a rather reach BPS structure hidden in the full model. First of all, there is the BPS submodel, referred to as the BPS Skyrme model, $\mathcal{L}_{\mathrm{BPS}} = \mathcal{L}_6 + \mathcal{L}_0$, which *is* a genuine BPS theory with a topological bound saturated by infinitely many solitons (BPS Skyrmions) in an arbitrary topological sector [32, 33]. An importance of this finding is related to a problem of too high binding energies in the original $\mathcal{L}_{24}$ model. The BPS theory has necessary zero classical binding energies, while small contributions can show up due to semiclassical quantisation and inclusion of the Coulomb interactions [34]. As the BPS Skyrme model is a point in the parameter space of the full model (*i.e.*, a limit, where coefficients, multiplying $\mathcal{L}_2$ and $\mathcal{L}_4$, vanish), one can use it as a starting point for a whole family of near-BPS Skyrme-type models with physically small classical binding energies. (For other Skyrme type model saturating a BPS bound, see [35, 36].)

Secondly, even the minimal part $\mathcal{L}_{24}$ enjoys an interesting BPS structure [28, 29]. To see this, we need to introduce explicit coordinates on $\mathrm{SU}(2) \cong \mathbb{S}^3$. Specifically, we use the standard parametrization of the $\mathrm{SU}(2)$ field U by one real scalar ξ and one three component isovector $\vec{n}$ of unit length

$$U = \exp\left(i\xi\vec{\tau}\cdot\vec{n}\,\right) , \tag{6}$$

where $\vec{\tau}$ are the Pauli matrices. Furthermore, $\vec{n}$ can be expressed by a complex scalar ω by the stereographic projection

$$\vec{n} = \left[\frac{\omega+\omega^*}{1+\omega\omega^*}, \frac{-i(\omega+\omega^*)}{1+\omega\omega^*}, \frac{1-\omega\omega^*}{1+\omega\omega^*}\right] .$$

Following [28] and [29], we write the two parts of the $\mathcal{L}_{24}$ Skyrme model as

$$\mathcal{L}_2 = \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} \,, \qquad \mathcal{L}_2^{(1)} = 4\frac{\sin^2(\xi)}{(1+\omega\omega^*)^2}\omega_\mu\omega^{*\mu} \,, \qquad \mathcal{L}_2^{(2)} = \xi_\mu\xi^\mu \,, \tag{7}$$

$$\mathcal{L}_4 = \mathcal{L}_4^{(1)} + \mathcal{L}_4^{(2)} \,, \qquad \mathcal{L}_4^{(1)} = 4\sin^2(\xi)\left(\xi_\mu\xi^\mu\frac{\omega_\mu\omega^{*\mu}}{(1+\omega\omega^*)^2} - \frac{\xi_\mu\omega^{*\mu}\xi_\nu\omega^\nu}{(1+\omega\omega^*)^2}\right) , \tag{8}$$

$$\mathcal{L}_4^{(2)} = 4\sin^4(\xi)\frac{(\omega_\mu\omega^{*\mu})^2 - \omega_\mu^2\omega_\nu^{*2}}{(1+\omega\omega^*)^2} \,. \tag{9}$$

Therefore,

$$\mathcal{L}_{24} = \left(\mathcal{L}_2^{(1)} + \mathcal{L}_4^{(1)}\right) + \left(\mathcal{L}_2^{(2)} + \mathcal{L}_4^{(2)}\right) \equiv \mathcal{L}^{(1)} + \mathcal{L}^{(2)} \,, \tag{10}$$

where each of the constituent submodels $\mathcal{L}^{(1)}, \mathcal{L}^{(2)}$, if taken separately, is a proper BPS model. Indeed, the static energy of the first BPS submodel can be written as

$$E^{(1)}=\int \mathrm{d}^3x\frac{4\sin^2\xi}{(1+\omega\omega^*)^2}\left[\omega_i\omega_i^*+\xi_j^2(\omega_i\omega_i^*)-(\xi_i\omega_i)\left(\xi_j\omega_j^*\right)\right]\geq 8\pi^2|B|\,, \tag{11}$$

where

$$B=\int B_0\mathrm{d}^3x=\frac{1}{\pi^2}\int \mathrm{d}^3x\frac{i\sin^2\xi}{\left(1+f^2\right)^2}\varepsilon_{ijk}\xi_i\omega_j\omega_k^*\,. \tag{12}$$

The bound is saturated for solution of the following Bogomolny equation:

$$\omega_i\pm i\varepsilon_{ijk}\xi_j\omega_k=0 \tag{13}$$

and its complex conjugation.

Analogously, the static energy for the second BPS submodel is

$$E^{(2)}=\int \mathrm{d}^3x\left(\xi_i^2+4\sin^4\xi\frac{1}{\left(1+|u|^2\right)^4}\left(i\varepsilon_{ijk}\omega_j\,\omega_k^*\right)^2\right)\geq 4\pi^2|B|\,, \tag{14}$$

where the corresponding Bogomolny equation is

$$\xi_i\mp\frac{2i\sin^2\xi}{(1+\omega\omega^*)^2}\varepsilon_{ijk}\omega_j\,\omega_k^*=0\,. \tag{15}$$

Together, both bounds provide the Faddeev bound. Moreover, there are no common solutions of these Bogomolny equations in $\mathbb{R}^3$ base space [37].

Note that the number of the independent equations resulting from the Bogomolny equations for the first BPS submodel (13), is twice as for the second BPS submodel (15). As a result, the Bogomolny equations of the first submodel, can be expressed as

$$\lambda_2=\pm\lambda_1\lambda_3\quad\text{and}\quad\lambda_3=\pm\lambda_1\lambda_2\,, \tag{16}$$

while for the second submodel, they are equivalent to

$$\lambda_1=\pm\lambda_2\lambda_3\,, \tag{17}$$

where λ_i^2 are the eigenvalues of the strain tensor $D_{ij}=-\frac{1}{2}\mathrm{Tr}(L_iL_j)$ [31, 37]. Such a nonequivalence in number of independent equations for the Bogomolny equations (13) and (15) leads to a question if it is possible to further decompose the $\mathcal{L}_{24}$ Skyrme to a collection of *three* BPS submodels such that each of them corresponds to one real scalar equation (related to one constrain on the eigenvalues λ_i). Then, each submodel would contribute to the total Faddeev bound in the same way.

Obviously, to find such a complete BPS structure of the $\mathcal{L}_{24}$ Skyrme model, it is enough to consider the first BPS submodel. Then, we further decompose the complex field by two real degrees of freedom $\omega = fe^{ig}$. Hence,

$$\begin{aligned} E^{(1)} &= 4\int \frac{\sin^2\xi}{(1+f^2)^2}\left[f_i^2 + f^2 g_i^2 + f^2(\varepsilon_{ijk}\xi_j g_k)^2 + (\varepsilon_{ijk}\xi_j f_k)^2\right] \mathrm{d}^3x \\ &= E_1^{(1)} + E_2^{(1)}\,, \end{aligned} \tag{18}$$

where

$$E_1^{(1)} = 4\int \frac{\sin^2\xi}{(1+f^2)^2}\left[f_i^2 + f^2(\varepsilon_{ijk}\xi_j g_k)^2\right] \mathrm{d}^3x \tag{19}$$

and

$$E_2^{(1)} = 4\int \frac{\sin^2\xi}{(1+f^2)^2}\left[f^2 g_i^2 + (\varepsilon_{ijk}\xi_j f_k)^2\right] \mathrm{d}^3x\,. \tag{20}$$

In the subsequent analysis, we will investigate the existence and properties of Bogomolny equations for these new subsectors of the $\mathcal{L}^{(1)}$ theory (*i.e.*, the first BPS Skyrme submodel). To accomplish this aim, we will apply the CSNC method. So, we begin with a short summary on this approach.

3. The concept of strong necessary conditions

The main idea of the concept of strong necessary conditions (shortly: CSNC) is such that instead of studying the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\mathcal{L}_{,\Phi^\nu_{,x^\mu}} - \mathcal{L}_{,\Phi^\nu} = 0\,, \tag{21}$$

we study the differential equations, generated by the strong necessary conditions [11–16]

$$\mathcal{L}_{,\Phi^\nu_{,x^\mu}} = 0\,, \tag{22}$$

$$\mathcal{L}_{,\Phi^\nu} = 0\,. \tag{23}$$

Obviously, the set of the solutions of these equations is a subset of the set of the solutions of equations (21). However, very often one can obtain rather trivial solutions of equations (22)–(23). On the other hand, we can prevent it by doing the following gauge transformation [11–16]:

$$\mathcal{L} \Longrightarrow \mathcal{L} + I\,, \tag{24}$$

where I is such a functional that $\delta I \equiv 0$.

After applying the strong necessary conditions (22)–(23), for the gauged Lagrangian (24), we get dual equations [11–16]. As one can see, the Euler–Lagrange equations (21) are invariant with respect to the gauge transformation (24), but the strong necessary conditions (22)–(23) are not invariant with respect to this transformation. Hence, we can extend the set of the solutions of the strong necessary conditions. Next, in order to obtain Bogomolny decomposition (Bogomolny equations, Bogomol'nyi equations), we need to make the dual equations self-consistent [17–21]. In the next sections will be shown how one can do it in the cases investigated in this paper.

4. Decomposition of the first BPS submodel

4.1. First subsubmodel

We derive strong necessary conditions for (19) with a generalization

$$
\begin{aligned}
\tilde{E}_1^{(1)} &= \int \tilde{\mathcal{H}}_1^{(1)} \mathrm{d}^3x \\
&= \int \left\{ G_0 \left[f_i^2 + f^2 (\varepsilon_{ijk}\xi_j g_k)^2 \right] + G_1 \varepsilon_{ijk}\xi_i f_j g_k + \sum_{p=1}^{3} D_p G_{p+1} \right\} \mathrm{d}^3x \,,
\end{aligned}
\tag{25}
$$

where $p = 1,2,3$ and $G_n = G_n(f,g,\xi),(n = 0,...,4)$ are certain functions, which are to be determined later.

The strong necessary conditions have the form of

$$
\begin{aligned}
\tilde{\mathcal{H}}_{1,f}^{(1)} &: G_{0,f} \left[f_{,i}^2 + f^2 \left(\varepsilon_{ijk}\xi_j g_k\right)^2 \right] + 2G_0 f \left(\varepsilon_{ijk}\xi_j g_k\right)^2 + G_{1,f}\varepsilon_{ijk}\xi_i f_j g_k \\
&\quad + \sum_{p=1}^{3} D_p G_{p+1,f} = 0 \,, \\
\tilde{\mathcal{H}}_{1,g}^{(1)} &: G_{0,g} \left[f_i^2 + f^2 \left(\varepsilon_{ijk}\xi_j g_k\right)^2 \right] + G_{1,g}\varepsilon_{ijk}\xi_i f_j g_k + \sum_{p=1}^{3} D_p G_{p+1,g} = 0 \,, \\
\tilde{\mathcal{H}}_{1,\xi}^{(1)} &: G_{0,\xi} \left[f_i^2 + f^2 \left(\varepsilon_{ijk}\xi_j g_k\right)^2 \right] + G_{1,\xi}\varepsilon_{ijk}\xi_i f_j g_k + \sum_{p=1}^{3} D_p G_{p+1,\xi} = 0 \,, \\
\tilde{\mathcal{H}}_{1,f,r}^{(1)} &: 2G_0 f_r + G_1 \varepsilon_{irk}\xi_i g_k + G_{r+1,f} = 0 \,, \\
\tilde{\mathcal{H}}_{1,g_r}^{(1)} &: 2G_0 f^2 \varepsilon_{mlr}\xi_l \left(\varepsilon_{ijk}\xi_j g_k\right) + G_1 \varepsilon_{ijr}\xi_i f_j + G_{r+1,g} = 0 \,, \\
\tilde{\mathcal{H}}_{1,\xi_r}^{(1)} &: 2G_0 f^2 \varepsilon_{mrl} g_l \left(\varepsilon_{ijk}\xi_j g_k\right) + G_1 \varepsilon_{rjk} f_j g_k + G_{r+1,\xi} = 0 \,.
\end{aligned}
\tag{26}
$$

As usually in the case of strong necessary conditions, in order to derive Bogomolny decomposition (Bogomolny equations), we need to make equations (26) to be self-consistent. This requires

$$G_1 = 2G_0 f\,, \tag{27}$$

$$G_{p+1} = \text{const.}, \qquad p = 1,2,3\,, \tag{28}$$

$$f_i - f\varepsilon_{ijk}\xi_j g_k = 0\,. \tag{29}$$

Then, three first equations are satisfied, and the Bogomolny decomposition has the form of

$$f_i - f\varepsilon_{ijk}\xi_j g_k = 0\,. \tag{30}$$

This Bogomolny equation can be used to find a topological bound on the energy of the first BPS submodel. Namely,

$$\begin{aligned} E_1^{(1)} &= 4\int \frac{\sin^2\xi}{(1+f^2)^2}\left[f_i \pm f\varepsilon_{ijk}\xi_j g_k\right]^2 \mathrm{d}^3x \mp 8\int \frac{\sin^2\xi}{(1+f^2)^2}\varepsilon_{ijk} f f_i \xi_j g_k \mathrm{d}^3x \\ &\geq \left| 8\int \frac{\sin^2\xi}{(1+f^2)^2}\varepsilon_{ijk} f f_i \xi_j g_k \mathrm{d}^3x \right| = 4\pi^2 |B|\,, \end{aligned} \tag{31}$$

which is saturated if and only if the Bogomolny equation (30) is obeyed.

4.2. Second subsubmodel

Now, we derive strong necessary conditions for (20) with a generalization

$$\begin{aligned} \tilde{E}_2^{(1)} &= \int \tilde{\mathcal{H}}_2^{(1)} \mathrm{d}^3x \\ &= \int \left\{ G_0\left[f^2 g_i^2 + (\varepsilon_{ijk}\xi_j f_k)^2\right] + G_1\varepsilon_{ijk}\xi_i f_j g_k + \sum_{p=1}^{3} D_p G_{p+1} \right\} \mathrm{d}^3x\,. \end{aligned} \tag{32}$$

The strong necessary conditions have the form of

$$\begin{aligned} \tilde{\mathcal{H}}_{2,f}^{(1)} \;:\; & G_{0,f}\left[f^2 g_i^2 + (\varepsilon_{ijk}\xi_j f_k)^2\right] + 2G_0 f g_k^2 + G_{1,f}\varepsilon_{ijk}\xi_i f_j g_k \\ & + \sum_{p=1}^{3} D_p G_{p+1,f} = 0\,, \end{aligned}$$

$$
\begin{aligned}
\tilde{\mathcal{H}}_{2,g}^{(1)} &: G_{0,g}\left[f^2 g_i^2 + \left(\varepsilon_{ijk}\xi_j f_k\right)^2\right] + G_{1,g}\varepsilon_{ijk}\xi_i f_j g_k + \sum_{p=1}^{3} D_p G_{p+1,g} = 0\,, \\
\tilde{\mathcal{H}}_{2,\xi}^{(1)} &: G_{0,\xi}\left[f^2 g_i^2 + \left(\varepsilon_{ijk}\xi_j f_k\right)^2\right] + G_{1,\xi}\varepsilon_{ijk}\xi_i f_j g_k + \sum_{p=1}^{3} D_p G_{p+1,\xi} = 0\,, \\
\tilde{\mathcal{H}}_{2,f_r}^{(1)} &: 2G_0\varepsilon_{mlr}\xi_l\left(\varepsilon_{ijk}\xi_j f_k\right) + G_1\varepsilon_{irk}\xi_i g_k + G_{r+1,f} = 0\,, \\
\tilde{\mathcal{H}}_{2,g_r}^{(1)} &: 2G_0 f^2 g_r + G_1\varepsilon_{ijr}\xi_i f_j + G_{r+1,g} = 0\,, \\
\tilde{\mathcal{H}}_{2,\xi_r}^{(1)} &: 2G_0\varepsilon_{mrl}f_l\left(\varepsilon_{ijk}\xi_j f_k\right) + G_1\varepsilon_{rjk}f_j g_k + G_{r+1,\xi} = 0\,.
\end{aligned} \tag{33}
$$

In order to make equations (33) self-consistent, we have to put

$$
G_1 = 2G_0 f\,, \tag{34}
$$

$$
G_{p+1} = \text{const.}, \qquad p = 1,2,3\,, \tag{35}
$$

$$
f g_i + \varepsilon_{ijk}\xi_j f_k = 0\,. \tag{36}
$$

In this case, the Bogomolny decomposition has the form of

$$
f g_i + \varepsilon_{ijk}\xi_j f_k = 0\,. \tag{37}
$$

Corresponding topological bound on the energy reads

$$
\begin{aligned}
E_2^{(1)} &= 4\int \frac{\sin^2\xi}{\left(1+f^2\right)^2}\left[f g_i \pm \varepsilon_{ijk}\xi_j f_k\right]^2 \mathrm{d}^3x \mp 8\int \frac{\sin^2\xi}{\left(1+f^2\right)^2}\varepsilon_{ijk} f f_i \xi_j g_k \mathrm{d}^3x \\
&\geq \left|8\int \frac{\sin^2\xi}{\left(1+f^2\right)^2}\varepsilon_{ijk} f f_i \xi_j g_k \mathrm{d}^3x\right| = 4\pi^2|B|\,,
\end{aligned} \tag{38}
$$

which now is saturated if and only if the Bogomolny equation (37) is obeyed.

As we see, the minimal Skyrme model $\mathcal{L}_{24}$ can be written as a sum of *three* BPS submodels. Each of them has the same topological bound $E \geq 4\pi|B|$, which however, is saturated for *different* field configurations as the corresponding Bogomolny equations are different. This provides a complete decomposition of the minimal Skyrme model as a sum of three coupled BPS submodels. Note also that none of them can be reached as a limit in the parameter space of the full model.

5. Static solutions of the new BPS submodels

In order to understand solutions with a nontrivial topology in the upper defined BPS submodels, we use the spherical coordinates and assume the following Ansatz

$$
\xi = \xi(r)\,, \qquad f = f(\theta)\,, \qquad g = n\phi\,. \tag{39}
$$

In addition, one has to impose the usual boundary conditions which guarantee a non-zero baryon charge (the whole $\mathbb{S}^3$ target space must be covered at least once)

$$\xi(r=0)=\pi\,, \qquad \xi(r=R)=0 \tag{40}$$

and

$$f(\theta=0)=0\,, \qquad f(\theta=\pi)=\infty\,. \tag{41}$$

Here, R is the geometric size of the soliton *i.e.*, value of the radial coordinate where the profile function reaches the vacuum. Then, the first Bogomolny equation (30) leads to solutions

$$\xi=\begin{cases} \pi-Cr\,, & r\le 1/C\,,\\ 0\,, & r\ge 1/C\,,\end{cases} \tag{42}$$

and

$$f=A\left(\tan\frac{\theta}{2}\right)^{Cn}, \tag{43}$$

where C is a positive constant. Furthermore, $A\in\mathbb{R}$. One can verify that such a solution has topological charge $B=n$. Note also that the size of solitons can be treated as a free parameter

$$R=1/C\,. \tag{44}$$

The same Ansatz applied to the second BPS submodel gives

$$\xi=\begin{cases} \pi-Dr\,, & r\le 1/D\,,\\ 0\,, & r\ge 1/D\,,\end{cases} \tag{45}$$

and

$$f=A\left(\tan\frac{\theta}{2}\right)^{\frac{n}{D}}. \tag{46}$$

Again, we find a one-parameter family of compact solutions (compact Skyrmions) with topological charge $B=n$ and radius

$$R=1/D\,. \tag{47}$$

Several comments are in order. First of all, the obtained solutions of our two new BPS submodels $E_1^{(1)}$ and $E_2^{(1)}$ are, in general, non-holomorphic configurations. Indeed, the angular part combines to holomorphic map only if $C=1$ or $D=1$, which is one of infinitely many possible solutions. Then,

$$\omega=fe^{ig}=A\left(\tan\frac{\theta}{2}\right)^{n}e^{in\phi}=Az^n\,. \tag{48}$$

It follows from this observation that, for $C = D = 1$, these BPS submodels do have a common solution which is exactly the compacton solution (with an arbitrary holomorphic map) of the $E^{(1)}$ BPS submodel [28]. In other words, the holomorphic map solutions of the first BPS submodel $E^{(1)}$ emerge as a mutual effect of a competition of $E_1^{(1)}$ and $E_2^{(1)}$. In addition, on the contrary to solutions of our submodels $E_1^{(1)}$ and $E_2^{(1)}$, solutions of $E^{(1)}$ have also a definite size.

Next, the solutions of $E_1^{(1)}$ and $E_2^{(1)}$ are of the same (lower) type of continuity as compactons of $E^{(1)}$ submodel. Again, the first derivative of the profile is not continuous at the boundary while physical observables as energy density as well as topological charge density are continuous.

6. Summary

In the present paper, a complete decomposition of the minimal Skyrme model is performed. We have found that the model can be written as a sum of three BPS submodels with identical topological bounds. These bounds are saturated if pertinent Bogomolny equations are obeyed, which are different for each submodel. Following that, there are no common solutions as it should be since the minimal Skyrme model does not saturate the Faddeev bound.

We also show how the rational maps (which are the main ingredient of the rational maps Ansatz of the Skyrme model [38–41]) emerge due to a mutual interplay between new derived BPS submodels $E_1^{(1)}$ and $E_2^{(1)}$.

On the other hand, the fact that each of the three BPS submodels $E_1^{(1)}$, $E_2^{(1)}$ and $E^{(2)}$ supports also non-holomorphic BPS solutions may perhaps shed some lights on the role of non-holomorphic contribution of Skyrmions. In fact, it was observed that the rational map approximated solutions can be improved if a small non-holomorphic term is included [42].

Finally, this hidden BPS structure of the full Skyrme model may be helpful in the construction of super-symmetric extensions of the Skyrme model [43–45].

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