SYNCHRONIZATION OF COMPLEX NETWORK BASED ON THE THEORY OF GRAVITATIONAL FIELD

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Based on the conception of gravitational field, the issue of synchronization of complex network turns to the interaction and motion of particles under a physical field. By the design of coupling factor based on velocity, the synchronization of complex network is obtained where the dynamics of those nodes may be discontinuous and different from each other. Unlike those common methods of synchronization, this new approach is not limited in any desired governing equation of motion. According to the idea of approximation, the conditions of network synchronization and the synchronous orbit equation in the gravitational field are pointed out. The speed of synchronization is positively related to the coefficient of gravity. Synchronization was obtained in complex network with 51 and 501 nodes of piecewise linear Chen systems, Sprott systems and Lorenz systems, which shows the effectiveness of the proposed method.

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1. Introduction

Complex dynamic networks have wide application in bio-technology, engineering technology or social fields. There are many synchronization methods including global synchronization based on the Lyapunov function, local synchronization based on the master stability function and global synchronization based on connection graph stability in time-varying network [1, 2].

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The traditional Lyapunov function method is often complicated and associated with the node equations [3–8], methods for local synchronization based on the master stability function mainly work for those networks with the same nodes [9–14]; while the synchronization method of connection graph stability in the time-varying network needs to combine the Lyapunov function with graph theory [15–18]. In this paper, the synchronization issue of complex networks is transformed into particle-motion synchronization in a gravitational field. Specifically in this paper, we solved the synchronization problem of complex networks, which contain nodes with different governing equations and even nodes with discontinuous equations at the right-hand sides. The coupling factor is only associated with the speed of the nodes, which has nothing to do with the motion equation. This method is essentially different from the design of the controller for synchronization and, therefore, has a strong generality. The conditions of network synchronization in the gravitational field were given according to the idea of approximation. The synchronous orbit equation was consequently obtained. It was found that the synchronization velocity is positively related to the coefficient of gravity. Simulations show the effectiveness of the proposed method.

2. Gravitation theory revised for synchronization and some mathematical preparation

2.1. Gravitation theory

Gravitation is the attraction between objects caused by the mass. Its mathematical expression is $\boldsymbol{F}(t) = G\frac{m_1 m_2}{r^2(t)}$, here $\boldsymbol{F}(t)$ is the attraction between two bodies at moment t, G is gravitational constant, m_1, m_2 are masses of two objects, respectively, $r(t)$ is the distance between two objects at the time t. According to Newton's second law, the acceleration of the object i at the time t can be calculated and, consequently, its velocity can be calculated ($i = 1, 2$). For multiple objects, the motion velocity caused by gravity can be similarly calculated.

Suppose $\boldsymbol{X}_i = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))$ is the location for the object i in n-dimensional space, $x_{ik}(t)$ is the position of the dimension k at time t ($k = 1, 2, \dots, n$). The masses of the particles are M_i ($i = 1, 2, \dots, m$). Therefore, at the moment t, the gravitational force between the particles i and j at the dimension k is $F^i_{jk}(t) = G\frac{M_i M_j}{\sum_{k=1}^{n}(x_{jk}(t)-x_{ik}(t))^2} \times \frac{x_{jk}(t)-x_{ik}(t)}{\|\boldsymbol{X}_j(t)-\boldsymbol{X}_i(t)\|}$ and, consequently, the velocity component of motion of the particle i caused by the resultant force from other particles in the dimension k is

$$v_{ik}(t) = v_{ik}(0) + \int_0^t \frac{1}{M_i} \sum_{j=1, j\neq i}^{m} F_{jk}^{i}(s)\mathrm{d}s$$
$$= v_{ik}(0) + G\int_0^t \sum_{j=1, j\neq i}^{m} \frac{M_j}{\sum_{k=1}^{n} (x_{jk}(s) - x_{ik}(s))^2} \times \frac{x_{jk}(s) - x_{ik}(s)}{\|\boldsymbol{X}_j(s) - \boldsymbol{X}_i(s)\|}\mathrm{d}s\,.$$

For the convenience of calculation, not limited to the laws of physics, here we define the velocity of motion of the particle i in the k dimension at the moment t, $v_{ik}(t) = G \times (\sum_{j=1}^{m} \frac{M_j}{R^{\gamma}(t)}(x_{jk}(t) - x_{ik}(t)))$, $(\gamma \geq 1)$. To decrease the complexity of calculation, we let $\gamma = 1$, $v_{ik}(t) = G \times (\sum_{j=1}^{m} \frac{M_j}{R(t)}(x_{jk}(t) - x_{ik}(t)))$, where

$$R(t) = \sqrt{\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m} (x_{jk}(t) - x_{ik}(t))^2}\,. \tag{1}$$

Definition 2.1. A physical field can be defined as a gravitational field if the particle i in a physical field has the velocity of motion in the k dimension at the moment t,

$$v_{ik}(t) = G \times (1 + W(t)) \left(\sum_{j=1}^{m} \frac{M_j}{R(t)}(x_{jk}(t) - x_{ik}(t)) \right), \tag{2}$$

here, G is a positive constant and $W(t) = \sum_{i=1}^{m}\sum_{k=1}^{n}(x_{ik}(t))^2$. We suppose each particle has a unit quality $M_i = 1$, $(i = 1, 2, \ldots, m)$.

2.2. Mathematical preparation

Mark $F(t,x) = Kf(t,x) = \bigcap_{\delta>0}\bigcap_{\mu N=0} \overline{\mathrm{co}} f(t, x^{\delta}\backslash N)$, here x^{δ} is the δ neighbourhood of the variable x, $\overline{\mathrm{co}}$ represents a convex closure of a set, $x \in R^n$, $f \in L_{\mathrm{loc}}^{\infty}(R \times R^n, R^n)$, $F(t,x)$ is a set-valued mapping.

Definition 2.2. Define the vector function $x(t)$ in the non-degenerate interval $I \subseteq R$ as the Filippov solution of the system $\dot{x} = f(t,x)$, if it is absolutely continuous in any compact subintervals of I and satisfy $\dot{x} \in F(t,x)$ for almost all $t \in I$.

Definition 2.3. $F(t,x)$ is upper semi-continuous if there is a neighbourhood V of (t_0, x_0) subjecting to $F(V) \subseteq U$, U is an open set containing $F(t_0, x_0)$.

Definition 2.4. The set-valued mapping $F(t,x)$ is said to satisfy basic condition, if in the region $\Delta \subseteq R \times R^n$, $F(t,x)$ is a non-empty compact convex set for any $(t,x) \in \Delta$, and is upper semi-continuous according to t and x.

Lemma 2.1. *Assume $F(t,x)$ satisfy basic condition in the region Δ, $(t_0,x_0) \in \Delta$. If there are positive constants b, c such that Δ contains a cylindrical domain $Z = \{(t,x)|t_0 \le t \le b, \| x - x_0 \| \le c\}$, the solution of differential equation with discontinuous right-hand side $\dot{x} = f(t,x)$ which satisfies the initial condition $x(t_0) = x_0$ exists on the interval $[t_0, t_0 + d]$, where $d = \min\{b - t_0, \frac{c}{m}\}$, $m = \sup\limits_{(t,x)\in Z} \| F(t,x) \|$ [19].*

Corollary 2.1. *For almost all $t \in [t_0, b]$ and any x subjecting to $\| x - x_0 \| \le c$, if the following conditions are satisfied: (i) $f(t,x)$ is piecewise continuous; (ii) $f(t,x)$ is measurable with respect to t for any x; (iii) there is an integrable function $m(t)$ subjecting to $\| f(t,x) \| \le m(t)$, then the Filippov solution of differential equation with discontinuous right-hand side $\dot{x} = f(t,x)$ which satisfies the initial condition $x(t_0) = x_0$ exists on the interval $[t_0, t_0 + d]$, where $d = \min\{b - t_0, \frac{c}{m}\}$, $m = \sup\limits_{(t,x)\in Z} \| F(t,x) \|$ [20].*

Lemma 2.2. *Suppose $F(t,x)$ satisfies the basic condition in open region Δ, $t_0 \in [a,b]$, $(t_0,x_0) \in \Delta$, and all the solutions $x(t)$ of the differential conclusion $\dot{x} \in F(t,x)$ with the initial condition $x(t_0) = x_0$ exist in $[a,b]$ and their images are all in Δ. If for any $\varepsilon > 0$, there exists $\delta > 0$ subjecting to $|t_0^* - t_0| \le \delta$, $\| x_0^* - x_0 \| \le \delta$, $F^*(t,x) \subseteq [\mathrm{co}F(t^\delta, x^\delta)]^\delta$ for any $t_0^* \in [a,b]$, x_0^* and $F^*(t,x)$ satisfying basic conditions, then all solutions $x^*(t,x)$ of the differential conclusion $\dot{x}^* \in F(t,x^*)$ satisfying the initial condition $x^*(t_0^*) = x_0^*$ exist on the interval $[a,b]$, and for each $x^*(t)$, there is a corresponding solution $x(t)$ of the differential inclusion $\dot{x} \in F(t,x)$ with the initial condition $x(t_0) = x_0$ which satisfies $\max\limits_{a \le t \le b} \| x(t) - x^*(t) \| \le \varepsilon$ [21].*

Lemma 2.3. *Assume scalar functions $u(t,x)$, $u_1(t,x)$ are continuous in the plane P and satisfy the inequation $u(t,x) \le u_1(t,x)$, $(t_0,x_0) \in P$. If $x = \varphi(t)$, $x = \Phi(t)$ are the unique solution $\begin{cases} \dot{x} = u(t,x) \\ x(t_0) = x_0 \end{cases}$ and $\begin{cases} \dot{x} = u_1(t,x) \\ x(t_0) = x_0 \end{cases}$ defined in $t \ge t_0$, then $\varphi(t) \le \Phi(t)$ when $t \ge t_0$.*

3. Synchronization based on the theory of gravitational field

Mark $R_+ = [0, +\infty)$, $\mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$ suppose there is complex network including m particles with the following equation of motion:

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \vdots \\ \dot{x}_{in} \end{pmatrix} = \begin{pmatrix} f_{i1}^{(1)} & f_{i1}^{(2)} & \cdots & f_{i1}^{(n)} \\ f_{i2}^{(1)} & f_{i2}^{(2)} & \cdots & f_{i2}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ f_{in}^{(1)} & f_{in}^{(2)} & \cdots & f_{in}^{(n)} \end{pmatrix} \begin{pmatrix} \mathrm{sgn}(x_{i1}) \\ \mathrm{sgn}(x_{i2}) \\ \vdots \\ \mathrm{sgn}(x_{in}) \end{pmatrix} + \begin{pmatrix} g_{i1} \\ g_{i2} \\ \vdots \\ g_{in} \end{pmatrix}, \qquad (i = 1, 2, \ldots, m)\,. \tag{3}$$

Here, $f_{ik}^{(j)} = f_{ik}^{(j)}(x_{i1}, x_{i2}, \ldots, x_{in})$, $g_{ik} = g_{ik}(x_{i1}, x_{i2}, \ldots, x_{in})$, $(i = 1, 2, \ldots, m;\ j, k = 1, 2, \ldots, n)$. Then, when $R(t) \neq 0$, the equation of motion of the complex network in the gravitational field is

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \vdots \\ \dot{x}_{in} \end{pmatrix} = \begin{pmatrix} f_{i1}^{(1)} & f_{i1}^{(2)} & \cdots & f_{i1}^{(n)} \\ f_{i2}^{(1)} & f_{i2}^{(2)} & \cdots & f_{i2}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ f_{in}^{(1)} & f_{in}^{(2)} & \cdots & f_{in}^{(n)} \end{pmatrix} \begin{pmatrix} \mathrm{sgn}(x_{i1}) \\ \mathrm{sgn}(x_{i2}) \\ \vdots \\ \mathrm{sgn}(x_{in}) \end{pmatrix} + \begin{pmatrix} g_{i1} \\ g_{i2} \\ \vdots \\ g_{in} \end{pmatrix} + \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{pmatrix}, \qquad (i = 1, 2, \ldots, m)\,. \tag{4}$$

Here, $f_{ik}^{(j)} = f_{ik}^{(j)}(x_{i1}, x_{i2}, \ldots, x_{in})$, $g_{ik} = g_{ik}(x_{i1}, x_{i2}, \ldots, x_{in})$, $v_{ik} = v_{ik}(t)$ and $M_i = 1$. $X_i^0 = (x_{i1}(0), x_{i2}(0), \ldots, x_{in}(0)) \neq 0$ is the initial condition of the node i, $(i = 1, 2, \ldots, m; j, k = 1, 2, \ldots, n)$.

To study the dynamics of complex network (4) when $R(t) = 0$ in the gravitational field, we introduce the parameter α, $(\alpha \in [0, +\infty))$. According to the approaching theory, considering a more general network,

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \vdots \\ \dot{x}_{in} \end{pmatrix} = \begin{pmatrix} f_{i1}^{(1)} & f_{i1}^{(2)} & \dots & f_{i1}^{(n)} \\ f_{i2}^{(1)} & f_{i2}^{(2)} & \dots & f_{i2}^{(n)} \\ \dots & \dots & \dots & \dots \\ f_{in}^{(1)} & f_{in}^{(2)} & \dots & f_{in}^{(n)} \end{pmatrix} \begin{pmatrix} \mathrm{sgn}(x_{i1}) \\ \mathrm{sgn}(x_{i2}) \\ \vdots \\ \mathrm{sgn}(x_{in}) \end{pmatrix} + \begin{pmatrix} g_{i1} \\ g_{i2} \\ \vdots \\ g_{in} \end{pmatrix} + \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{pmatrix}, \qquad (i = 1, 2, \dots, m). \tag{5}$$

Here, $f_{ik}^{(j)} = f_{ik}^{(j)}(x_{i1}, x_{i2}, \dots, x_{in})$, $g_{ik} = g_{ik}(x_{i1}, x_{i2}, \dots, x_{in})$, $v_{ik} = v_{ik}(t, \alpha)$ $= G \times (1 + W(t))(\sum_{j=1}^{m} \frac{1}{R(t)+\alpha}(x_{jk}(t) - x_{ik}(t)))$, $f_{ik}(x_{i1}, x_{i2}, \dots, x_{in})$ and the initial condition $X_i^0(0, \alpha) = (x_{i1}(0), x_{i2}(0), \dots, x_{in}(0))$ are equal to complex network (4), $(i = 1, 2, \dots, m; j, k = 1, 2, \dots, n; \alpha \in [0, +\infty))$.

Hypothesis 3.1. Assume complex networks (3), (4), (5) satisfy: $f_{ik}^{(j)}(x_1, x_2, \dots, x_n)$, $g_{ik}(x_1, x_2, \dots, x_n)$ are continuous functions in R^n, there exist positive constants $A_{ik}^{(j)} > 0$, $B_{ik}^{(j)} > 0$, $A_{ik} > 0$, $B_{ik} > 0$ and continuous functions in R^{2n} $p_{ik,l}^{(j)}(x_1, x_2, \dots, x_{2n})$, $q_{ik,l}(x_1, x_2, \dots, x_{2n})$ such that

$$\begin{aligned}
&\left| f_{ik}^{(j)}(x_1, x_2, \dots, x_n) \right| \le B_{ik}^{(j)} + A_{ik}^{(j)} \sum_{l=1}^{n} (x_l)^2 , \\
&|g_{ik}(x_1, x_2, \dots, x_n)| \le B_{ik} + A_{ik} \sum_{l=1}^{n} (x_l)^2 , \\
&\left| f_{ik}^{(j)}(x_1, x_2, \dots, x_n) - f_{ik}^{(j)}(y_1, y_2, \dots, y_n) \right| \\
&\le \sum_{l=1}^{n} p_{ik,l}^{(j)}(|x_1|, |x_2|, \dots, |x_n|, |y_1|, |y_2|, \dots, |y_n|)|x_l - y_l| , \\
&|g_{ik}(x_1, x_2, \dots, x_n) - g_{ik}(y_1, y_2, \dots, y_n)| \\
&\le \sum_{l=1}^{n} q_{ik,l}(|x_1|, |x_2|, \dots, |x_n|, |y_1|, |y_2|, \dots, |y_n|)|x_l - y_l| , \\
&(i = 1, 2, \dots, m; j = 1, 2, \dots, n; k = 1, 2, \dots, n) .
\end{aligned}$$

Obviously, according to Hypothesis 3.1 and Corollary 2.1, for complex networks (3), (4), (5), there exists Filippov solution in their domain of definition.

Mark $A = \max\{A_{ik}^{(j)}, A_{ik}\}$, $B = \max\{(m-1)n(n+1)A, (m-1)\sum_{k=1}^{n}\sum_{i=1}^{m}(B_{ik} + \sum_{j=1}^{n} B_{ik}^{(j)})\}$, $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; k = 1, 2, \ldots, n)$.

Mark $x_{ik} = x_{ik}(t, \alpha)$ as the solution of equation of node i in complex network (5), $(i = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$, mark $V(t, \alpha) = \frac{1}{2}\sum\limits_{k=1}^{n}\sum\limits_{i=1}^{m}(x_{ik})^2$, obviously for an arbitrary $\alpha \in [0, +\infty)$, $x_{ik}(0, \alpha) = x_{ik}(0)$, $V(0) = V(0, \alpha) = \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}(x_{ik}(0))^2$ are constant.

Theorem 3.1. *For the complex network (5), there exists T, $0 < \frac{\pi}{2B} - \frac{1}{B}\arctan\sqrt{2V(0)} \le T \le +\infty$, when $t \in [0, T)$, for any arbitrary $\alpha \in (0, +\infty)$, $V(t, \alpha)$ is continuous in $[0, T)$, and $0 \le V(t, \alpha) \le \frac{1}{2}\tan^2(Bt + \arctan\sqrt{2V(0)}) < +\infty$. When $\alpha = 0$, if each node in complex network (5) has solutions in $[0, T_1) \subset [0, T)$, then $V(t, \alpha)$ is continuous in $[0, T_1)$, and $0 \le V(t, \alpha) \le \frac{1}{2}\tan^2(Bt + \arctan\sqrt{2V(0)}) < +\infty$.*

Proof. Obviously, there exists measurable function $\gamma_{ik}^{(j)}(t, \alpha) \in K[\mathrm{sgn}(x_{ik}^{(j)}(t, \alpha))]$ subjecting to

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \sum_{k=1}^{n}\sum_{i=1}^{m} x_{ik}\dot{x}_{ik} = \sum_{k=1}^{n}\sum_{i=1}^{m} x_{ik}\left(\sum_{l=1}^{n} f_{ik}^{(l)}(x_{i1}, x_{i2}, \ldots, x_{in})\gamma_{ik}^{(l)}(t)\right. \\
&\left. + g_{ik}(x_{i1}, x_{i2}, \ldots, x_{in}) + v_{ik}\right) \\
&= \sum_{k=1}^{n}\sum_{i=1}^{m} x_{ik}\left(\sum_{l=1}^{n} f_{ik}^{(l)}(x_{i1}, x_{i2}, \ldots, x_{in})\gamma_{ik}^{(l)}(t) + g_{ik}(x_{i1}, x_{i2}, \ldots, x_{in})\right) \\
&+ \sum_{k=1}^{n}\sum_{i=1}^{m} x_{ik}\frac{G(1+W)}{R+\alpha}\left(\sum_{l=1}^{m}(x_{lk} - x_{ik})\right) \\
&= \sum_{k=1}^{n}\sum_{i=1}^{m} x_{ik}\left(\sum_{l=1}^{n} f_{ik}^{(l)}(x_{i1}, x_{i2}, \ldots, x_{in})\gamma_{ik}^{(l)}(t) + g_{ik}(x_{i1}, x_{i2}, \ldots, x_{in})\right) \\
&- \frac{G(1+W)}{R+\alpha}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk} - x_{ik})^2 \\
&\le \sqrt{2V}\sqrt{\sum_{k=1}^{n}\sum_{i=1}^{m}\left(\sum_{l=1}^{n} f_{ik}^{(l)}(x_{i1}, x_{i2}, \ldots, x_{in})\gamma_{ik}^{(l)}(t) + g_{ik}(x_{i1}, x_{i2}, \ldots, x_{in})\right)^2} \\
&- \frac{G(1+W)}{R+\alpha}R^2
\end{aligned}$$

$$
\begin{aligned}
&\le \sqrt{2V}\sum_{k=1}^{n}\sum_{i=1}^{m}\left|\sum_{l=1}^{n} f_{ik}^{(l)}(x_{i1},x_{i2},\ldots,x_{in})\gamma_{ik}^{(l)}(t)+g_{ik}(x_{i1},x_{i2},\ldots,x_{in})\right| \\
&-\frac{G(1+W)}{R+\alpha}R^2 \\
&\le \sqrt{2V}\sum_{k=1}^{n}\sum_{i=1}^{m}\left(B_{ik}+A_{ik}\sum_{l=1}^{n}(x_{il})^2+\sum_{j=1}^{n}\left(B_{ik}^{(j)}+A_{ik}^{(j)}\sum_{l=1}^{n}(x_{il})^2\right)\right) \\
&-\frac{G(1+W)}{R+\alpha}R^2 \\
&\le \sqrt{2V}\sum_{k=1}^{n}\sum_{i=1}^{m}\left(A_{ik}\sum_{l=1}^{n}(x_{il})^2+\sum_{j=1}^{n}\left(A_{ik}^{(j)}\sum_{l=1}^{n}(x_{il})^2\right)\right) \\
&+\sqrt{2V}\sum_{k=1}^{n}\sum_{i=1}^{m}\left(B_{ik}+\sum_{j=1}^{n}B_{ik}^{(j)}\right)-\frac{G(1+W)}{R+\alpha}R^2 \\
&\le 2n(n+1)AV\sqrt{2V}+\sqrt{2V}\sum_{k=1}^{n}\sum_{i=1}^{m}\left(B_{ik}+\sum_{j=1}^{n}B_{ik}^{(j)}\right)-\frac{G(1+W)}{R+\alpha}R^2 \\
&\le \sqrt{2V}B(1+2V)-\frac{G(1+W)}{R+\alpha}R^2 \le \sqrt{2V}B(1+2V)\,.
\end{aligned}
$$

Let $P=\sqrt{2V}$, thus

$$
\frac{\mathrm{d}P}{\mathrm{d}t} \le B\left(1+P^2\right)\,, \tag{6}
$$

then $\arctan P \le Bt+\arctan P(0)$. Therefore, when $t<\frac{\pi}{2B}-\frac{1}{B}\arctan P(0)$, $P\le\tan(Bt+\arctan P(0))$, *i.e.*, $V\le\frac{1}{2}\tan^2(Bt+\arctan P(0))<+\infty$. Thus, there exists T satisfying $0<\frac{\pi}{2B}-\frac{1}{B}\arctan P(0)\le T\le+\infty$, when $t\in[0,T)$, $0\le V(t)\le\frac{1}{2}\tan^2(Bt+\arctan P(0))=\frac{1}{2}\tan^2(Bt+\arctan\sqrt{2V(0)}\,)<+\infty$. □

For complex network (5), mark $H(t,\alpha)=\frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}(t)-x_{ik}(t))^2$, obviously $H(0)=H(0,\alpha)=\frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}(0)-x_{ik}(0))^2$ is constant, $(\alpha\in[0,+\infty))$.

Lemma 3.1. *For complex network (4), there exists an inherent constant G_0 associated only with network (3), when $G>(1+\frac{2\sqrt{2H(0)}}{\pi-2\arctan\sqrt{2H(0)}})G_0$, there exist $0<t_G\le\frac{\sqrt{2H(0)}}{m(G-G_0)}<T$ and point X_0, when $t\to t_G^-$, $X_i=(x_{i1}(t),x_{i2}(t),\ldots,x_{in}(t))\to X_0$, $(i=1,2,\ldots,m)$.*

The proof for this will be given in Appendix A.

Mark $X_0 = (x_1(t_G), x_2(t_G), \dots, x_n(t_G)) = (x_1(t_G, 0), x_2(t_G, 0), \dots, x_n(t_G, 0))$.

Corollary 3.1. *For complex network (5), for any arbitrary constant $\varepsilon > 0$, there exists sufficiently small $\alpha > 0$ subjecting to*

$$\sum_{k=1}^{n}\sum_{i=1}^{m}(x_{ik}(t_G,\alpha) - x_k(t_G,0))^2 < \varepsilon\,, \quad \textit{and}$$
$$\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}(t_G,\alpha) - x_{ik}(t_G,\alpha))^2 < 2m\sqrt{\varepsilon}\,.$$

From Lemmas 2.2 and 3.1, Corollary 3.1 can be proved easily.

Theorem 3.2. *For complex network (5) and sufficiently small $\alpha > 0$, if there exists the time $t_1 \in (0, T)$ subjecting to $\frac{\sqrt{2}G_0}{2(G-G_0)}\alpha < \sqrt{H(t_1)} \le \sqrt{H(0)}$, then when*

$$t \in \left(t_1 + \frac{\sqrt{2H(t_1)}}{m(G-G_0)}, T\right),$$
$$0 \le H(t) \le \frac{1}{2}\left[\frac{G_0}{G-G_0} + \frac{G \times e^{\frac{1}{2}\ln H(0) - \frac{G_0}{G}}}{(G-G_0)\left(\frac{\sqrt{2}m}{2}(G-G_0)(t-t_1) - \sqrt{H(t_1)}\right)}\right]^2 \alpha^2\,.$$

Here, G_0 is only related with system (3) but does not depend on α.

Proof. For complex network (5), from an analogous deduction like (18),

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &\le -\frac{mG(1+W)}{R+\alpha}R^2 + R(1+W)B = -\left(mG\frac{R}{R+\alpha} - B\right)(1+W)R \\ &\le -\left(mG\frac{R}{R+\alpha} - B\right)R = -\left(mG\frac{R}{R+\alpha} - B\right)\sqrt{2H}\,,\end{aligned}$$

then

$$\frac{\mathrm{d}Q}{\mathrm{d}t} \le -\frac{\sqrt{2}}{2}\left(mG\frac{Q}{Q+\beta} - B\right) = -\frac{\sqrt{2}m}{2}\left(G\frac{Q}{Q+\beta} - G_0\right), \qquad (7)$$

here, $A = \max\{A_{ik}^{(j)}, A_{ik}\}$, $B = \max\{(m-1)n(n+1)A, (m-1)\sum_{k=1}^{n}\sum_{i=1}^{m}(B_{ik} + \sum_{j=1}^{n} B_{ik}^{(j)})\}$, $\beta = \frac{\sqrt{2}}{2}\alpha$, $Q = \sqrt{H}$, $G_0 = \frac{B}{m}$, $(i = 1, 2, \dots, m;$

$j = 1, 2, \ldots, n; k = 1, 2, \ldots, n$). When $G > G_0$, we investigate differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\sqrt{2}m}{2}\left(G\frac{y}{y+\beta} - G_0\right) \tag{8}$$

satisfying $y(t_1) = Q(t_1) = \sqrt{H(t_1)}$.

For a sufficiently small α, from the known condition $y(t_1) > \frac{G_0}{G-G_0}\beta$, the solution of (8) is

$$y + \frac{G}{G-G_0}\beta \ln\left(y - \frac{G_0}{G-G_0}\beta\right) = -\frac{\sqrt{2}m}{2}(G-G_0)t + c_1\,, \tag{9}$$

where $c_1 = \frac{\sqrt{2}m}{2}(G-G_0)t_1 + y(t_1) + \frac{G}{G-G_0}\beta \ln(y(t_1) - \frac{G_0}{G-G_0}\beta)$. From Eq. (9), y is monotonically decreasing and when $t \to +\infty$, y gets close to $\frac{G_0}{G-G_0}\beta$, so $\frac{G_0}{G-G_0}\beta < y \leq y(t_1)$. From Eq. (9), we can also get

$$\begin{aligned}
&\frac{G}{G-G_0}\beta \ln\left(y - \frac{G_0}{G-G_0}\beta\right) = -\frac{\sqrt{2}m}{2}(G-G_0)t + c_1 - y \\
&\leq -\frac{\sqrt{2}m}{2}(G-G_0)t + c_1 - \frac{G_0}{G-G_0}\beta \\
&\leq -\frac{\sqrt{2}m}{2}(G-G_0)(t-t_1) + y(t_1) + \frac{G}{G-G_0}\beta \ln(y(t_1)) - \frac{G_0}{G-G_0}\beta\,,
\end{aligned}$$

so

$$\begin{aligned}
\ln\left(y - \frac{G_0}{G-G_0}\beta\right) \;\leq\; & -\frac{G-G_0}{G}\left(\frac{\sqrt{2}m}{2}(G-G_0)(t-t_1) - y(t_1)\right)\frac{1}{\beta} \\
& + \ln(y(t_1)) - \frac{G_0}{G}\,,
\end{aligned}$$

then

$$y \leq \frac{G_0}{G-G_0}\beta + e^{-\frac{G-G_0}{G}\left(\frac{\sqrt{2}m}{2}(G-G_0)(t-t_1) - y(t_1)\right)\frac{1}{\beta} + \ln(y(t_1)) - \frac{G_0}{G}}\,.$$

From Eq. (7) and Lemma 2.3,

$$Q \leq y \leq \frac{G_0}{G-G_0}\beta + e^{-\frac{G-G_0}{G}\left(\frac{\sqrt{2}m}{2}(G-G_0)(t-t_1) - y(t_1)\right)\frac{1}{\beta} + \ln(y(t_1)) - \frac{G_0}{G}}\,.$$

When β is sufficiently small,

$$\begin{aligned}\sqrt{H} &\le \frac{G_0}{G-G_0}\beta + e^{\ln(y(t_1))-\frac{G_0}{G}} \frac{1}{\frac{G-G_0}{G}\left(\frac{\sqrt{2}m}{2}(G-G_0)(t-t_1)-y(t_1)\right)\frac{1}{\beta}} \\ &= \left[\frac{G_0}{G-G_0} + \frac{Ge^{\ln(y(t_1))-\frac{G_0}{G}}}{(G-G_0)\left(\frac{\sqrt{2}m}{2}(G-G_0)(t-t_1)-y(t_1)\right)}\right]\beta\,.\end{aligned}$$

Since $y(t_1)=\sqrt{H(t_1,\alpha)}\le\sqrt{H(0)}$, $\beta=\frac{\sqrt{2}}{2}\alpha$, so

$$0\le H(t,\alpha)\le\frac{1}{2}\left[\frac{G_0}{G-G_0}+\frac{Ge^{\frac{1}{2}\ln(H(0))-\frac{G_0}{G}}}{(G-G_0)\left(\frac{\sqrt{2}m}{2}(G-G_0)(t-t_1)-\sqrt{H(t_1)}\right)}\right]^2\alpha^2\,.$$

□

Corollary 3.2. *For complex network (5) and sufficiently small $0<\varepsilon<1$, there exists $0<\delta\le\varepsilon$ when $0<\alpha\le\delta$, (i) $H(t,\alpha)<D\varepsilon$ in $[t_G,T)$ except a small region I with the length less than $C\sqrt{\varepsilon}$, here C,D are positive constants, which are only related with G,G_0 and m. (ii) For arbitrary b, $(0<t_G<b<T)$, there exists a positive constant E, which is only related with G,G_0 and m,b subjecting to $H(t,\alpha)<E\varepsilon$ in the region $[t_G,b]\subset[t_G,T)$.*

Proof will be shown in Appendix B.

Suppose $x_{ik}=x_{ik}(t,\alpha)$ is the Filippov solution of the node i of complex network (5), $(i=1,2,\ldots,m;k=1,2,\ldots,n)$, then

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sum_{i=1}^m x_{i1}}{m}\right)=\frac{1}{m}\sum\limits_{j=1}^n\sum\limits_{i=1}^m f_{i1}^{(j)}\mathrm{sgn}(x_{ij})+\frac{1}{m}\sum\limits_{i=1}^m g_{i1}\,,\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sum_{i=1}^m x_{i2}}{m}\right)=\frac{1}{m}\sum\limits_{j=1}^n\sum\limits_{i=1}^m f_{i2}^{(j)}\mathrm{sgn}(x_{ij})+\frac{1}{m}\sum\limits_{i=1}^m g_{i2}\,,\\ \quad\cdots\quad\cdots\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sum_{i=1}^m x_{in}}{m}\right)=\frac{1}{m}\sum\limits_{j=1}^n\sum\limits_{i=1}^m f_{in}^{(j)}\mathrm{sgn}(x_{ij})+\frac{1}{m}\sum\limits_{i=1}^m g_{in}\,.\end{cases}$$

Here, $f_{ik}^{(j)}=f_{ik}^{(j)}(x_{i1},x_{i2},\ldots,x_{in})$, $g_{ik}=g_{ik}(x_{i1},x_{i2},\ldots,x_{in})$, $(i=1,2,\ldots,m;$ $j,k=1,2,\ldots,n)$. Obviously, $(\frac{\sum_{i=1}^m x_{i1}}{m},\frac{\sum_{i=1}^m x_{i2}}{m},\ldots,\frac{\sum_{i=1}^m x_{in}}{m},\alpha)^T$ is the Filippov solution of system (10) with the initial condition
$y_0^*(t_G)=(\frac{\sum_{i=1}^m x_{i1}(t_G,\alpha)}{m},\frac{\sum_{i=1}^m x_{i2}(t_G,\alpha)}{m},\ldots,\frac{\sum_{i=1}^m x_{in}(t_G,\alpha)}{m},\alpha)^T$.

$$\begin{cases} \frac{\mathrm{d}y_1}{\mathrm{d}t} = \frac{1}{m}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m} f_{i1}^{(j)}(y_1, y_2, \ldots, y_n)\mathrm{sgn}(y_j - \beta_{ij}(t, y_{n+1})) \\ \qquad\quad +\frac{1}{m}\sum\limits_{i=1}^{m} g_{i1}(y_1, y_2, \ldots, y_n) + \gamma_1(t, y_{n+1})\,, \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} = \frac{1}{m}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m} f_{i2}^{(j)}(y_1, y_2, \ldots, y_n)\mathrm{sgn}(y_j - \beta_{ij}(t, y_{n+1})) \\ \qquad\quad +\frac{1}{m}\sum\limits_{i=1}^{m} g_{i2}(y_1, y_2, \ldots, y_n) + \gamma_2(t, y_{n+1})\,, \\ \quad \cdots \quad \cdots \\ \frac{\mathrm{d}y_n}{\mathrm{d}t} = \frac{1}{m}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m} f_{in}^{(j)}(y_1, y_2, \ldots, y_n)\mathrm{sgn}(y_j - \beta_{ij}(t, y_{n+1})) \\ \qquad\quad +\frac{1}{m}\sum\limits_{i=1}^{m} g_{in}(y_1, y_2, \ldots, y_n) + \gamma_n(t, y_{n+1})\,, \\ \frac{\mathrm{d}y_{n+1}}{\mathrm{d}t} = 0\,. \end{cases} \tag{10}$$

Here,

$$\begin{aligned} &\beta_{ij}(t, y_{n+1}) = \frac{\sum_{l=1}^{m} x_{lj}(t, y_{n+1})}{m} - x_{ij}(t, y_{n+1})\,, \\ &\gamma_k(t, y_{n+1}) = -\frac{1}{m}\sum_{j=1}^{n}\sum_{i=1}^{m}\Bigg[f_{ik}^{(j)}\Bigg(\frac{\sum_{i=1}^{m} x_{i1}(t, y_{n+1})}{m}, \frac{\sum_{i=1}^{m} x_{i2}(t, y_{n+1})}{m}, \ldots, \\ &\frac{\sum_{i=1}^{m} x_{in}(t, y_{n+1})}{m}\Bigg) - f_{ik}^{(j)}(x_{i1}(t, y_{n+1}), x_{i2}(t, y_{n+1}), \ldots, x_{in}(t, y_{n+1}))\Bigg] \\ &\times\mathrm{sgn}(x_{ij}(t, y_{n+1})) \\ &-\frac{1}{m}\sum_{i=1}^{m}\Bigg[g_{ik}\Bigg(\frac{\sum_{i=1}^{m} x_{i1}(t, y_{n+1})}{m}, \frac{\sum_{i=1}^{m} x_{i2}(t, y_{n+1})}{m}, \ldots, \frac{\sum_{i=1}^{m} x_{in}(t, y_{n+1})}{m}\Bigg) \\ &-g_{ik}(x_{i1}(t, y_{n+1}), x_{i2}(t, y_{n+1}), \ldots, x_{in}(t, y_{n+1}))\Bigg], \\ &(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; k = 1, 2, \ldots, n)\,. \end{aligned}$$

Mark $y = (y_1, y_2, \ldots, y_{n+1})^T$, $f^*(t, y) = (f_1^*(t, y), f_2^*(t, y), \ldots, f_n^*(t, y), 0)^T$, $f_k^*(t, y) = \frac{1}{m}\sum_{j=1}^{n}\sum_{i=1}^{m} f_{ik}^{(j)}(y_1, y_2, \ldots, y_n)\mathrm{sgn}(y_j - \beta_{ij}(t, y_{n+1})) + \frac{1}{m}\sum_{i=1}^{m} g_{ik}(y_1, y_2, \ldots, y_n) + \gamma_k(t, y_{n+1})$, $(k = 1, 2, \ldots, n)$, then the solution $y(t)$ of system (10) satisfies the ordinary differential inclusion, $\dot{y} \in F^*(t, y)$, $F^*(t, y) = Kf^*(t, y) = \bigcap\limits_{\delta>0}\bigcap\limits_{\mu N=0} \overline{\mathrm{co}} f^*(t, y^\delta \backslash N)$.

Let us consider a system

$$\begin{cases} \frac{\mathrm{d}y_1}{\mathrm{d}t} = \frac{1}{m}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m} f_{i1}^{(j)}(y_1, y_2, \ldots, y_n)\mathrm{sgn}(y_j) + \frac{1}{m}\sum\limits_{i=1}^{m} g_{i1}(y_1, y_2, \ldots, y_n)\,, \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} = \frac{1}{m}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m} f_{i2}^{(j)}(y_1, y_2, \ldots, y_n)\mathrm{sgn}(y_j) + \frac{1}{m}\sum\limits_{i=1}^{m} g_{i2}(y_1, y_2, \ldots, y_n)\,, \\ \quad \cdots \quad \cdots \\ \frac{\mathrm{d}y_n}{\mathrm{d}t} = \frac{1}{m}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m} f_{in}^{(j)}(y_1, y_2, \ldots, y_n)\mathrm{sgn}(y_j) + \frac{1}{m}\sum\limits_{i=1}^{m} g_{in}(y_1, y_2, \ldots, y_n)\,, \\ \frac{\mathrm{d}y_{n+1}}{\mathrm{d}t} = 0\,, \end{cases} \tag{11}$$

with an initial condition $y_0(t_G) = (x_1(t_G), x_2(t_G), \ldots, x_n(t_G), 0)^T$. Mark

$$\begin{aligned} y &= (y_1, y_2, \ldots, y_{n+1})^T\,, \\ f(t, y) &= (f_1(t, y), f_2(t, y), \ldots, f_n(t, y), 0)^T\,, \\ f_k(t, y) &= \frac{1}{m}\sum_{j=1}^{n}\sum_{i=1}^{m} f_{ik}^{(j)}(y_1, y_2, \ldots, y_n)\mathrm{sgn}(y_j) + \frac{1}{m}\sum_{i=1}^{m} g_{ik}(y_1, y_2, \ldots, y_n)\,, \\ &\quad (k = 1, 2, \ldots, n)\,, \end{aligned}$$

then the solution $y(t)$ of system (11) satisfies the ordinary differential inclusion, $\dot{y} \in F(t, y), F(t, y) = Kf(t, y) = \bigcap\limits_{\delta>0}\bigcap\limits_{\mu N=0} \overline{\mathrm{co}} f(t, y^{\delta}\backslash N)$. For any arbitrary b, $(0 < t_G < b < T)$, for system (11), from a similar proof of Theorem 3.1, $\frac{1}{1+P_1^2}\frac{\mathrm{d}P_1}{\mathrm{d}t} \le \frac{1}{m}B \le B$ in $[t_G, b] \subset [t_G, T)$, then $\arctan P_1 \le Bt - Bt_G + \arctan P_1(t_G)$ in $[t_G, b] \subset [t_G, T)$, so

$$P_1 \le \tan(Bt - Bt_G + \arctan P_1(t_G)) < +\infty\,. \tag{12}$$

From Theorem 3.1,

$$P_1(t_G) \le \tan(Bt_G + \arctan P(0)) < +\infty\,. \tag{13}$$

Substituting (13) into (12), $P_1 \le \tan(Bt + \arctan P(0)) < +\infty$, $P_1 = \sqrt{\sum_{k=1}^{n}(y_k)^2}$, *i.e.*, $\sum_{k=1}^{n}(y_k)^2 \le \tan^2(Bt + \arctan P(0)) = \tan^2(Bt + \arctan\sqrt{2V(0)}) < +\infty$, then $\sum_{k=1}^{n+1}(y_k)^2 = \sum_{k=1}^{n}(y_k)^2 + (y_{n+1})^2 \le \tan^2(Bt + \arctan\sqrt{2V(0)}) < +\infty$. So, for any arbitrary b, $(0 < t_G < b < T)$, and sufficiently small $\varepsilon > 0$, the initial condition $y_0(t_G) = (x_1(t_G), x_2(t_G), \ldots, x_n(t_G), 0)^T$, there are solutions satisfying the ordinary differential inclusion in $[t_G, b] \subset [t_G, T)$, and all the solutions in the open region $\Delta = \{(t, y)|t_G - \varepsilon < t < b + \varepsilon, \sum_{k=1}^{n+1}(y_k)^2 < \tan^2(B(b + \varepsilon) + \arctan\sqrt{2V(0)}) < +\infty\}$.

Lemma 3.2. *$F^*(t,y)$ and $F(t,y)$ meet the basic conditions in the open domain Δ.*

Detail of proof will be shown in Appendix C.

Definition 3.1. For complex network (4), if there exists a system

$$\begin{cases} \dot{x}_1 = h_1(x_1, x_2, \ldots, x_n)\,, \\ \dot{x}_2 = h_2(x_1, x_2, \ldots, x_n)\,, \\ \quad \cdots \quad \cdots \\ \dot{x}_n = h_n(x_1, x_2, \ldots, x_n) \end{cases} \tag{14}$$

for any arbitrary $\varepsilon > 0$, if there exists $\delta > 0$, when $0 < \alpha \leq \delta$, for any solution of the node i of complex network (5) $x_i(t)$, there exists a solution of system (14) $x'(t)$ subjecting to $\max\limits_{t_G \leq t \leq b} \parallel x_i(t) - x'(t) \parallel \leq \varepsilon$, $(i = 1, 2, \ldots, m)$, then the complex network (4) under gravity field synchronizes to system (14) in the solution set.

Theorem 3.3. *There exist $G_1 > 0$ and $0 < T \leq +\infty$, when $G > G_1$ and there exists t_G, when $t \in (t_G, T]$, complex network (4) synchronizes to the system with initial condition of $X_0 = (x_1(t_G), x_2(t_G), \ldots, x_n(t_G))^T$ according to the solution set*

$$\begin{cases} \dot{x}_1 = \frac{1}{m} \sum\limits_{j=1}^{n} \sum\limits_{i=1}^{m} f_{i1}^{(j)}(x_1, x_2, \ldots, x_n) \mathrm{sgn}(x_j) + \frac{1}{m} \sum\limits_{i=1}^{m} g_{i1}(x_1, x_2, \ldots, x_n)\,, \\ \dot{x}_2 = \frac{1}{m} \sum\limits_{j=1}^{n} \sum\limits_{i=1}^{m} f_{i2}^{(j)}(x_1, x_2, \ldots, x_n) \mathrm{sgn}(x_j) + \frac{1}{m} \sum\limits_{i=1}^{m} g_{i2}(x_1, x_2, \ldots, x_n)\,, \\ \quad \cdots \quad \cdots \\ \dot{x}_n = \frac{1}{m} \sum\limits_{j=1}^{n} \sum\limits_{i=1}^{m} f_{in}^{(j)}(x_1, x_2, \ldots, x_n) \mathrm{sgn}(x_j) + \frac{1}{m} \sum\limits_{i=1}^{m} g_{in}(x_1, x_2, \ldots, x_n)\,. \end{cases} \tag{15}$$

The larger is the G, the faster is the synchronization. Specifically, if the solution of system (15) $x'(t)$ is unique, when $t \in (t_G, T)$, complex network (4) synchronizes to $x'(t)$.

Proof. From Lemma 3.1, for complex network (4), there exist a constant $G_1 = (1 + \frac{2\sqrt{2H(0)}}{\pi - 2\arctan\sqrt{2V(0)}})G_0$ and T, when $G > G_1$, there exist $0 < t_G \leq \frac{\sqrt{2H(0)}}{m(G-G_0)} < T$ and the point $X_0 = (x_1(t_G), x_2(t_G), \ldots, x_n(t_G))^T$, when $t \to t_G^-$, $X_i = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \to X_0$. Suppose $y(t) = (x_1(t), x_2(t), \ldots, x_n(t), 0)^T$ is the arbitrary solution of the ordinary differential inclusion $\dot{y} \in F(t, y)$ satisfying the initial data $y_0(t_G) = (x_1(t_G), x_2(t_G),$

$\ldots, x_n(t_G), 0)^T$. For system (10), when $t \in [t_G, b] \subset [t_G, T)$, for any arbitrary sufficient small $\varepsilon > 0$, from Hypothesis 3.1 and Corollary 3.2, there exist $\delta_1 > 0$, $\delta_2 > 0$, when

$$
\begin{aligned}
& t = t_G \qquad \text{and} \qquad \| \, y_0^*(t_G) - y_0(t_G) \, \| \leq \delta_1 \,, \\
& 0 < \alpha \leq \sqrt{\delta_2} \,, \\
& |\beta_{ij}(t, y_{n+1})| = \left| \frac{\sum_{l=1}^{m} x_{lj}(t, y_{n+1})}{m} - x_{ij}(t, y_{n+1}) \right| \\
& \leq \frac{\sum_{l=1}^{m} |x_{lj}(t, y_{n+1}) - x_{ij}(t, y_{n+1})|}{m} \leq \delta_2 \,, \\
& |\gamma_k(t, y_{n+1})| = \Bigg| - \frac{1}{m} \sum_{j=1}^{n} \sum_{i=1}^{m} \\
& \times \left[f_{ik}^{(j)} \left(\frac{\sum_{i=1}^{m} x_{i1}(t, y_{n+1})}{m}, \frac{\sum_{i=1}^{m} x_{i2}(t, y_{n+1})}{m}, \ldots, \frac{\sum_{i=1}^{m} x_{in}(t, y_{n+1})}{m} \right) \right. \\
& \left. - f_{ik}^{(j)}(x_{i1}(t, y_{n+1}), x_{i2}(t, y_{n+1}), \ldots, x_{in}(t, y_{n+1})) \right] \times \operatorname{sgn}(x_{ij}(t, y_{n+1})) \\
& - \frac{1}{m} \sum_{i=1}^{m} \left[g_{ik} \left(\frac{\sum_{i=1}^{m} x_{i1}(t, y_{n+1})}{m}, \frac{\sum_{i=1}^{m} x_{i2}(t, y_{n+1})}{m}, \ldots, \frac{\sum_{i=1}^{m} x_{in}(t, y_{n+1})}{m} \right) \right. \\
& \left. - g_{ik}(x_{i1}(t, y_{n+1}), x_{i2}(t, y_{n+1}), \ldots, x_{in}(t, y_{n+1})) \right] \Bigg| \\
& \leq \frac{1}{m} \sum_{j=1}^{n} \sum_{i=1}^{m} \\
& \times \left| f_{ik}^{(j)} \left(\frac{\sum_{i=1}^{m} x_{i1}(t, y_{n+1})}{m}, \frac{\sum_{i=1}^{m} x_{i2}(t, y_{n+1})}{m}, \ldots, \frac{\sum_{i=1}^{m} x_{in}(t, y_{n+1})}{m} \right) \right. \\
& \left. - f_{ik}^{(j)}(x_{i1}(t, y_{n+1}), x_{i2}(t, y_{n+1}), \ldots, x_{in}(t, y_{n+1})) \right| \\
& + \frac{1}{m} \sum_{i=1}^{m} \left| g_{ik} \left(\frac{\sum_{i=1}^{m} x_{i1}(t, y_{n+1})}{m}, \frac{\sum_{i=1}^{m} x_{i2}(t, y_{n+1})}{m}, \ldots, \frac{\sum_{i=1}^{m} x_{in}(t, y_{n+1})}{m} \right) \right. \\
& \left. - g_{ik}(x_{i1}(t, y_{n+1}), x_{i2}(t, y_{n+1}), \ldots, x_{in}(t, y_{n+1})) \right| \leq \delta_2 \,, \\
& (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; k = 1, 2, \ldots, n) \,.
\end{aligned}
$$

Let $\delta = \max\{\delta_1, \delta_2\}$, like in the proof of Lemma 3.2 (*cf.* Appendix C), $F^*(t,y) \subseteq [\mathrm{co}F(t^\delta, y^\delta)]^\delta$ and from Lemma 3.2, $F^*(t,y)$ and $F(t,y)$ satisfy the basic condition in the open region Δ. Therefore, from Lemma 2.2, for sufficiently small $\varepsilon > 0$, there exists $\delta > 0$, when $0 < \alpha \leq \delta$, any solution $y^*(t)$ of the ordinary differential inclusion $\dot{y} \in F^*(t,y)$ satisfying the initial condition $y_0^*(t_G) = (\frac{\sum_{i=1}^m x_{i1}(t_G,\alpha)}{m}, \frac{\sum_{i=1}^m x_{i2}(t_G,\alpha)}{m}, \ldots, \frac{\sum_{i=1}^m x_{in}(t_G,\alpha)}{m}, \alpha)^T$ exists in the region $[t_G, b]$, and for each solution $y^*(t)$ there exists a solution $y(t)$ of the ordinary differential inclusion $\dot{y} \in F(t,y)$ with the initial data $y_0(t_G) = (x_1(t_G), x_2(t_G), \ldots, x_n(t_G), 0)^T$, subjecting to $\max\limits_{t_G \leq t \leq b} \parallel y(t) - y^*(t) \parallel \leq \varepsilon$. Since $y_1^*(t) = (\frac{\sum_{i=1}^m x_{i1}(t,\alpha)}{m}, \frac{\sum_{i=1}^m x_{i2}(t,\alpha)}{m}, \ldots, \frac{\sum_{i=1}^m x_{in}(t,\alpha)}{m}, \alpha)^T$ is the solution of the ordinary differential inclusion $\dot{y} \in F^*(t,y)$ satisfying the initial data $y_0^*(t_G) = (\frac{\sum_{i=1}^m x_{i1}(t_G,\alpha)}{m}, \frac{\sum_{i=1}^m x_{i2}(t_G,\alpha)}{m}, \ldots, \frac{\sum_{i=1}^m x_{in}(t_G,\alpha)}{m}, \alpha)^T$, then for sufficiently small $\varepsilon > 0$, there exists $\delta_3 > 0$, when $0 < \alpha \leq \delta_3$, $\max\limits_{t_G \leq t \leq b} \parallel y(t) - y_1^*(t) \parallel \leq \frac{1}{2}\varepsilon$.

Mark $x_i^*(t) = (x_{i1}(t,\alpha), x_{i2}(t,\alpha), \ldots, x_{in}(t,\alpha), \alpha)^T$, from Corollary 3.2 (*ii*), there exists $\delta_4 > 0$, when $0 < \alpha \leq \delta_4$, $\max\limits_{t_G \leq t \leq b} \parallel x_i^*(t) - y_1^*(t) \parallel \leq \frac{1}{2}\varepsilon$. Mark $\delta = \{\delta_3, \delta_4\}$, when $0 < \alpha \leq \delta$, $\max\limits_{t_G \leq t \leq b} \parallel x_i^*(t) - y(t) \parallel \leq \varepsilon$. Mark $y'(t) = (y_1, y_2, \ldots, y_n)^T$, $x_i(t) = (x_{i1}(t,\alpha), x_{i2}(t,\alpha), \ldots, x_{in}(t,\alpha))^T$, than $\max\limits_{t_G \leq t \leq b} \parallel x_i(t) - y'(t) \parallel \leq \varepsilon$. Here, $y'(t)$ is the solution of system (15), $x_i'(t)$ is the solution of the node i of complex network (5) $(i = 1, 2, 3, \ldots, m)$, therefore, complex network (4) synchronizes to system (15) according to the solution set. From Lemma 3.1, the larger G leads to a faster synchronization. □

Note: In the proof of Theorem 3.3, to show the existence of the constant G_1, let $G_1 = (1 + \frac{2\sqrt{2H(0)}}{\pi - 2\arctan\sqrt{2V(0)}})G_0$. However, generally $G_1 < (1 + \frac{2\sqrt{2H(0)}}{\pi - 2\arctan\sqrt{2V(0)}})G_0$. In addition, if there exists a $f_{ik}^{(j)}(x_1, x_2, \ldots, x_n)$, $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; k = 1, 2, \ldots, n)$ equaling zero, the above theory is still true.

4. Numerical simulation

Let us consider piecewise linear Chen systems (a) with the number of m_1, piecewise linear Sprott systems (b) with the number of m_2, piecewise Lorenz systems (c) with the number of m_3,

$$\text{(a)}\begin{cases}\dot{x}_{i1}=-1.18x_{i1}+1.18x_{i2}\,,\\ \dot{x}_{i2}=(1-x_{i3})\mathrm{sgn}(x_{i1})+0.7x_{i2}\,,\\ \dot{x}_{i3}=x_{i1}\mathrm{sgn}(x_{i2})-0.168x_{i3}\,,\end{cases}$$

$$\text{(b)}\begin{cases}\dot{x}_{i1}=x_{i3}\mathrm{sgn}(x_{i2})\,,\\ \dot{x}_{i2}=x_{i1}-x_{i2}\,,\\ \dot{x}_{i3}=1-|x_{i2}|\,,\end{cases}$$

$$\text{(c)}\begin{cases}\dot{x}_{i1}=0.9(-x_{i1}+x_{i2})\,,\\ \dot{x}_{i2}=(2-x_{i3})\mathrm{sgn}(x_{i1})\,,\\ \dot{x}_{i3}=|x_{i1}|-0.1x_{i3}\,,\end{cases} \tag{16}$$

as the nodes with the number of $m=m_1+m_2+m_3$ in complex network in the gravity field. Here, $v_{ik}=G\times(1+W(t))(\sum_{j=1}^{m}\frac{1}{R}(x_{jk}(t)-x_{ik}(t)))$, $W(t)=\sum_{i=1}^{m}\sum_{k=1}^{3}(x_{ik}(t))^2$, $R(t)=\sqrt{\sum_{k=1}^{3}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}(t)-x_{ik}(t))^2}$, $x_i^0=(x_{i1}(0),x_{i2}(0),\ldots,x_{in}(0))^T\neq 0$, ($x_i^0$ is of inequality) is the initial position of the node i of complex networks (5) and (10) $(i=1,2,\ldots,m;k=1,2,3)$.

Theorem 4.1. *There exists a constant G_1, when $G>G_1$, then there exists t_G, when $t>t_G$, complex network (16) synchronize to the following system according to a solution set:*

$$\begin{cases}\dot{x}_1=\frac{m_1}{m}(-1.18x_1+1.18x_2)+\frac{m_2}{m}(x_3\mathrm{sgn}(x_2))+\frac{0.9m_3}{m}(-x_1+x_2)\,,\\ \dot{x}_2=\frac{m_1}{m}[(1-x_3)\mathrm{sgn}(x_1)+0.7x_2]+\frac{m_2}{m}(x_1-x_2)+\frac{m_3}{m}[(2-x_3)\mathrm{sgn}(x_1)]\,,\\ \dot{x}_3=\frac{m_1}{m}(x_1\mathrm{sgn}(x_2)-0.618x_3)+\frac{m_2}{m}(1-|x_2|)+\frac{m_3}{m}(|x_1|-0.1x_3)\,,\end{cases} \tag{17}$$

here, $m=m_1+m_2+m_3$, the parameter G can speed the synchronization.

This can be proven from Theorem 3.3.

4.1. Case A: $m_1=21$, $m_2=18$, $m_3=12$, $G=0.05$

Here, the initial position X_i^0 is random. The effect of synchronization is shown in Fig. 1. Here, $e_1=x_{11}-x_{i1}$, $e_2=x_{12}-x_{i2}$, $(i=2,3,\ldots,51)$.

4.2. Case B: $m_1=21$, $m_2=18$, $m_3=12$, $G=1$

The synchronization effect is shown in Fig. 2. Here, $e_1=x_{11}-x_{i1}$, $e_2=x_{12}-x_{i2}$, $(i=2,3,\ldots,51)$.

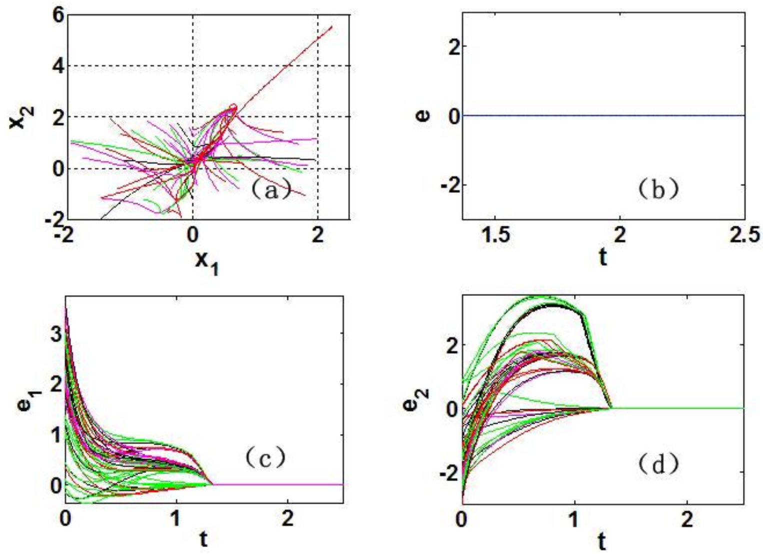

Fig. 1. When $m_1 = 21$, $m_2 = 18$, $m_3 = 12$, $G = 0.05$, (a) network operation state, (b) the synchronization error between network and system (17), (c) the synchronization error among the first coordinates at different node positions, (d) the synchronization error among the second coordinates at different node positions.

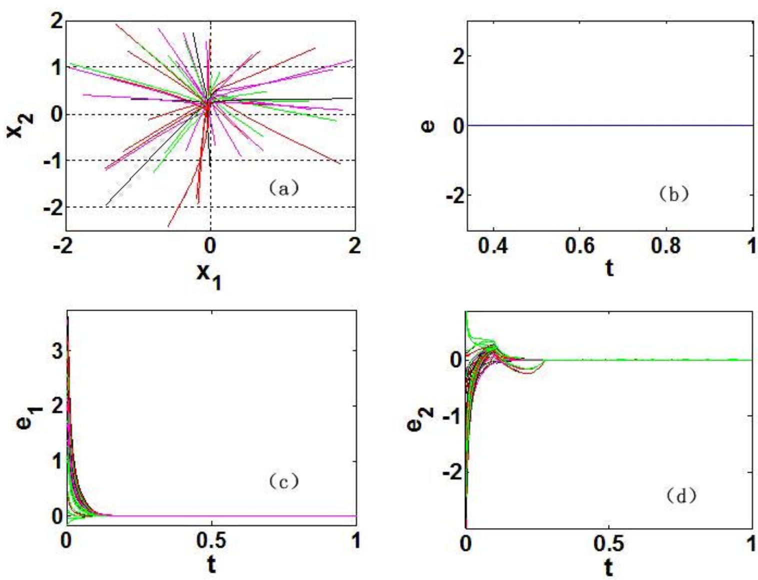

Fig. 2. When $m_1 = 21$, $m_2 = 18$, $m_3 = 12$, $G = 1$, (a) network operation state, (b) the synchronization error between network and system (17), (c) the synchronization error among the first coordinates at different node positions, (d) the synchronization error among the second coordinates at different node positions.

4.3. Case C: $m_1 = 171$, $m_2 = 168$, $m_3 = 172$, $G = 2000$

The effect of synchronization is shown in Fig. 3. Here, $e_1 = x_{11} - x_{i1}$, $e_2 = x_{12} - x_{i2}$, $(i = 2, 3, \ldots, 501)$.

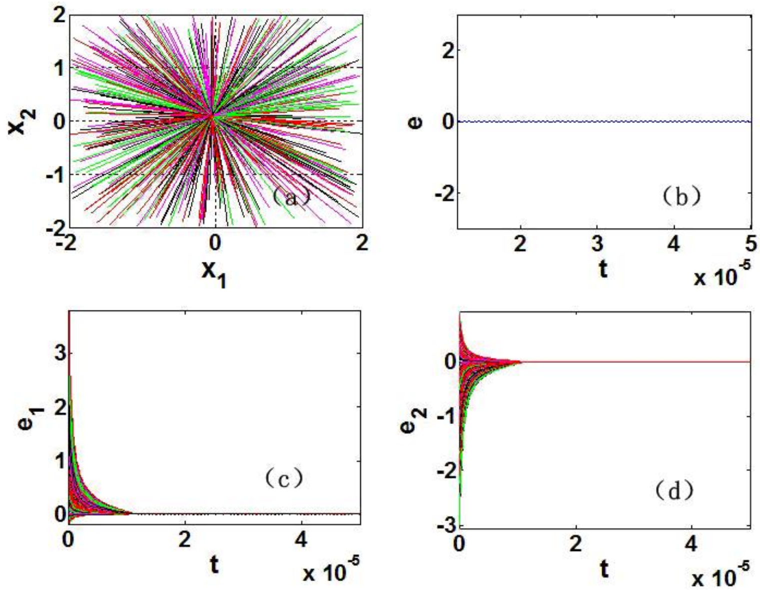

Fig. 3. When $m_1 = 171$, $m_2 = 168$, $m_3 = 172$, $G = 2000$, (a) network operation state, (b) the synchronization error between network and system (17), (c) the synchronization error among the first coordinates at different node positions, (d) the synchronization error among the second coordinates at different node positions.

Comparing figure 1 and figure 2, when G is bigger, one can see that the speed of synchronization is faster.

5. Conclusion and discussions

By introducing the concept of gravity field, the issue of synchronization of complex network turns to be the synchronization in gravity field. In this paper, we solved the question of synchronization where the dynamical equation of each node may be different and even the differential equations have discontinuous right-hand side. According to the general definition of the velocity (the coupling term), since it is independent of the equation of motion, which endues the synchronization method with more generality than other control methods. Simulations based on the complex network with 51 and 501 nodes of piecewise linear Chen systems, piecewise Sprott systems and piecewise Lorenz systems agree with the theoretic analysis.

Appendix A

Proof for Lemma 3.1

$$
\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}-x_{ik})(\dot{x}_{jk}-\dot{x}_{ik})\\
&= \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}-x_{ik})\Bigg[\sum_{h=1}^{n} f_{jk}^{(h)}(x_{j1},x_{j2},\ldots,x_{jn})\gamma_{jk}^{(h)}(t)\\
&\quad + g_{jk}(x_{j1},x_{j2},\ldots,x_{jn}) + v_{jk} - \sum_{h=1}^{n} f_{ik}^{(h)}(x_{i1},x_{i2},\ldots,x_{in})\gamma_{ik}^{(h)}(t)\\
&\quad - g_{ik}(x_{i1},x_{i2},\ldots,x_{in}) - v_{ik}\Bigg]\\
&= \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}-x_{ik})\Bigg[\sum_{h=1}^{n} f_{jk}^{(h)}(x_{j1},x_{j2},\ldots,x_{jn})\gamma_{jk}^{(h)}(t)\\
&\quad + g_{jk}(x_{j1},x_{j2},\ldots,x_{jn}) - \sum_{h=1}^{n} f_{ik}^{(h)}(x_{i1},x_{i2},\ldots,x_{in})\gamma_{ik}^{(h)}(t)\\
&\quad - g_{ik}(x_{i1},x_{i2},\ldots,x_{in})\Bigg]\\
&\quad + \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}\left\{(x_{jk}-x_{ik})\frac{G(1+W)}{R}\left[\sum_{l=1}^{m}(x_{lk}-x_{jk}) - \sum_{l=1}^{m}(x_{lk}-x_{ik})\right]\right\}\\
&= \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}-x_{ik})\Bigg[\sum_{h=1}^{n} f_{jk}^{(h)}(x_{j1},x_{j2},\ldots,x_{jn})\gamma_{jk}^{(h)}(t)\\
&\quad + g_{jk}(x_{j1},x_{j2},\ldots,x_{jn}) - \sum_{h=1}^{n} f_{ik}^{(h)}(x_{i1},x_{i2},\ldots,x_{in})\gamma_{ik}^{(h)}(t)\\
&\quad - g_{ik}(x_{i1},x_{i2},\ldots,x_{in})\Bigg] - m\times\frac{G(1+W)}{R}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}-x_{ik})^2\\
&= -mG(1+W)R\\
&\quad + \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}-x_{ik})\Bigg[\sum_{h=1}^{n} f_{jk}^{(h)}(x_{j1},x_{j2},\ldots,x_{jn})\gamma_{jk}^{(h)}(t)
\end{aligned}
$$

$$
\begin{aligned}
&+g_{jk}(x_{j1},x_{j2},\ldots,x_{jn})-\sum_{h=1}^{n}f_{ik}^{(h)}(x_{i1},x_{i2},\ldots,x_{in})\gamma_{ik}^{(h)}(t)\\
&\left.-g_{ik}(x_{i1},x_{i2},\ldots,x_{in})\right]\\
&\le -mG(1+W)R\\
&+R\sqrt{\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}\left(\sum_{h=1}^{n}f_{jk}^{(h)}\gamma_{jk}^{(h)}+g_{jk}-\sum_{h=1}^{n}f_{ik}^{(h)}\gamma_{ik}^{(h)}-g_{ik}\right)^2}\\
&\le -mG(1+W)R\\
&+R\sqrt{\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}\left(\left|\sum_{h=1}^{n}f_{jk}^{(h)}\gamma_{jk}^{(h)}+g_{jk}\right|+\left|\sum_{h=1}^{n}f_{ik}^{(h)}\gamma_{ik}^{(h)}+g_{ik}\right|\right)^2}\\
&\le -mG(1+W)R\\
&+R\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}\left(\left|\sum_{h=1}^{n}f_{jk}^{(h)}\gamma_{jk}^{(h)}+g_{jk}\right|+\left|\sum_{h=1}^{n}f_{ik}^{(h)}\gamma_{ik}^{(h)}+g_{ik}\right|\right)\\
&= -mG(1+W)R+(m-1)R\sum_{k=1}^{n}\sum_{i=1}^{m}\left|\sum_{h=1}^{n}f_{ik}^{(h)}\gamma_{ik}^{(h)}+g_{ik}\right|\\
&\le -mG(1+W)R+(m-1)R\sum_{k=1}^{n}\sum_{i=1}^{m}\left[B_{ik}+A_{ik}\sum_{l=1}^{n}(x_{il})^2\right.\\
&\left.+\sum_{h=1}^{n}\left(B_{ik}^{(h)}+A_{ik}^{(h)}\sum_{l=1}^{n}(x_{il})^2\right)\right]\\
&\le -mG(1+W)R\\
&+(m-1)R\sum_{k=1}^{n}\sum_{i=1}^{m}\left[A_{ik}\sum_{l=1}^{n}(x_{il})^2+\sum_{h=1}^{n}\left(A_{ik}^{(h)}\sum_{l=1}^{n}(x_{il})^2\right)\right]\\
&+(m-1)R\sum_{k=1}^{n}\sum_{i=1}^{m}\left(B_{ik}+\sum_{h=1}^{n}B_{ik}^{(h)}\right)\\
&\le -mG(1+W)R+(m-1)n(n+1)ARW\\
&+(m-1)R\sum_{k=1}^{n}\sum_{i=1}^{m}\left(B_{ik}+\sum_{h=1}^{n}B_{ik}^{(h)}\right)\\
&\le -mG(1+W)R+R(1+W)B\\
&= -(mG-B)(1+W)R=-\sqrt{2}(mG-B)(1+W)\sqrt{H}\,.
\end{aligned}
$$

Let $G_0 = \frac{B}{m}$, when $G > G_0 + \frac{2B\sqrt{2H(0)}}{m(\pi - 2\arctan\sqrt{2V(0)})}$,

$$\frac{\mathrm{d}H}{\mathrm{d}t} \leq -\sqrt{2}(mG - B)\sqrt{H} = -\sqrt{2}m(G - G_0)\sqrt{H}\,. \tag{18}$$

From Eq. (18), $\sqrt{H} \leq \sqrt{H(0)} - \frac{\sqrt{2}}{2}m(G - G_0)t$.

Let $[0, a)$ be the largest region subjecting to positive $H(t)$, obviously $H(t)$ decrease monotonously in $[0, a)$, and $a \leq \frac{\sqrt{2H(0)}}{m(G-G_0)} < T$. In the following, we prove $\lim\limits_{t\to a^-} H(t) = 0$.

Suppose $\lim\limits_{t\to a^-} H(t) \neq 0$, then $\lim\limits_{t\to a^-} H(t) > 0$. For any arbitrary $\varepsilon > 0$, there exists $0 < \delta < \varepsilon$, when $|t_1 - a| < \delta, |t_2 - a| < \delta, (0 < t_1, t_2 < a$, may as well think $t_1 < t_2$, obviously $|t_1 - t_2| < \delta < \varepsilon)$

$$\begin{aligned}
&|x_{ik}(t_1) - x_{ik}(t_2)| \leq \\
&\int_{t_1}^{t_2} \left| \sum_{h=1}^{n} f_{ik}^{(h)}(x_{i1}, x_{i2}, \ldots, x_{in})\gamma_{ik}^{(h)}(t) + g_{ik}(x_{i1}, x_{i2}, \ldots, x_{in}) + v_{ik} \right| \mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \left(\left| \sum_{h=1}^{n} f_{ik}^{(h)}(x_{i1}, x_{i2}, \ldots, x_{in})\gamma_{ik}^{(h)}(t) + g_{ik}(x_{i1}, x_{i2}, \ldots, x_{in})| + |v_{ik} \right| \right) \mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \left(\sum_{h=1}^{n} \left| f_{ik}^{(h)}(x_{i1}, x_{i2}, \ldots, x_{in})\gamma_{ik}^{(h)}(t) \right| + |g_{ik}(x_{i1}, x_{i2}, \ldots, x_{in})| + |v_{ik}| \right) \mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \Bigg[B_{ik} + A_{ik} \sum_{l=1}^{n} (x_{il})^2 + \sum_{h=1}^{n} \left(B_{ik}^{(h)} + A_{ik}^{(h)} \sum_{l=1}^{n} (x_{il})^2 \right) \\
&+ \sqrt{m}G(1 + W(t)) \Bigg] \mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \left[B_{ik} + \sum_{h=1}^{n} B_{ik}^{(h)} + \sqrt{m}G + \left(A_{ik} + \sum_{h=1}^{n} A_{ik}^{(h)} + \sqrt{m}G \right) W(t) \right] \mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \Bigg\{ B_{ik} + \sum_{h=1}^{n} B_{ik}^{(h)}
\end{aligned}$$

$$+\sqrt{m}G+\left(A_{ik}+\sum_{h=1}^{n}A_{ik}^{(h)}+\sqrt{m}G\right)\tan^2\left[\frac{B\sqrt{2H(0)}}{m(G-G_0)}+\arctan\sqrt{2V(0)}\right]\Bigg\}\mathrm{d}t$$

$$\leq\Bigg\{B_{ik}+\sum_{h=1}^{n}B_{ik}^{(h)}+\sqrt{m}G$$

$$+\left(A_{ik}+\sum_{h=1}^{n}A_{ik}^{(h)}+\sqrt{m}G\right)\tan^2\left[\frac{B\sqrt{2H(0)}}{m(G-G_0)}+\arctan\sqrt{2V(0)}\right]\Bigg\}(t_2-t_1)$$

$$<\Bigg\{B_{ik}+\sum_{h=1}^{n}B_{ik}^{(h)}+\sqrt{m}G$$

$$+\left(A_{ik}+\sum_{h=1}^{n}A_{ik}^{(h)}+\sqrt{m}G\right)\tan^2\left[\frac{B\sqrt{2H(0)}}{m(G-G_0)}+\arctan\sqrt{2V(0)}\right]\Bigg\}\varepsilon$$

i.e.

$$|x_{ik}(t_1)-x_{ik}(t_2)| < \Bigg\{B_{ik}+\sum_{h=1}^{n}B_{ik}^{(h)}+\sqrt{m}G+\left(A_{ik}+\sum_{h=1}^{n}A_{ik}^{(h)}+\sqrt{m}G\right) \times\tan^2\left[\frac{B\sqrt{2H(0)}}{m(G-G_0)}+\arctan\sqrt{2V(0)}\right]\Bigg\}\varepsilon \quad (19)$$

$(i=1,2,\ldots,m; k=1,2,\ldots,n)$. Therefore, $\lim_{t\to a^-} x_{ik}(t)$ exists, mark $\lim_{t\to a^-} x_{ik}(t) = x_{ik}(a)$, $(i=1,2,\ldots,m; k=1,2,\ldots,n)$, then

$$\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}(a)-x_{ik}(a))^2 = \lim_{t\to a^-}\left[\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=i+1}^{m}(x_{jk}(t)-x_{ik}(t))^2\right] = 2\lim_{t\to a^-}H(t)>0\,.$$

Thus $x_{ik}(a)$ $(i=1,2,\ldots,m;k=1,2,\ldots,n)$ are not all equal. Suppose $\boldsymbol{X}_i(a)$ is the initial data, from the continuity of solution of complex network (4), here we can extend the solution to the region $[0, a+\delta_G(a)]$ subjecting to $H(t)>0$ in $[0,a+\delta_G(a)]$, $(\delta_G(a)>0)$. This is in conflict with the conclusion that the region $[0,a)$ is the biggest region for positive $H(t)$, therefore, $\lim_{t\to a^-} H(t)=0$.

Let $t_G=a$, then $0<t_G\leq\frac{\sqrt{2H(0)}}{m(G-G_0)}<T$, like in the proof for Eq. (19), there exists a point X_0, when $t\to t_G^-$, $\boldsymbol{X}_i=(x_{i1}(t),x_{i2}(t),\ldots,x_{in}(t))\to\boldsymbol{X}_0$, $(i=1,2,\ldots,m)$.

Obviously, when $\alpha = 0$, complex network (5) turns to be (4). Thus for complex network (5) when $\alpha = 0$, there exists a constant G_0, which is only related to network (3) when $G > (1 + \frac{2\sqrt{2H(0)}}{\pi - 2\arctan\sqrt{2V(0)}})G_0$, and there exist $0 < t_G \le \frac{\sqrt{2H(0)}}{m(G-G_0)} < T$ and point X_0, when $t \to t_G^-$, $\boldsymbol{X}_i = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t)) \to \boldsymbol{X}_0$, $(i = 1, 2, \dots, m)$.

Appendix B

Proof for Corollary 3.2

(i) If $y(t_G) = \sqrt{H(t_G, \alpha)} > \frac{G_0}{G-G_0}\beta$, from the poof for Theorem 3.2, when $t_G + \frac{\sqrt{2H(t_G,\alpha)}}{m(G-G_0)} + \sqrt{\alpha} \le t < T$,

$$\begin{aligned} 0 \le\ & H(t,\alpha) \\ \le\ & \frac{1}{2}\left[\frac{G_0}{G-G_0} + \frac{Ge^{\frac{1}{2}\ln(H(0)) - \frac{G_0}{G}}}{(G-G_0)\left(\frac{\sqrt{2}m}{2}(G-G_0)(t-t_G) - \sqrt{H(t_G,\alpha)}\right)}\right]^2 \alpha^2 \\ \le\ & \frac{1}{2}\left[\frac{G_0}{G-G_0} + \frac{\sqrt{2}Ge^{\frac{1}{2}\ln(H(0)) - \frac{G_0}{G}}}{m(G-G_0)^2\sqrt{\alpha}}\right]^2 \alpha^2 \\ \le\ & \frac{1}{2}\left[\frac{G_0}{G-G_0} + \frac{\sqrt{2}Ge^{\frac{1}{2}\ln(H(0)) - \frac{G_0}{G}}}{m(G-G_0)^2}\right]^2 \alpha\,. \end{aligned}$$

If $y(t_G) = \sqrt{H(t_G, \alpha)} \le \frac{G_0}{G-G_0}\beta$, and if there exists $\xi \in (t_G, T)$ subjecting to $\sqrt{H(t_G, \alpha)} > \frac{G_0}{G-G_0}\beta$, let ξ^* be the minimal value of ξ, from the continuity of $\sqrt{H(t_G, \alpha)}$, there exist $t_1 \in (t_G, T)$ and $\xi^* < t_1 < \xi^* + \frac{1}{2}\sqrt{\alpha}$ subjecting to $\frac{G_0}{G-G_0}\beta < \sqrt{H(t_1, \alpha)} < \frac{2G_0}{G-G_0}\beta$, *i.e.*, $\frac{G_0}{G-G_0}\alpha < \sqrt{2H(t_1, \alpha)} < \frac{2G_0}{G-G_0}\alpha$. From the above discussion, when $t_1 + \frac{\sqrt{2H(t_1,\alpha)}}{m(G-G_0)} + \frac{1}{2}\sqrt{\alpha} \le t < T$,

$$\begin{aligned} 0 \le\ & H(t,\alpha) \\ \le\ & \frac{1}{2}\left[\frac{G_0}{G-G_0} + \frac{Ge^{\frac{1}{2}\ln(H(0)) - \frac{G_0}{G}}}{(G-G_0)\left(\frac{\sqrt{2}m}{2}(G-G_0)(t-t_1) - \sqrt{H(t_1)}\right)}\right]^2 \alpha^2 \\ \le\ & \frac{1}{2}\left[\frac{G_0}{G-G_0} + \frac{2\sqrt{2}Ge^{\frac{1}{2}\ln(H(0)) - \frac{G_0}{G}}}{m(G-G_0)^2\sqrt{\alpha}}\right]^2 \alpha^2 \\ \le\ & \frac{1}{2}\left[\frac{G_0}{G-G_0} + \frac{2\sqrt{2}Ge^{\frac{1}{2}\ln(H(0)) - \frac{G_0}{G}}}{m(G-G_0)^2}\right]^2 \alpha\,. \end{aligned}$$

From Corollary 3.1 for an arbitrary small $0 < \varepsilon < 1$, there exists $0 < \delta \le \varepsilon$, when $0 < \alpha \le \delta$, $\sqrt{2H(t_G, \alpha)} < \sqrt{\varepsilon}$. Mark

$$C = \max\left\{\frac{1}{m(G-G_0)} + 1, \frac{2G_0}{m(G-G_0)^2} + \frac{1}{2}\right\},$$
$$D = \max\left\{\frac{1}{2}\left[\frac{G_0}{G-G_0} + \frac{\sqrt{2}Ge^{\frac{1}{2}\ln(H(0)) - \frac{G_0}{G}}}{m(G-G_0)^2}\right]^2,\right.$$
$$\left.\frac{1}{2}\left[\frac{G_0}{G-G_0} + \frac{2\sqrt{2}Ge^{\frac{1}{2}\ln(H(0)) - \frac{G_0}{G}}}{m(G-G_0)^2}\right]^2\right\},$$

then for complex network (5) and sufficient small $0 < \varepsilon < 1$, there exists $0 < \delta \le \varepsilon$, when $0 < \alpha \le \delta$, $H(t, \alpha) \le D\varepsilon$ in those regions in $[t_G, T)$ without the region I with length smaller than $C\sqrt{\varepsilon}$.

(ii) Can be obtained from Theorem 3.1 and *(i)*.

Appendix C

Proof for Lemma 3.2

Here, we prove only to show that $F^*(t, y)$ satisfies the basic conditions in the open region Δ, and $F(t, y)$ can be proved similarly. According to the definition $F^*(t, y)$, obviously, for arbitrary $(t, y) \in \Delta$, $F^*(t, y)$ is a compact convex set. For any fixed $t^{(0)}$ and point $y^{(0)} = (y_1^{(0)}, y_2^{(0)}, \ldots, y_n^{(0)})^T$, *(i)* when $y_j^{(0)} - \beta_{ij}(t^{(0)}, y_{n+1}^{(0)}) \neq 0$ and $x_{ij}(t^{(0)}, y_{n+1}^{(0)}) \neq 0$, from the continuity of $f_{ik}^{(j)}(x_1, x_2, \ldots, x_n)$, $g_{ik}(x_1, x_2, \ldots, x_n)$, $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; k = 1, 2, \ldots, n)$ and Hypothesis 3.1, we know $F^*(t, y)$ is upper semi-continuous in (t, y). *(ii)* If $y_j^{(0)} - \beta_{ij}(t^{(0)}, y_{n+1}^{(0)}) = 0$, $(i \in \{i_1, i_2, \ldots, i_{m_1}\}, j \in \{j_1, j_2, \ldots, j_{n_1}\}, 1 \le i_1 < i_2 < \cdots < i_{m_1} \le m, 1 \le j_1 < j_2 < \cdots < j_{n_1} \le n)$, or $x_{ij}(t^{(0)}, y_{n+1}^{(0)}) = 0$, $(i \in \{p_1, p_2, \ldots, p_{m_2}\}, j \in \{q_1, q_2, \ldots, q_{n_2}\}, 1 \le p_1 < p_2 < \cdots < p_{m_2} \le m, 1 \le q_1 < q_2 < \cdots < q_{n_2} \le n)$. Without losing generality, let us suppose here $y_1^{(0)} - \beta_{11}(t^{(0)}, y_{n+1}^{(0)}) = 0$, $x_{11}(t^{(0)}, y_{n+1}^{(0)}) = 0$. Mark

$$f_1^{**}\left(t^{(0)}, y^{(0)}\right) = \frac{1}{m}\sum_{j=2}^{n}\sum_{i=2}^{m} f_{i1}^{(j)}\left(y_1^{(0)}, y_2^{(0)}, \ldots, y_n^{(0)}\right)$$
$$\times \mathrm{sgn}\left(y_j^{(0)} - \beta_{ij}\left(t^{(0)}, y_{n+1}^{(0)}\right)\right) + \frac{1}{m}\sum_{i=1}^{m} g_{i1}\left(y_1^{(0)}, y_2^{(0)}, \ldots, y_n^{(0)}\right)$$

$$-\frac{1}{m}\sum_{j=2}^{n}\sum_{i=2}^{m}\left[f_{i1}^{(j)}\left(\frac{\sum_{i=1}^{m}x_{i1}\left(t^{(0)},y_{n+1}^{(0)}\right)}{m},\frac{\sum_{i=1}^{m}x_{i2}\left(t^{(0)},y_{n+1}^{(0)}\right)}{m},\ldots,\right.\right.$$

$$\left.\frac{\sum_{i=1}^{m}x_{in}\left(t^{(0)},y_{n+1}^{(0)}\right)}{m}\right)-f_{i1}^{(j)}\left(x_{i1}\left(t^{(0)},y_{n+1}^{(0)}\right),x_{i2}\left(t^{(0)},y_{n+1}^{(0)}\right),\ldots,\right.$$

$$\left.\left.x_{in}\left(t^{(0)},y_{n+1}^{(0)}\right)\right)\right]\operatorname{sgn}\left(x_{ij}\left(t^{(0)},y_{n+1}^{(0)}\right)\right)$$

$$-\frac{1}{m}\sum_{i=1}^{m}\left[g_{i1}\left(\frac{\sum_{i=1}^{m}x_{i1}\left(t^{(0)},y_{n+1}^{(0)}\right)}{m},\frac{\sum_{i=1}^{m}x_{i2}\left(t^{(0)},y_{n+1}^{(0)}\right)}{m},\ldots,\right.\right.$$

$$\left.\frac{\sum_{i=1}^{m}x_{in}\left(t^{(0)},y_{n+1}^{(0)}\right)}{m}\right)$$

$$\left.-g_{i1}\left(x_{i1}\left(t^{(0)},y_{n+1}^{(0)}\right),x_{i2}\left(t^{(0)},y_{n+1}^{(0)}\right),\ldots,x_{in}\left(t^{(0)},y_{n+1}^{(0)}\right)\right)\right],$$

$$a^*(t,y)=\frac{1}{m}f_{11}^{(1)}\left(y_1^{(0)},y_2^{(0)},\ldots,y_n^{(0)}\right),$$

$$b^*\left(t^{(0)},y^{(0)}\right)$$

$$=\frac{1}{m}\left[f_{11}^{(1)}\left(\frac{\sum_{i=1}^{m}x_{i1}\left(t^{(0)},y_{n+1}^{(0)}\right)}{m},\frac{\sum_{i=1}^{m}x_{i2}\left(t^{(0)},y_{n+1}^{(0)}\right)}{m},\ldots,\right.\right.$$

$$\left.\frac{\sum_{i=1}^{m}x_{in}\left(t^{(0)},y_{n+1}^{(0)}\right)}{m}\right)$$

$$\left.-f_{11}^{(1)}\left(x_{i1}\left(t^{(0)},y_{n+1}^{(0)}\right),x_{i2}\left(t^{(0)},y_{n+1}^{(0)}\right),\ldots,x_{in}\left(t^{(0)},y_{n+1}^{(0)}\right)\right)\right],$$

here, $F^*(t^{(0)},y^{(0)})$ is the closed convex polyhedron with vertices

$$\begin{aligned}J_1\left(t^{(0)},y^{(0)}\right) = &\left(f_1^{**}\left(t^{(0)},y^{(0)}\right)+a^*\left(t^{(0)},y^{(0)}\right)\right.\\ &\left.+b^*\left(t^{(0)},y^{(0)}\right),f_2^*\left(t^{(0)},y^{(0)}\right),\ldots,f_n^*\left(t^{(0)},y^{(0)}\right),0\right)^T,\end{aligned}$$

$$\begin{aligned} J_2\left(t^{(0)}, y^{(0)}\right) &= \Big(f_1^{**}\left(t^{(0)}, y^{(0)}\right) + a^*\left(t^{(0)}, y^{(0)}\right) \\ &\quad -b^*\left(t^{(0)}, y^{(0)}\right), f_2^*\left(t^{(0)}, y^{(0)}\right), \ldots, f_n^*\left(t^{(0)}, y^{(0)}\right), 0\Big)^T, \\ J_3\left(t^{(0)}, y^{(0)}\right) &= \Big(f_1^{**}\left(t^{(0)}, y^{(0)}\right) - a^*\left(t^{(0)}, y^{(0)}\right) \\ &\quad +b^*\left(t^{(0)}, y^{(0)}\right), f_2^*\left(t^{(0)}, y^{(0)}\right), \ldots, f_n^*\left(t^{(0)}, y^{(0)}\right), 0\Big)^T, \\ J_4\left(t^{(0)}, y^{(0)}\right) &= \Big(f_1^{**}\left(t^{(0)}, y^{(0)}\right) - a^*\left(t^{(0)}, y^{(0)}\right) \\ &\quad -b^*\left(t^{(0)}, y^{(0)}\right), f_2^*\left(t^{(0)}, y^{(0)}\right), \ldots, f_n^*\left(t^{(0)}, y^{(0)}\right), 0\Big)^T. \end{aligned}$$

For any open set including $F^*(t^{(0)}, y^{(0)})$, there exists $\varepsilon_0 > 0$ subjecting to $(F^*(t^{(0)}, y^{(0)}))^{\varepsilon_0} \subset U$, there exist $\delta_1 > 0$, $\delta_2 > 0$, when $\|(t, y) - (t^{(0)}, y^{(0)})\| < \delta_1, y_j - \beta_{ij}(t, y_{n+1}) \neq 0$ and $x_{ij}(t, y_{n+1}) \neq 0$, $(i = 2, 3, \ldots, m; j = 2, 3, \ldots, n)$; when

$$\begin{aligned} \left\|(t, y) - \left(t^{(0)}, y^{(0)}\right)\right\| &< \delta_2, \\ \left\|J_1(t, y) - J_1\left(t^{(0)}, y^{(0)}\right)\right\| &< \varepsilon_0, \\ \left\|J_2(t, y) - J_2\left(t^{(0)}, y^{(0)}\right)\right\| &< \varepsilon_0, \\ \left\|J_3(t, y) - J_3\left(t^{(0)}, y^{(0)}\right)\right\| &< \varepsilon_0, \\ \left\|J_4(t, y) - J_4\left(t^{(0)}, y^{(0)}\right)\right\| &< \varepsilon_0. \end{aligned}$$

Let $\delta_0 = \min\{\delta_1, \delta_2\}$, $V = \{(t, y) | \parallel (t, y) - (t^{(0)}, y^{(0)}) \parallel < \delta_0\}$, then $F^*(V) \subset (F^*(t^{(0)}, y^{(0)}))^{\varepsilon_0} \subset U$. Thus, $F^*(t, y)$ is upper semi-continuous with (t, y). Therefore, we conclude that $F^*(t, y)$ satisfies the basic conditions in the open region Δ.

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REFERENCES

[1] T.P. Chen, W.L. Lu, *Theory of Coordination in Complex Networks*, Beijing, Higher Education Press, 2013.

[2] J.A. Lu, H. Liu, J. Chen, *Synchronization in Complex Dynamical Networks*, Beijing, Higher Education Press, 2016.

[3] X.F. Wang, G.R. Chen, *Int. J. Bifur. Chaos* **12**, 187 (2002).

[4] J. Chen, J.A. Lu, X.Q. Wu, *J. Systems Sci. Complex.* **24**, 433 (2011).

[5] J. Zhou, J.A. Lu, J.H. Lü, *IEEE Trans. Automatic Control* **51**, 652 (2006).

[6] J. Zhou, J.A. Lu, J.H. Lü, *Automatica* **44**, 996 (2008).

[7] X.P. Han, J.A. Lu, X.Q. Wu, *Int. J. Bifur. Chaos* **18**, 1539 (2008).

[8] J. Zha, C. Li, B. Song, W. Hu, *Int. J. Systems Sci.* **47**, 3952 (2016).

[9] L.M. Pecora, T.L. Carroll, *Phys. Rev. Lett.* **64**, 821 (1990).

[10] L.M. Pecora, T.L. Carroll, *Phys. Rev. Lett.* **80**, 2109 (1998).

[11] A. Stefański, P. Perlikowski, T. Kapitaniak, *Phys. Rev. E* **75**, 016210 (2007).

[12] Z.S. Duan, G.R. Chen, *J. Phys A: Math. Theor.* **44**, 205101 (2011).

[13] M. Barahona, L. Pecora, *Phys. Rev. Lett.* **89**, 054101 (2002).

[14] L.K. Tang *et al.*, *Int. J. Bifur. Chaos* **22**, 1250282 (2012).

[15] V.N. Belykh, I.V. Belykh, M. Hasler, *Physica D* **195**, 159 (2004).

[16] V.N. Belykh, I.V. Belykh, M. Hasler, *Physica D* **224**, 42 (2006).

[17] H. Liu, M. Cao, C.W. Wu, *IEEE Trans. Circuits Syst.-I* **61**, 1520 (2014).

[18] H. Liu *et al.*, *IEEE Trans. Circuits Syst.-I* **62**, 1185 (2015).

[19] J.P. Aubin, A. Cellina, *Theory of Coordination in Complex Networks*, Berlin, Springer, 1984.

[20] J.L. Davy, *Bull. Austral. Math. Soc.* **6**, 379 (1972).

[21] A.F. Filippov, *Mathematics and Its Applications*, Boston, Kluwer Academic, 1988.