ON EINSTEIN–CARTAN'S THEORY IN THE STATIC SPHERICALLY SYMMETRIC CASE WITH AN ANTISYMMETRIC TORSION TENSOR AND BREAKING OF THE WEAK GAUSS LAW

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We consider Einstein–Cartan's theory in the static spherically symmetric case with a completely antisymmetric torsion tensor. We show, in particular, that the weak Gauss law of general relativity is broken.

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1. Introduction

In reference [1], there is shown a particular example of Einstein–Cartan's theory [2–4] in the static spherically symmetric case, *i.e.* the Kottler or Schwarzschild–de Sitter model with torsion that the weak Gauss law is broken: The interior mass M_{i}, the real mass, and the exterior mass M_{e}, defining the strength of the gravitational field outside the mass distribution, do not coincide.

In the example shown, the torsion tensor does not possess a completely antisymmetric irreducible part (only the vectorial and mixed irreducible parts are present). The fact that the torsion tensor is completely antisymmetric is much appealing [5]. In particular, when this happens, the connection geodesics coincide with Christoffel's geodesics.

In this paper, we will show that the torsion tensor possesses, in general, the three irreducible parts. Requiring the irreducible vectorial and mixed parts to vanish results in a completely antisymmetric torsion tensor. Then,

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we will show that, in general, the weak Gauss law is broken when the torsion tensor is completely antisymmetric. We obtain, in particular, a mass formula for the external mass $M_{\rm e}$.

In comparison to reference [1], we can, under some conditions, go further and solve completely the problem, that is, we determine the metric tensor, together with the energy-momentum tensor. In the absence of torsion, we confirm previous results [6–8]. The paper is constructed in a similar manner as reference [1] and uses results of references [1, 9].

The paper is organized as follows: In Section 2, we determine the general form of the connection after imposing invariance under time translations and rotations around the 3 axes. In Section 3, we establish general formulae for the curvature and torsion tensors. Then, we compute the torsion tensor and decompose it into its irreducible parts for the static spherically symmetric case. In Section 4, we require the torsion tensor to be completely antisymmetric and then write the corresponding curvature tensor, the Ricci tensor and the scalar curvature. Finally, we write Einstein–Cartan's equations. In Section 5, we consider the case of a Schwarzschild star, *i.e.* the case with constant mass and spin densities. We then determine the mass formula, *i.e.* the external mass $M_{\rm e}$ in term of the internal mass $M_{\rm i}$. We are lead, naturally, to distinguish between time reversal even and time reversal odd cases. In the more interesting case from the point of view of cosmology, we can, under some assumption, go further and determine completely the remaining component of the metric tensor, together with the energy-momentum tensor components. In Section 6 we discuss our findings, and Section 7 is devoted to conclusions.

2. General form of the metric connection in the static spherically symmetric case

Einstein's theory admits the metric g as a fundamental variable. Einstein –Cartan's theory has in addition to the metric, another independent fundamental variable, the metric connection, not to be confused with Christoffel's connection. The presence of symmetries impose constraints on the metric as well as on the connection.

In the static spherically symmetric case, it is well-known that the metric $g_{\mu\nu}$ may be on the one handput, without loss of generality, in the form of

$$g_{\mu\nu} = \begin{pmatrix} B(r) & 0 & 0 & 0 \\ 0 & -A(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} . \tag{1}$$

On the other hand, we have to solve the connection Killing equation

$$\xi^{\alpha}\frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\alpha}}-\frac{\partial\xi^{\lambda}}{\partial x^{\bar{\lambda}}}\Gamma^{\bar{\lambda}}_{\mu\nu}+\frac{\partial\xi^{\bar{\mu}}}{\partial x^{\mu}}\Gamma^{\lambda}_{\bar{\mu}\nu}+\frac{\partial\xi^{\bar{\nu}}}{\partial x^{\nu}}\Gamma^{\lambda}_{\mu\bar{\nu}}+\frac{\partial^2\xi^{\lambda}}{\partial x^{\mu}\partial x^{\nu}}=0\,, \tag{2}$$

for Killing vector fields

$$\xi=\xi^{\alpha}\frac{\partial}{\partial x^{\alpha}} \tag{3}$$

corresponding to time translation

$$\xi=\frac{\partial}{\partial t}\,, \tag{4}$$

and rotations

$$\xi = -\sin\varphi\frac{\partial}{\partial\theta}-\cos\varphi\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\,, \tag{5}$$

$$\xi = \cos\varphi\frac{\partial}{\partial\theta}-\sin\varphi\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\,, \tag{6}$$

$$\xi = \frac{\partial}{\partial\varphi}\,. \tag{7}$$

One gets the following non-vanishing components of the connection:

$$\begin{aligned}
\Gamma^{a}_{bc} &= X^{a}_{bc}(r)\,, & & a,b,c\in\{t,r\}\,,\\
\Gamma^{a}_{\theta\theta} &= \frac{\Gamma^{a}_{\varphi\varphi}}{\sin^2\theta}=E^{a}(r)\,, & & \Gamma^{a}_{\theta\varphi}=-\Gamma^{a}_{\varphi\theta}=\sin\theta F^{a}(r)\,,\\
\Gamma^{\theta}_{a\theta} &= \Gamma^{\varphi}_{a\varphi}=C_a(r)\,, & & \Gamma^{\theta}_{\theta a}=\Gamma^{\varphi}_{\varphi a}=Y_a(r)\,,\\
\Gamma^{\theta}_{\varphi a} &= -\sin^2\theta\Gamma^{\varphi}_{\theta a}=-\sin\theta Z_a(r)\,, & & \Gamma^{\theta}_{a\varphi}=-\sin^2\theta\Gamma^{\varphi}_{a\theta}=\sin\theta G_a(r)\,,\\
\Gamma^{\theta}_{\varphi\varphi} &= -\sin\theta\cos\theta\,, & & \Gamma^{\varphi}_{\theta\varphi}=\Gamma^{\varphi}_{\varphi\theta}=\frac{\cos\theta}{\sin\theta}\,.
\end{aligned} \tag{8}$$

In comparison to [1], we have additional non-vanishing components parametrized by four new functions of r: $Z_t(r)$, $Z_r(r)$, $G_t(r)$, $G_r(r)$.

Next, requiring the metricity condition to be satisfied

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}-\Gamma^{\bar{\mu}}_{\mu\lambda}g_{\bar{\mu}\nu}-\Gamma^{\bar{\nu}}_{\nu\lambda}g_{\mu\bar{\nu}}=0\,, \tag{9}$$

one obtains the following non-vanishing components of the connection:

$$\begin{aligned}
\Gamma^t_{rt} &= D_t \frac{A}{B}\,, & \Gamma^r_{tt} &= D_t\,, \\
\Gamma^t_{rr} &= D_r \frac{A}{B}\,, & \Gamma^r_{tr} &= D_r\,, \\
\Gamma^t_{tr} &= \frac{B'}{2B}\,, & \Gamma^r_{rr} &= \frac{A'}{2A}\,, \\
\Gamma^t_{\theta\theta} &= \frac{r^2}{B} C_t\,, & \Gamma^r_{\theta\theta} &= -\frac{r^2}{A} C_r\,, \\
\Gamma^t_{\varphi\varphi} &= \frac{r^2}{B}\sin^2\theta C_t\,, & \Gamma^r_{\varphi\varphi} &= -\frac{r^2}{A}\sin^2\theta C_r\,, \\
\Gamma^\theta_{t\theta} &= C_t\,, & \Gamma^\theta_{r\theta} &= C_r\,, \\
\Gamma^\varphi_{t\varphi} &= C_t\,, & \Gamma^\varphi_{r\varphi} &= C_r\,, \\
\Gamma^\theta_{\theta r} &= \frac{1}{r}\,, & \Gamma^\varphi_{\varphi r} &= \frac{1}{r}\,, \\
\Gamma^t_{\theta\varphi} &= \frac{r^2}{B}\sin\theta G_t\,, & \Gamma^r_{\theta\varphi} &= -\frac{r^2}{A}\sin\theta G_r\,, \\
\Gamma^t_{\varphi\theta} &= -\frac{r^2}{B}\sin\theta G_t\,, & \Gamma^r_{\varphi\theta} &= \frac{r^2}{A}\sin\theta G_r\,, \\
\Gamma^\varphi_{t\theta} &= -\frac{G_t}{\sin\theta}\,, & \Gamma^\varphi_{r\theta} &= -\frac{G_r}{\sin\theta}\,, \\
\Gamma^\theta_{t\varphi} &= G_t \sin\theta\,, & \Gamma^\theta_{r\varphi} &= G_r \sin\theta\,, \\
\Gamma^\varphi_{\theta t} &= \frac{Z_t}{\sin\theta}\,, & \Gamma^\varphi_{\theta r} &= \frac{Z_r}{\sin\theta}\,, \\
\Gamma^\theta_{\varphi t} &= -\sin\theta Z_t\,, & \Gamma^\theta_{\varphi r} &= -\sin\theta Z_r\,, \\
\Gamma^\theta_{\varphi\varphi} &= -\sin\theta\cos\theta\,, & \Gamma^\varphi_{\theta\varphi} &= \Gamma^\varphi_{\varphi\theta} = \frac{\cos\theta}{\sin\theta}
\end{aligned} \tag{10}$$

parametrized by eight functions of r: $C_t(r)$, $C_r(r)$, $D_t(r)$, $D_r(r)$, $Z_t(r)$, $Z_r(r)$, $G_t(r)$, $G_r(r)$. In comparison to [1], there are additional non-vanishing components parametrized by the four functions of r: $Z_t(r)$, $Z_r(r)$, $G_t(r)$, $G_r(r)$. In fact, our solution is the most general one corresponding to the static spherically symmetric case. The solution of [1] is less general, corresponding to a particular case of ours, obtained by setting $Z_t(r) = Z_r(r) = G_t(r) = G_r(r) = 0$. We will show in the following that the four functions $Z_t(r)$, $Z_r(r)$, $G_t(r)$, $G_r(r)$ are crucial in order to obtain a completely antisymmetric torsion tensor.

From now on, we will work in an orthonormal frame e^a related to the holonomic frame $\mathrm{d}x^\mu$ by

$$\mathrm{e}^a = \mathrm{e}^a_\mu \mathrm{d}x^\mu \,, \tag{11}$$

where $\mathrm{e}^a_\mu \in \mathrm{GL}(4)$. From (11), we get

$$g_{\mu\nu}(x) = \mathrm{e}^a_\mu(x)\mathrm{e}^b_\nu(x)\eta_{ab} \,. \tag{12}$$

Since the metric tensor is given by (1), one deduces that

$$\mathrm{e}^a_\mu = \begin{pmatrix} \sqrt{B} & 0 & 0 & 0 \\ 0 & \sqrt{A} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix} . \tag{13}$$

The components of the same connection with respect to the orthonormal frame ω and with respect to the holonomic frame Γ are linked by a gauge transformation

$$\omega^a_{b\mu} = \mathrm{e}^a_\alpha \Gamma^\alpha_{\beta\mu} \left(\mathrm{e}^{-1}\right)^\beta_b + \mathrm{e}^a_\alpha \frac{\partial}{\partial x^\mu} \left(\mathrm{e}^{-1}\right)^\alpha_b \,. \tag{14}$$

The non-vanishing components of the connection ω are

$$\begin{aligned}
\omega^t_{rt} &= \sqrt{\frac{A}{B}} D_t \,, & \omega^\theta_{\varphi t} &= -Z_t \,, \\
\omega^t_{rr} &= \sqrt{\frac{A}{B}} D_r \,, & \omega^\theta_{\varphi r} &= -Z_r \,, \\
\omega^t_{\theta\theta} &= \frac{r}{\sqrt{B}} C_t \,, & \omega^t_{\varphi\theta} &= -\frac{r}{\sqrt{B}} G_t \,, \\
\omega^r_{\theta\theta} &= -\frac{r}{\sqrt{A}} C_r \,, & \omega^r_{\varphi\theta} &= \frac{r}{\sqrt{A}} G_r \,, \\
\omega^t_{\theta\varphi} &= \frac{r}{\sqrt{B}} \sin\theta G_t \,, & \omega^t_{\varphi\varphi} &= \frac{r}{\sqrt{B}} \sin\theta C_t \,, \\
\omega^r_{\theta\varphi} &= -\frac{r}{\sqrt{A}} \sin\theta G_r \,, & \omega^r_{\varphi\varphi} &= -\frac{r}{\sqrt{A}} \sin\theta C_r \,, \\
\omega^\theta_{\varphi\varphi} &= -\cos\theta \,.
\end{aligned} \tag{15}$$

Let us notice that in comparison to [1], there are additional non-vanishing components expressed in terms of the four new parametrizing functions $Z_t(r)$, $Z_r(r)$, $G_t(r)$, $G_r(r)$.

3. Formulae for the curvature and torsion tensors. Computation of the torsion tensor

Thanks to the structure equations for the curvature R^a_b and the torsion T^a

$$\begin{aligned} R^a_b &= d\omega^a_b + \omega^a_c \omega^c{}_b\,, & (16)\\ T^a &= d\mathrm{e}^a + \omega^a_c \mathrm{e}^c\,, & (17)\end{aligned}$$

one can compute the curvature and torsion tensors defined respectively by

$$R^{ab} = \tfrac{1}{2} R^{ab}_{cd} \mathrm{e}^c\, \mathrm{e}^d \tag{18}$$

and

$$T^a = \tfrac{1}{2} \eta^{aa'} T_{a'bc} \mathrm{e}^b\, \mathrm{e}^c\,. \tag{19}$$

It is a straightforward calculation to obtain explicit formulae for the curvature and torsion tensors

$$\begin{aligned} R^a_{bcd} &= \left(\partial_\mu \omega^a_{b\nu} - \partial_\nu \omega^a_{b\mu} + \omega^a_{\bar{b}\mu}\omega^{\bar{b}}_{b\nu} - \omega^a_{\bar{b}\nu}\omega^{\bar{b}}_{b\mu}\right)\left(\mathrm{e}^{-1}\right)^\mu_c \left(\mathrm{e}^{-1}\right)^\nu_d\,, & (20)\\ T^a_{bc} &= \left(\partial_\mu \mathrm{e}^a_\nu - \partial_\nu \mathrm{e}^a_\mu + \omega^a_{k\mu}\mathrm{e}^k_\nu - \omega^a_{k\nu}\mathrm{e}^k_\mu\right)\left(\mathrm{e}^{-1}\right)^\mu_b \left(\mathrm{e}^{-1}\right)^\nu_c\,, & (21)\end{aligned}$$

where $(\mathrm{e}^{-1})^\mu_b$ is the inverse of e^a_μ, that is

$$\mathrm{e}^a_\mu \left(\mathrm{e}^{-1}\right)^\mu_b = \eta^a_b\,. \tag{22}$$

Thus,

$$\left(\mathrm{e}^{-1}\right)^\mu_a = \begin{pmatrix} \frac{1}{\sqrt{B}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{A}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}. \tag{23}$$

As stressed in the introduction, we are especially interested in the case where the torsion tensor is completely antisymmetric. We will proceed in three steps: We must first compute the torsion tensor, then decompose it into its irreducible completely antisymmetric A_{abc}, vectorial V_a, and mixed M_{abc} parts and finally, require the vectorial and mixed parts to vanish, *i.e.* require the torsion tensor to be completely antisymmetric. The non-vanishing components of the torsion tensor are

$$
\begin{aligned}
T_{ttr} &= \left(D_t - \frac{1}{2}\frac{B'}{A}\right)\frac{\sqrt{A}}{B}\,, && (24)\\
T_{rtr} &= \frac{D_r}{\sqrt{B}}\,, && (25)\\
T_{\theta t\theta} &= T_{\varphi t\varphi} = \frac{C_t}{\sqrt{B}}\,, && (26)\\
T_{\varphi t\theta} &= -T_{\theta t\varphi} = -\frac{1}{\sqrt{B}}\left(Z_t + G_t\right)\,, && (27)\\
T_{\theta r\theta} &= T_{\varphi r\varphi} = \frac{1}{\sqrt{A}}\left(C_r - \frac{1}{r}\right)\,, && (28)\\
T_{\varphi r\theta} &= T_{\theta\varphi r} = -\frac{1}{\sqrt{A}}\left(Z_r + G_r\right)\,, && (29)\\
T_{t\theta\varphi} &= -\frac{2}{\sqrt{B}}G_t\,, && (30)\\
T_{r\theta\varphi} &= -\frac{2}{\sqrt{A}}G_r\,. && (31)
\end{aligned}
$$

In addition to the non-vanishing components of [1], we have additional non-vanishing components expressed in terms of $Z_t(r)$, $Z_r(r)$, $G_t(r)$, $G_r(r)$. Let us now decompose the torsion tensor into its irreducible completely anti-symmetric A_{abc}, vectorial V_a and mixed M_{abc} parts

$$T_{abc} = A_{abc} + \eta_{ab}V_c - \eta_{ac}V_b + M_{abc}\,, \qquad (32)$$

with

$$
\begin{aligned}
A_{abc} &= \frac{1}{3}\left(T_{abc} + T_{bca} + T_{cab}\right)\,, && (33)\\
V_c &= \frac{1}{3}\eta^{ab}T_{abc}\,, && (34)\\
M_{abc} &= T_{abc} - A_{abc} - \eta_{ab}V_c + \eta_{ac}V_b\,. && (35)
\end{aligned}
$$

For the non-vanishing components, one obtains the following expressions:

$$
\begin{aligned}
A_{t\theta\varphi} &= \frac{-2}{3\sqrt{B}}\left(Z_t + 2G_t\right) , \\
A_{r\theta\varphi} &= \frac{-2}{3\sqrt{A}}\left(Z_r + 2G_r\right) , \\
V_t &= \frac{1}{3\sqrt{B}}\left(D_r + 2C_t\right) , \\
V_r &= \frac{1}{3\sqrt{A}}\left(\frac{-B'}{2B} - \frac{2}{r} + 2C_r + \frac{A}{B}D_t\right) , \\
M_{ttr} &= \frac{-2}{3\sqrt{A}}\left(\frac{B'}{2B} - \frac{1}{r} + C_r - \frac{A}{B}D_t\right) , \\
M_{t\theta\varphi} &= \frac{2}{3\sqrt{B}}\left(Z_t - G_t\right) , \\
M_{rtr} &= \frac{2}{3\sqrt{B}}\left(D_r - C_t\right) , \\
M_{r\theta\varphi} &= \frac{2}{3\sqrt{A}}\left(Z_r - G_r\right) , \\
M_{\theta t\theta} &= -\frac{1}{2}M_{rtr} = \frac{1}{3\sqrt{B}}\left(C_t - D_r\right) , \\
M_{\theta t\varphi} &= \frac{1}{3\sqrt{B}}\left(Z_t - G_t\right) , \\
M_{\theta r\theta} &= \frac{1}{2}M_{trt} = \frac{1}{3\sqrt{A}}\left(\frac{B'}{2B} - \frac{1}{r} + C_r - \frac{A}{B}D_t\right) , \\
M_{\theta r\varphi} &= \frac{1}{3\sqrt{A}}\left(Z_r - G_r\right) , \\
M_{\varphi t\theta} &= \frac{-1}{3\sqrt{B}}\left(Z_t - G_t\right) , \\
M_{\varphi t\varphi} &= M_{\theta t\theta} = -\frac{1}{2}M_{rtr} = \frac{1}{3\sqrt{B}}\left(C_t - D_r\right) . \qquad (36)
\end{aligned}
$$

4. Completely antisymmetric torsion tensor case

Finally, requiring the vectorial and mixed parts of the torsion tensor to vanish, one obtains the constraints

$$
C_t = 0\,, \qquad C_r = \frac{1}{r}\,, \qquad D_t = \frac{B'}{2A}\,, \quad D_r = 0\,, \qquad G_t = Z_t\,, \qquad G_r = Z_r\,. \tag{37}
$$

The completely antisymmetric torsion tensor obtained in this way has two independent components $A_{t\theta\varphi}$ and $A_{r\theta\varphi}$ related respectively to the parametrizing functions Z_t and Z_r

$$A_{t\theta\varphi} = -2\frac{Z_t}{\sqrt{B}}\,, \tag{38}$$

$$A_{r\theta\varphi} = -2\frac{Z_r}{\sqrt{A}}\,. \tag{39}$$

In the completely antisymmetric torsion tensor case, the expressions of the components of ω simplify to

$$\begin{aligned}
\omega^t_{rt} &= \frac{B'}{2\sqrt{AB}}\,, & \omega^\theta_{\varphi t} &= -Z_t\,,\\
\omega^\theta_{\varphi r} &= -Z_r\,, & \omega^t_{\varphi\theta} &= -\frac{r}{\sqrt{B}}Z_t\,,\\
\omega^r_{\theta\theta} &= -\frac{1}{\sqrt{A}}\,, & \omega^r_{\varphi\theta} &= \frac{r}{\sqrt{A}}Z_r\,,\\
\omega^t_{\theta\varphi} &= \frac{r}{\sqrt{B}}\sin\theta Z_t\,, & \omega^r_{\theta\varphi} &= -\frac{r}{\sqrt{A}}\sin\theta Z_r\,,\\
\omega^r_{\varphi\varphi} &= -\frac{\sin\theta}{\sqrt{A}}\,, & \omega^\theta_{\varphi\varphi} &= -\cos\theta\,.
\end{aligned} \tag{40}$$

We can now compute the curvature tensor components R^{ab}_{cd} using formula (20). We have the following non-vanishing components:

$$\begin{aligned}
R^{tr}_{tr} &= \frac{1}{\sqrt{AB}}\left(\frac{B'}{2\sqrt{AB}}\right)'\,, & R^{\theta\varphi}_{tr} &= -\frac{1}{\sqrt{AB}}Z'_t\,,\\
R^{t\theta}_{t\theta} &= \frac{1}{B}\left(\frac{B'}{2rA} - Z_t^2\right)\,, & R^{t\varphi}_{t\theta} &= \frac{-B'}{2AB}Z_r\,,\\
R^{r\theta}_{t\theta} &= \frac{Z_r}{\sqrt{A}}\frac{Z_t}{\sqrt{B}}\,, & R^{r\varphi}_{t\theta} &= \frac{1}{\sqrt{AB}}\left(\frac{B'}{2B}+\frac{1}{r}\right)Z_t\,,\\
R^{t\theta}_{t\varphi} &= \frac{B'}{2AB}Z_r\,, & R^{t\varphi}_{t\varphi} &= \frac{1}{B}\left(\frac{B'}{2rA} - Z_t^2\right)\,,\\
R^{r\theta}_{t\varphi} &= -\frac{1}{\sqrt{AB}}\left(\frac{B'}{2B}+\frac{1}{r}\right)Z_t\,, & R^{r\varphi}_{t\varphi} &= \frac{Z_r}{\sqrt{A}}\frac{Z_t}{\sqrt{B}}\,,
\end{aligned}$$

$$
\begin{aligned}
R^{t\theta}_{r\theta} &= -\frac{Z_r}{\sqrt{A}}\frac{Z_t}{\sqrt{B}}\,, & R^{r\varphi}_{t\varphi} &= -\frac{1}{r\sqrt{A}}\frac{Z_t}{\sqrt{B}}\,,\\
R^{r\theta}_{r\theta} &= \frac{1}{r\sqrt{A}}\left(\frac{1}{\sqrt{A}}\right)' + \frac{Z_r^2}{A}\,, & R^{t\theta}_{r\varphi} &= -\frac{1}{r\sqrt{A}}\frac{Z_t}{\sqrt{B}}\,,\\
R^{t\varphi}_{r\varphi} &= -\frac{Z_r}{\sqrt{A}}\frac{Z_t}{\sqrt{B}}\,, & R^{r\varphi}_{r\varphi} &= \frac{1}{r\sqrt{A}}\left(\frac{1}{\sqrt{A}}\right)' + \frac{Z_r^2}{A}\,,\\
R^{tr}_{\theta\varphi} &= \frac{2}{r\sqrt{A}}\frac{Z_t}{\sqrt{B}}\,, & R^{\theta\varphi}_{\theta\varphi} &= -\frac{1}{r^2} - \frac{Z_t^2}{B} + \frac{1}{r^2 A} + \frac{Z_r^2}{A}\,,
\end{aligned}
\tag{41}
$$

where the $'$ denotes derivative with respect to r.

The non-vanishing components of the Ricci tensor $\mathcal{R}^a_b$, defined by

$$
\mathcal{R}^a_b = R^{ca}_{cb} \tag{42}
$$

are

$$
\begin{aligned}
\mathcal{R}^t_t &= \frac{1}{\sqrt{AB}}\left(\frac{B'}{2\sqrt{AB}}\right)' + \frac{B'}{rAB} - \frac{2}{B}Z_t^2\,,\\
\mathcal{R}^r_r &= \frac{1}{\sqrt{AB}}\left(\frac{B'}{2\sqrt{AB}}\right)' + \frac{2}{\sqrt{A}}\left(\frac{1}{r\sqrt{A}}\right)' - \frac{2}{A}\left(\frac{1}{r^2} + Z_r^2\right)\,,\\
\mathcal{R}^\theta_\theta &= \mathcal{R}^\varphi_\varphi = \frac{1}{B}\left(\frac{B'}{2rA} - 2Z_t^2\right) + \frac{1}{\sqrt{A}}\left(\frac{1}{r\sqrt{A}}\right)' + \frac{2}{A}\left(\frac{1}{r^2} + Z_r^2\right) - \frac{1}{r^2}\,,\\
\mathcal{R}^t_r &= -2\frac{Z_r}{\sqrt{A}}\frac{Z_t}{\sqrt{B}}\,,\\
\mathcal{R}^r_t &= 2\frac{Z_r}{\sqrt{A}}\frac{Z_t}{\sqrt{B}}\,,\\
\mathcal{R}^\theta_\varphi &= -\mathcal{R}^\varphi_\theta = \frac{B'}{2AB}Z_r\,,
\end{aligned}
\tag{43}
$$

and the scalar curvature $\mathcal{R}$, defined by

$$
\mathcal{R} = \mathcal{R}^a_a = \mathcal{R}^t_t + \mathcal{R}^r_r + \mathcal{R}^\theta_\theta + \mathcal{R}^\varphi_\varphi \tag{44}
$$

is given by

$$
\begin{aligned}
\mathcal{R} &= \frac{2}{\sqrt{AB}}\left(\frac{B'}{2\sqrt{AB}}\right) + \frac{4}{\sqrt{A}}\left(\frac{1}{r\sqrt{A}}\right)' + \frac{2}{A}\left(\frac{2}{rA} + \frac{1}{r^2} + 3Z_r^2\right)\\
&\quad + \frac{2}{B}\left(\frac{B'}{rA} - 3Z_t^2\right) - \frac{2}{r^2}\,.
\end{aligned}
\tag{45}
$$

The Einstein equation reads [1, 9, 10]

$$G_{ab} - \Lambda\eta_{ab} = 8\pi G\tau_{ba}\,, \tag{46}$$

where G_{ab} is the Einstein tensor, related to the Ricci tensor $\mathcal{R}_{ab}$ and scalar curvature $\mathcal{R}$ by

$$G_{ab} = \mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}\eta_{ab}\,. \tag{47}$$

In the orthonormal frame, the most general energy momentum tensor reads

$$\tau_{ab} = \left(\begin{array}{cccc} \rho(r) & q(r) & 0 & 0 \\ -o(r) & p_r(r) & 0 & 0 \\ 0 & 0 & p_a(r) & u(r) \\ 0 & o & -u(r) & p_a(r) \end{array}\right)\,. \tag{48}$$

Let us justify that (48) is the more general form of the energy momentum tensor. It is clear that the vanishing of a component of $G_{ab} - \Lambda\eta_{ab}$ implies the vanishing of the corresponding component τ_{ba} of the energy momentum tensor. Since the components $t\theta$, θt, $t\varphi$, φt, $r\theta$, θr, $r\varphi$, φr of $G_{ab} - \Lambda\eta_{ab}$ are vanishing, this on the one hand implies that $\tau_{\theta t} = 0$, $\tau_{t\theta} = 0$, $\tau_{\varphi t} = 0$, $\tau_{t\varphi} = 0$, $\tau_{\theta r} = 0$, $\tau_{r\theta} = 0$, $\tau_{\varphi r} = 0$, $\tau_{r\varphi} = 0$. On the other hand, $G_{\varphi\theta} - \Lambda\eta_{\varphi\theta} = -(G_{\theta\varphi} - \Lambda\eta_{\theta\varphi})$ implies that $\tau_{\varphi\theta} = -\tau_{\theta\varphi}$ and $G_{\theta\theta} - \Lambda\eta_{\theta\theta} = G_{\varphi\varphi} - \Lambda\eta_{\varphi\varphi}$ implies that $\tau_{\theta\theta} = \tau_{\varphi\varphi}$. Finally, since $G_{ab} - \Lambda\eta_{ab}$ is a function only of r, the same is true for the components of the energy momentum tensor τ_{ab}. Hence, we can, without loss of generality, parametrize the energy momentum tensor as in (48). τ_{tt} is interpreted as a density of matter ρ, and τ_{rr} and $\tau_{\theta\theta} = \tau_{\varphi\varphi}$ as pressures: $\tau_{rr} = p_r$ as radial pressure, and $\tau_{\theta\theta} = \tau_{\varphi\varphi} = p_a$ as azimuthal pressure. Using (43), (45) and (48), we can explicit the components of the Einstein equation

$$\begin{aligned}
&-\frac{2}{\sqrt{A}}\left(\frac{1}{r\sqrt{A}}\right)' - \frac{1}{A}\left(\frac{3}{r^2} + 3Z_r^2\right) + \frac{Z_t^2}{B} + \frac{1}{r^2} - \Lambda = 8\pi G\rho\left(r\right)\,, \\
&\frac{1}{A}\left(\frac{1}{r^2} + Z_r^2\right) + \frac{1}{B}\left(\frac{B'}{rA} - 3Z_t^2\right) - \frac{1}{r^2} + \Lambda = 8\pi G p_r\left(r\right)\,, \\
&\frac{1}{\sqrt{AB}}\left(\frac{B'}{2\sqrt{AB}}\right)' + \frac{1}{\sqrt{A}}\left(\frac{1}{r\sqrt{A}}\right)' + \frac{1}{A}\left(\frac{1}{r^2} + Z_r^2\right) \\
&+\frac{1}{B}\left(\frac{B'}{2rA} - Z_t^2\right) + \Lambda = 8\pi G p_a\left(r\right)\,,
\end{aligned} \tag{49}$$

$$\begin{aligned}
&\frac{Z_t}{\sqrt{B}}\frac{Z_r}{\sqrt{A}} = 4\pi G o\left(r\right) \ , \\
&\frac{Z_t}{\sqrt{B}}\frac{Z_r}{\sqrt{A}} = -4\pi G q\left(r\right) \ , \\
&-\frac{1}{\sqrt{A}}\left(\frac{Z_r}{\sqrt{A}}\right)^{'} - \frac{1}{B}\left(\frac{B^{'}}{2A}Z_r\right) = -8\pi G u\left(r\right) \ .
\end{aligned} \tag{50}$$

In the case of a completely antisymmetric torsion tensor, Cartan's equations [1] reduce to

$$A_{abc} = -8\pi G a_{abc} \,, \tag{51}$$

where a_{abc} is the completely antisymmetric irreducible part of the spin tensor s_{abc}: $a_{abc} = \frac{1}{3}\left(s_{abc} + s_{cab} + s_{bca}\right)$ [1]. Here, since the completely antisymmetric torsion tensor has two independent components (38), (39), we have two Cartan equations

$$\begin{aligned}
\frac{Z_t}{\sqrt{B}} &= 4\pi G a_{t\theta\varphi} \,, \\
\frac{Z_r}{\sqrt{A}} &= 4\pi G a_{r\theta\varphi} \,.
\end{aligned} \tag{52}$$

5. Schwarzschild star

Now, let us consider the case of a Schwarzschild star with the constant matter density ρ and constant spin densities $a_{t\theta\varphi}$ and $a_{r\theta\varphi}$. Then, it is clear that

$$a_{t\theta\varphi} = \omega_t \rho \tag{53}$$

and

$$a_{r\theta\varphi} = \omega_r \rho \,, \tag{54}$$

where ω_t and ω_r are constants. In the following, ρ, ω_t and ω_r will be treated as parameters. Then, Einstein's equations simplify to

$$\begin{aligned}
&-\frac{2}{\sqrt{A}}\left(\frac{1}{r\sqrt{A}}\right)^{'} - \frac{3}{r^2A} - 3\left(4\pi G\omega_r\rho\right)^2 + \left(4\pi G\omega_t\rho\right)^2 + \frac{1}{r^2} - \Lambda = 8\pi G\rho \,, \\
&\frac{1}{r^2A} + \left(4\pi G\omega_r\rho\right)^2 + \frac{B^{'}}{rAB} - 3\left(4\pi G\omega_t\rho\right)^2 - \frac{1}{r^2} + \Lambda = 8\pi G p_r\left(r\right) \ , \\
&\frac{1}{\sqrt{AB}}\left(\frac{B^{'}}{2\sqrt{AB}}\right)^{'} + \frac{1}{\sqrt{A}}\left(\frac{1}{r\sqrt{A}}\right)^{'} + \frac{1}{r^2A} + \left(4\pi G\omega_r\rho\right)^2 \\
&+\frac{B^{'}}{2rAB} - \left(4\pi G\omega_t\rho\right)^2 + \Lambda = 8\pi G p_a\left(r\right) \ ,
\end{aligned} \tag{55}$$

$$
\begin{aligned}
o &= -q = 4\pi G\omega_r\omega_t\rho^2\,, \\
u\,(r) &= \frac{B'}{4B\sqrt{A}}\omega_r\rho\,,
\end{aligned}
\tag{56}
$$

and Cartan's equations reduce to

$$
\frac{Z_t}{\sqrt{B}} = 4\pi G\omega_t\rho
$$

and

$$
\frac{Z_r}{\sqrt{A}} = 4\pi G\omega_r\rho\,. \tag{57}
$$

The tt-component of Einstein's equation (55) may be put in the form of

$$
-\left(\frac{r}{A}\right)' - 3\left(4\pi G\omega_r\rho\right)^2 r^2 + \left(4\pi G\omega_t\rho\right)^2 r^2 + 1 - \Lambda r^2 = 8\pi G\rho r^2\,. \tag{58}
$$

Integrating the last equation, one obtains

$$
-\frac{r}{A} - \left(4\pi G\omega_r\rho\right)^2 r^3 + \frac{1}{3}\left(4\pi G\omega_t\rho\right)^2 r^3 + r - \frac{\Lambda}{3}r^3 - 8\pi G\rho\frac{r^3}{3} = K\,, \tag{59}
$$

where K is a constant of integration to be determined. Evaluating for $r=0$ and taking into account that $A\,(0)>0$, one gets $K=0$. Then,

$$
\frac{r}{A\,(r)} + \left(4\pi G\omega_r\rho\right)^2 r^3 - \frac{1}{3}\left(4\pi G\omega_t\rho\right)^2 r^3 - r + \Lambda\frac{r^3}{3} + 8\pi G\rho\frac{r^3}{3} = 0\,. \tag{60}
$$

Finally,

$$
A\,(r) = \left[1 - \left(\Lambda + 8\pi G\rho + 48\pi^2G^2\omega_r^2\rho^2 - 16\pi^2G^2\omega_t^2\rho^2\right)\frac{r^2}{3}\right]^{-1}\,. \tag{61}
$$

For $r=R$, we have

$$
A^{-1}\,(R) = 1 - \left(\Lambda + 8\pi G\rho + 48\pi^2G^2\omega_r^2\rho^2 - 16\pi^2G^2\omega_t^2\rho^2\right)\frac{R^2}{3}\,. \tag{62}
$$

However,

$$
A^{-1}\,(R) = 1 - \frac{2GM_{\mathrm{e}}}{R} - \frac{\Lambda R^2}{3}\,, \tag{63}
$$

where M_{e} is the external mass [1] defining the strength of the gravitational field. Then, by comparison,

$$
2GM_{\mathrm{e}} = 3\left(3\omega_r^2 - \omega_t^2\right)\frac{G^2M_{\mathrm{i}}^2}{R^3} + 2GM_{\mathrm{i}}\,, \tag{64}
$$

where $M_{\rm i}$ is the internal mass defined by

$$M_{\rm i} = \frac{4}{3}\pi\rho R^3 \,. \tag{65}$$

In this way, we obtain a mass formula for the external mass

$$M_{\rm e} = M_{\rm i}\left[1 + \frac{3}{2}\left(3\omega_r^2 - \omega_t^2\right)\frac{GM_{\rm i}}{R^3}\right] \,, \tag{66}$$

quadratic in ω_r^2 and ω_t^2, which shows that the external mass is different from the internal mass, *i.e.* that the weak Gauss law does not hold, except when $\frac{\omega_r^2}{\omega_t^2} = \frac{1}{3}$.

For $\frac{\omega_r^2}{\omega_t^2} > \frac{1}{3}$, the external mass is greater than the internal mass and for $\frac{\omega_r^2}{\omega_t^2} < \frac{1}{3}$, it is the internal mass which is greater. Solution (61) is valid inside the mass distribution that is, for $r \leq R$, where R is the radius of the mass distribution. Outside the mass distribution, *i.e.* for $r \geq R$, the solution is the well-known Kottler, also named Schwarzschild–de Sitter, vacuum solution

$$A(r) = \left[1 - \frac{2GM_{\rm e}}{r} - \frac{\Lambda r^2}{3}\right]^{-1} \,. \tag{67}$$

Equation (64), with (62) and (63), is nothing but the continuity condition. Thus, we have an exact analytical solution for $A(r)$ valid for all values of r

$$A(r) = \begin{cases} \left[1 - \left(\Lambda + 8\pi G\rho + 48\pi^2 G^2\omega_r^2\rho^2 - 16\pi^2 G^2\omega_t^2\rho^2\right)\frac{r^2}{3}\right]^{-1} & r \leq R \,, \\ \left[1 - \frac{2GM_{\rm e}}{r} - \frac{\Lambda r^2}{3}\right]^{-1} & r \geq R \,. \end{cases} \tag{68}$$

Using the condition

$$\Gamma^{\lambda}_{\mu\nu}(x) = \left(\Lambda^{-1}\right)^{\lambda}_{\bar\lambda}(x)\Lambda^{\bar\mu}_{\mu}(x)\Lambda^{\bar\nu}_{\nu}(x)\Gamma^{\bar\lambda}_{\bar\mu\bar\nu}\left(\varphi(x)\right) - \left(\Lambda^{T}\right)^{\bar\nu}_{\nu}(x)\frac{\partial}{\partial x^{\mu}}\left(\Lambda^{-1}\right)^{\lambda}_{\bar\nu}(x) \,, \tag{69}$$

where $\Lambda^{\bar\mu}_{\mu}(x)$ is the Jacobian matrix

$$\Lambda^{\bar\mu}_{\mu}(x) = \frac{\partial\varphi^{\bar\mu}}{\partial x^{\mu}}(x) \,,$$

which applies to continuous as well as to discreet isometries preserving the metric, it is easy to see that Z_t and Z_r are, respectively, odd and even under time reversal. This allows us to treat the two cases separately $Z_t \neq 0$, $Z_r = 0$ ($\omega_t \neq 0$, $\omega_r = 0$) and $Z_t = 0$, $Z_r \neq 0$ ($\omega_t = 0$, $\omega_r \neq 0$). We would like to have

$M_{\mathrm{e}} > M_{\mathrm{i}}$, in view of the resolution of some dark matter problems. From now on, let us take $\omega_t = 0$ so the mass formula (66) reduces to

$$M_{\mathrm{e}} = M_{\mathrm{i}} \left[1 + \frac{9}{2} \omega_r^2 \frac{GM_{\mathrm{i}}}{R^3} \right] , \tag{70}$$

and since $Z_t = 0$, (56) reduces to

$$\begin{aligned} o(r) &= 0 , \qquad q(r) = 0 , \\ u(r) &= \frac{B'}{4B\sqrt{A}} \omega_r \rho \end{aligned} \tag{71}$$

and the expression of $A(r)$ (68) reduces to

$$A(r) = \begin{cases} \left[1 - \left(\Lambda + 8\pi G\rho + 48\pi^2 G^2 \omega_r^2 \rho^2 \right) \frac{r^2}{3} \right]^{-1} & r \le R , \\ \left[1 - \frac{2GM_{\mathrm{e}}}{r} - \frac{\Lambda r^2}{3} \right]^{-1} & r \ge R . \end{cases} \tag{72}$$

To simplify further, we take $p_r = p_a = p$ that is, we consider only one pressure. The rr and $\theta\theta$ components of the Einstein equation (55) reduce to

$$\begin{aligned} &\frac{1}{r^2 A} + \left(4\pi G \omega_r \rho\right)^2 + \frac{B^{'}}{rAB} - \frac{1}{r^2} + \Lambda = 8\pi G p\left(r\right) , \\ &\frac{1}{\sqrt{AB}} \left(\frac{B^{'}}{2\sqrt{AB}} \right)^{'} + \frac{1}{\sqrt{A}} \left(\frac{1}{r\sqrt{A}} \right)^{'} + \frac{1}{r^2 A} + \left(4\pi G \omega_r \rho\right)^2 \\ &+ \frac{B'}{2rAB} + \Lambda = 8\pi G p\left(r\right) . \end{aligned} \tag{73}$$

Since the expression of $A(r)$ is already known, (73) may be considered as a coupled nonlinear system of two equations with two unknowns $B(r)$ and $p(r)$. The solution of (73) satisfying the continuity conditions

$$B(R) = 1 - \frac{2GM_{\mathrm{e}}}{R} - \frac{\Lambda R^2}{3} , \qquad p(R) = 0 \tag{74}$$

is [11]

$$\begin{aligned} \sqrt{B(r)} &= \widetilde{\beta} \sqrt{1 - \widetilde{\gamma} r^2} + \widetilde{\alpha} \sqrt{1 - \widetilde{\gamma} R^2} \\ &= \left(1 - \widetilde{\alpha}\right) \sqrt{1 - \widetilde{\gamma} r^2} + \widetilde{\alpha} \sqrt{1 - \widetilde{\gamma} R^2} , \end{aligned} \tag{75}$$

$$p(r) = \left(\rho + 4\pi G \omega_r^2 \rho^2\right) \left(\frac{\sqrt{1 - \widetilde{\gamma} r^2}}{\sqrt{1 - \widetilde{\gamma} R^2}} - 1 \right) , \tag{76}$$

where we have introduced notations similar to those of [8]

$$\begin{aligned}
\widetilde{\gamma} &= \frac{1}{3}\left(\Lambda + 8\pi G\rho + 48\pi^2 G^2 \omega_r^2 \rho^2\right) , && (77)\\
\widetilde{\alpha} &= \frac{12\pi G\left(\rho + 4\pi G\omega_r^2 \rho^2\right)}{3\widetilde{\gamma}} , && (78)\\
\widetilde{\beta} &= \frac{\Lambda - 4\pi G\rho}{3\widetilde{\gamma}} = 1 - \widetilde{\alpha} . && (79)
\end{aligned}$$

The knowledge of $A(r)$ (72), and of $B(r)$ (75) allows us to determine the expression of $u(r)$

$$u(r) = -\frac{1}{2}\frac{(1-\widetilde{\alpha})\,\widetilde{\gamma}\omega_r \rho r}{(1-\widetilde{\alpha})\sqrt{1-\widetilde{\gamma}r^2} + \widetilde{\alpha}\sqrt{1-\widetilde{\gamma}R^2}} . \tag{80}$$

Let us notice that the expressions of $B(r)$ (75), $p(r)$ (76), and $u(r)$ (80) hold inside the mass ditribution, *i.e.* for $r \leq R$. For $r \geq R$, the solution is well-known

$$B(r) = A(r)^{-1} = 1 - \frac{2GM_{\mathrm{e}}}{r} - \frac{\Lambda r^2}{3} , \qquad p(r) = 0 , \qquad u(r) = 0 . \tag{81}$$

It is worthwhile to stress that the determination of $A(r)$ and $B(r)$, which is equivalent to the determination of the metric tensor, is essential. For instance, $A(r)$ and $B(r)$ are both involved in the geodesic equations

$$\begin{aligned}
\ddot{t} + \frac{B'(r)}{B(r)}\dot{t}\dot{r} &= 0 , && (82)\\
\ddot{r} + \frac{B'(r)}{2A(r)}\dot{t}^2 + \frac{A'(r)}{2A(r)}\dot{r}^2 - \frac{r}{A(r)}\dot{\varphi}^2 &= 0 , && (83)\\
\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} &= 0 , && (84)
\end{aligned}$$

where the dot denotes derivation with respect to an affine parameter p and the $'$ derivation with respect to r. Thus, to study geodesics, it is essential to have explicit expressions for $A(r)$ and $B(r)$.

6. Discussion

Let us now discuss our results and compare them to those of reference [1]. First, it is worthwhile to notice that in comparison to reference [1], we have, to parametrize the non-vanishing components of the connection, four additional functions of r: G_t, G_r, Z_t, Z_r. In reference [1], due to the lack of these functions, the torsion tensor does not possess an irreducible, completely antisymmetric part and thus, it was not possible to construct completely antisymmetric torsion tensors. In the present paper, in the general case, the torsion tensor has an irreducible, completely antisymmetric part and it was possible to construct completely antisymmetric torsion tensors, by requiring the irreducible vector and mixed parts to vanish. This results in torsion tensors parametrized by two functions Z_t and Z_r. These functions transform differently under time reversal. Thus, we can treat the time reversal even case corresponding to $Z_t = 0$ and the time reversal odd case corresponding to $Z_r = 0$ separately. In both cases, the weak Gauss law is broken, that is, the external $M_{\rm e}$ and the internal mass $M_{\rm i}$ differ. In the time reversal even case, the external mass $M_{\rm e}$ is greater than the internal mass $M_{\rm i}$, while in the time reversal odd case, the external mass $M_{\rm e}$ is smaller than the internal mass $M_{\rm i}$. In the light of these findings, the torsion, in the time reversal even case, may be an alternative to the dark matter by choosing suitable values of the parameter ω_r. One can interpret the internal mass $M_{\rm i}$ as the ordinary matter mass, that is, the real mass enclosed inside the sphere of radius R. However, the observer outside the sphere of mass felt a different mass, the external mass $M_{\rm e}$ greater than the internal mass $M_{\rm i}$. All happens as if the external observer feels a supplementary mass $M_{\rm e} - M_{\rm i}$, which is interpreted as dark matter mass. Using the mass formula (70), one gets

$$\frac{M_{\rm e}}{M_{\rm i}} = 1 + \frac{9}{2}\omega_r^2 \frac{GM_{\rm i}}{R^3}\,. \tag{85}$$

From (85), one can notice that the ratio $M_{\rm e}/M_{\rm i}$ is independent of the cosmological constant Λ and is a sum of two terms: 1 and a positive quadratic term in ω_r. This contrasts with the result of [1], where the ratio $M_{\rm e}/M_{\rm i}$ is a sum of three terms: 1, a linear term in ω and a positive quadratic term in ω

$$\frac{M_{\rm e}}{M_{\rm i}} = 1 + \frac{6\omega}{R^3}\int\limits_0^R \frac{\tilde{r}{\rm d}\tilde{r}}{\sqrt{A\left(\tilde{r}\right)}} + \frac{3\omega^2 GM_{\rm i}}{2R^3}\,, \tag{86}$$

from formula (58) of reference [1]. Furthermore, in this case, the ratio $M_{\rm e}/M_{\rm i}$ presents a dependence, although weak, on the cosmological constant Λ coming from the linear term in ω. For illustration purposes, let us determine, as in reference [1], the values of ω_r for the sun and for a cluster of galaxies for a ratio $M_{\rm e}/M_{\rm i} = 5$ corresponding roughly to the ratio of the ordinary matter mass to the total matter mass. From (85), one obtains

$$\omega_r = \left(\frac{8R^3}{9GM}\right)^{\frac{1}{2}}\,. \tag{87}$$

In the case of our sun, whose mass and radius are receptively: $M_{\rm i} = M_\odot = 1.9884 \times 10^{30}$ kg and $R = 7 \times 10^8$ m, we have $\omega_r = 1.52 \times 10^3$ s. For a cluster of galaxies, such that $M_{\rm i} = 10^{15} M_\odot$ and $R = 3 \times 10^{23}$, one obtains $\omega_r = 4.26 \times 10^{17}$ s. For the particular example studied by the authors of reference [1], $\omega = 3.1$ s for the sun and $\omega = 1.33 \times 10^{15}$ s for the cluster of galaxies. On the other hand, the authors of reference [9] have shown that the Hubble diagram of super novae can be fitted with the Einstein–Cartan theory with $\omega = 10^{17}$ and no dark matter. Although the value of ω_r obtained in this paper is closer to the value of ω obtained in reference [9] than that obtained in reference [1], the three values are not very far from each other. However, they are very far from the naive microscopic value

$$\omega = \frac{\hbar/2}{m_{\rm proton} c^2} \sim 10^{-25}\ {\rm s}\,. \tag{88}$$

7. Conclusion

We consider Einstein–Cartan's theory in the static spherical case with a completely antisymmetric torsion tensor. In the case of Schwarzschild star with constant mass density ρ and constant spin densities $a_{t\theta\varphi}$ and $a_{r\theta\varphi}$, *i.e.* $a_{t\theta\varphi} = \omega_t \rho$ and $a_{r\theta\varphi} = \omega_r \rho$, with ρ, ω_t, ω_r constants, we integrate the tt-component of the Einstein equation obtaining for $A(r)$ an expression showing that the weak Gauss law is broken. In the case of $\omega_t = 0$, $\omega_r \neq 0$, the torsion may be an alternative to dark matter. In this case, if we assume only one pressure p, *i.e.* $p_r = p_a = p$, we can completely solve the problem, that is, we can determine $B(r)$ and the components of the energy momentum tensor $q(r)$, $o(r)$, $p(r)$ and $u(r)$. In the $\omega_r = 0$ limit, *i.e.* in the absence of torsion, we recover the interior Kottler solution [6–8].

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