FERROMAGNETIC TRANSITION IN A SIMPLE VARIANT OF THE ISING MODEL ON MULTIPLEX NETWORKS WITH PARTIAL OVERLAP

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Multiplex networks consist of a fixed set of nodes connected by several sets of edges which are generated separately and correspond to different networks ("layers"). In this paper, the Ising model is considered on multiplex networks with two layers with partial overlap, *i.e.*, sharing only a part of nodes, with spins located in the nodes and edges corresponding to non-zero exchange integrals of ferromagnetic interactions. Critical temperature for the ferromagnetic transition is evaluated using heterogeneous mean-field approximation and the replica method, from the replica-symmetric solution. The results are valid for layers in the form of general complex networks, in particular for heterogeneous scale-free networks. The size of the overlap and the correlation between the degrees within different layers of nodes belonging to the overlap significantly influence the critical temperature. It is also argued that in typical cases, the size of the overlap does not influence the critical exponent for the ferromagnetic transition. Analytic predictions are partly confirmed by numerical simulations.

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1. Introduction

In the last two decades, investigation of complex networks and their role in natural sciences, technology and social life have formed a rapidly developing branch of statistical physics [1, 2]. A significant part of this research is devoted to the physics of interacting systems on complex networks, in particular those exhibiting various kinds of collective phenomena including phase transitions [3, 4]. For example, ferromagnetic (FM) and spin glass phase transitions in the generic Ising model on complex, possibly heterogeneous networks were studied analytically using, *e.g.*, the heterogeneous mean-field (MF) approximation [5–8], the replica method [6, 9–11], the belief propagation algorithm [12, 13], the effective field approach [14, 15] and numerically via Monte Carlo (MC) simulations [16–20]. In the context of recent interest in even more complex structures ("networks of networks"), much attention has been devoted to multiplex networks (MNs) which consist of a fixed set of nodes connected by various sets of edges [21–23]; a set of nodes together with one set of edges forms a network called a layer of an MN. MNs emerge in various social systems (*e.g.*, transportation or communications networks), and interacting systems on MNs exhibit rich variety of collective behaviors and critical phenomena. For example, percolation transition [24–27], cascading failures [28], threshold cascades [29, 30], diffusion processes [31, 32], epidemic spreading [33, 34], FM and spin glass phase transitions in the Ising model [35, 36] and in the related Ashkin–Teller model [37], ordering transition in nonequilibrium models for the opinion formation [38–41] *etc.*, were studied on MNs.

An important subclass of MNs consists of MNs with partial overlap, in which each layer shares only part of its nodes with other layers, while the sets of edges are generated separately (Fig. 1). The size and other details of

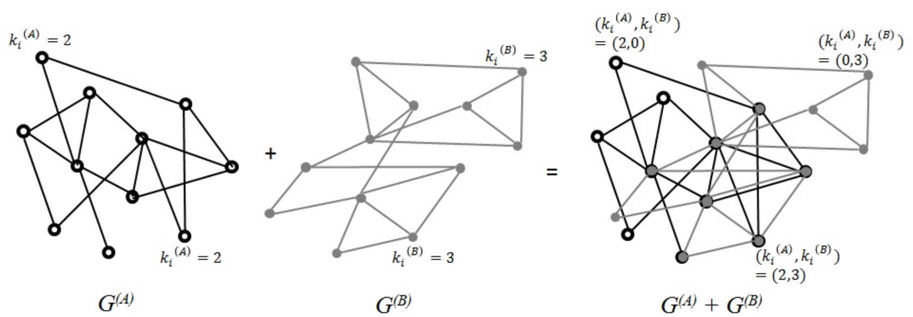

Fig. 1. Schematic illustration of generation of a MN with two layers (duplex network) with partial overlap, as described in Sec. 2.1. First, two layers are generated separately, $G^{(A)}$ with $N^{(A)} = 11$ nodes (empty circles) connected with edges (black lines) and $G^{(B)}$ with $N^{(B)} = 12$ nodes (gray points) connected with edges (gray lines). Then, $n = 7$ nodes out of $N^{(A)}$ ($N^{(B)}$) nodes belonging to the layer $G^{(A)}$ ($G^{(B)}$) are identified with each other and assumed to belong to the overlapping part of the MN. The resulting MN, shown on the right, consists of $N = N^{(A)} + N^{(B)} - n = 16$ nodes: the n nodes belonging to the overlapping part are shown as circles filled with gray. The nodes $i = 1, 2, \ldots n$ within the layer $G^{(A)}$ ($G^{(B)}$) are characterized by the degrees $k_i^{(A)}$ ($k_i^{(B)}$): for the nodes belonging to the overlap, there is $k_i^{(A)} > 0$, $k_i^{(B)} > 0$, while for the nodes belonging only to the layer $G^{(A)}$ ($G^{(B)}$), there is $k_i^{(A)} > 0$, $k_i^{(B)} = 0$ ($k_i^{(A)} = 0$, $k_i^{(B)} > 0$). In order to define the Ising model on this MN, spins are located on the nodes, and the edges in the form of black (gray) lines correspond to exchange interactions $J^{(A)} > 0$ ($J^{(B)} > 0$) in Hamiltonian (1).

the overlap play an important role, *e.g.*, in the formation of the giant mutual component in the percolation process on MNs [42, 43], in the robustness of MNs against attacks [44], epidemics spreading on MNs [45], phase transitions in nonequilibrium models of social dynamics [39, 41], *etc.* The aim of this paper is to investigate the effect of partial overlap of layers on the FM transition in the Ising model on MNs. For this purpose, a simple variant of the Ising model is studied, with spins placed on a fixed set of nodes and with separately generated sets of edges within layers corresponding to FM exchange interactions; the layers have a form of complex, possibly heterogeneous networks. The critical behavior of the model is investigated in the heterogeneous MF approximation and by means of the replica method, using the replica symmetric (RS) solution [46, 47]. In the previous studies of this model on MNs with full overlap, the possibility of the occurrence of the FM transition and strong dependence of the critical temperature on the correlation between degrees of nodes (numbers of attached edges) within different layers were demonstrated [35]. In this paper, it is shown that the size of the overlap also strongly influences the critical temperature for the FM transition. Besides, the critical temperature is again influenced by the correlation between degrees within different layers of nodes belonging to the overlapping parts of the layers. However, it is argued that in typical cases, the size of the overlap does not influence the critical exponent for the magnetization. Theoretical findings are partly confirmed by comparison with results of MC simulations.

2. The model

2.1. Multiplex networks with partial overlap

MNs consist of a fixed set of nodes connected by several sets of edges; the set of nodes with each set of edges forms a network which is called a layer of a MN [22, 23]. In the following, for simplicity, MNs with only two layers denoted as $G^{(A)}$, $G^{(B)}$ will be considered; such MNs are often referred to as duplex networks (Fig. 1). In the case of MNs with full overlap, all nodes belong to both layers, *i.e.*, each node has at least one edge from each layer attached to it. In the case of MNs with partial overlap, only part of nodes belongs to both layers, and the remaining nodes belong only to one of the layers $G^{(A)}$ ($G^{(B)}$), *i.e.*, they have at least one edge from the layer $G^{(A)}$ ($G^{(B)}$) attached to them but they have not edges from the other layer $G^{(B)}$ ($G^{(A)}$) attached to them. Henceforth, the total number of nodes in the MN is denoted as N and the nodes are labeled $i = 1, 2, \ldots N$, the number of nodes belonging to the layer $G^{(A)}$ ($G^{(B)}$) is denoted as $N^{(A)}$ ($N^{(B)}$) and the number of nodes belonging to both layers as n so that $N = N^{(A)} + N^{(B)} - n$; besides, if necessary, the nodes belonging to the layer $G^{(A)}$ ($G^{(B)}$) can be also separately labeled $l = 1, 2, \ldots N^{(A)}$ ($l' = 1, 2, \ldots N^{(B)}$). The overlap of the layer $G^{(A)}$ ($G^{(B)}$) with the other layer is $R^{(A)} = n/N^{(A)}$ ($R^{(B)} = n/N^{(B)}$), thus $R^{(A)} = R^{(B)} = 1$ corresponds to full overlap and $R^{(A)} = R^{(B)} = 0$ to two disjoint layers forming separate networks. The sets of edges of the two layers are generated separately and, possibly, independently. As a result, multiple connections between nodes are not allowed within the same layer, but the same nodes belonging to both layers can be connected by multiple edges belonging to different layers.

The nodes $i = 1, 2, \ldots N$ are characterized by their degrees $k_i^{(A)}$, $k_i^{(B)}$ within each layer, *i.e.*, the number of edges attached to them within each layer. For the nodes belonging to both layers, there is $k_i^{(A)} > 0$, $k_i^{(B)} > 0$, while for the nodes belonging only to one layer, say $G^{(A)}$, there is $k_i^{(A)} > 0$, $k_i^{(B)} = 0$. The, possibly heterogeneous, joint probability distribution of the degrees of nodes within both layers, *i.e.*, probability distribution that a node belonging to the MN has degrees $k^{(A)}$, $k^{(B)}$, is denoted as $p_{k^{(A)},k^{(B)}}$ and the averages with respect to this probability distribution are denoted as $\langle\cdot\rangle$. For example, the mean degree within the layer $G^{(A)}$ is $\langle k^{(A)}\rangle = N^{-1}\sum_{i=1}^{N} k_i^{(A)} = \sum_{k^{(A)},k^{(B)}} k^{(A)} p_{k^{(A)},k^{(B)}}$ (note that for MNs with partial overlap some terms in these sums can be zero). Besides, separate probability distributions of the degrees of nodes within each layer, *i.e.*, probability distributions that the node belonging to the layer $G^{(A)}$ ($G^{(B)}$) has degree $k^{(A)}$ ($k^{(B)}$) are denoted as $p_{k^{(A)}}$ ($p_{k^{(B)}}$); by definition, a necessary condition for $p_{k^{(A)}} > 0$ ($p_{k^{(B)}} > 0$) is $k^{(A)} > 0$ ($k^{(B)} > 0$). The respective averages are denoted as $\langle\cdot\rangle_A$ ($\langle\cdot\rangle_B$), *e.g.*, the mean degree of nodes in the layer $G^{(A)}$ is $\langle k^{(A)}\rangle_A = (N^{(A)})^{-1}\sum_{l=1}^{N^{(A)}} k_l^{(A)} = \sum_{k^{(A)}} k^{(A)} p_{k^{(A)}}$ (note that all terms in these sums are nonzero). In the simplest case of an MN with full overlap and with separately and independently generated layers, there is $p_{k^{(A)},k^{(B)}} = p_{k^{(A)}} p_{k^{(B)}}$, and this case was considered in Ref. [35].

2.2. The Hamiltonian

In a simple version of the Ising model on an MN with two layers considered in this paper, two-state spins $s_i = \pm 1$ are located in the nodes $i = 1, 2 \ldots N$ and edges within the layers $G^{(A)}$, $G^{(B)}$ connecting pairs of nodes i, j correspond to FM exchange interactions with integrals $J^{(A)} > 0$, $J^{(B)} > 0$, respectively. The Hamiltonian of the model is

$$H = -J^{(A)} \sum_{(i,j)\in G^{(A)}} s_i s_j - J^{(B)} \sum_{(i,j)\in G^{(B)}} s_i s_j \,, \tag{1}$$

where the sums are over all different edges belonging to the layer $G^{(A)}$ ($G^{(B)}$). It should be emphasized that in the model under study, there is only one spin s_i located in node i which interacts with all its neighbors within all layers; thus, interaction between different layers in Hamiltonian (1) is provided by n spins belonging to both layers $G^{(A)}$, $G^{(B)}$.

At a first glance, the model under study seems trivial since Hamiltonian (1) is equivalent to that of the Ising model on a network being a superposition of the two layers (a super-network), in which nodes belonging to both layers are simply more densely connected than the remaining ones. However, analytic study of the Ising model on such super-network can be difficult and hardly conclusive. For example, the consecutive layers can be uncorrelated networks, *i.e.*, networks without correlations between the degrees of nodes within layers; this group of networks comprises many random graphs and a broad class of heterogeneous scale-free (SF) networks. Then, for each layer, it is a simple task to evaluate a basic quantity, the probability that a node is connected to a node with a given degree within a layer. As a result, it is possible to derive the MF approximation for the model on an MN in which two magnetization-like order parameters, related to the two layers of the MN, occur in a natural way, and to evaluate the critical temperature for the possible FM phase transition [35]. In contrast, as a rule, the corresponding super-network is a correlated network, and evaluation of the correlations between the degrees of nodes of this super-network is not straightforward. In turn, in particular in the case of strongly heterogeneous layers, neglecting these correlations in the MF approximation for the Ising model on a super-network yields critical temperatures for the FM transition which noticeably deviate from the more correct values obtained in the above-mentioned MF approximation which takes into account the layered structure of the network [35]. Similarly, if the replica method is used to investigate the FM transition in a certain variant of the model under study, it is relatively easy to evaluate statistical sums over replicated spin configurations separately for each layer; again, two sets of order parameters (magnetization, spin glass order parameter, *etc.*) related to the two layers occur then in a natural way [35, 36]. Moreover, in the approaches based on the MN, it is possible to observe certain phenomena, *e.g.*, dependence of the critical temperature for the FM transition on the correlation between the degrees of nodes within different layers [35], which can easily be overlooked in the approach based on the super-network. So far, the above-mentioned approaches based on the MF approximation and the replica method have been applied to the Ising model on different MNs with full overlap. In this paper, the effect of partial overlap of layers of an MN on the FM transition in the model described by Hamiltonian (1) is investigated using similar methods.

3. Investigation of the ferromagnetic transition using the mean-field approximation

3.1. The model for a multiplex network with partial overlap

In this section, heterogeneous MF approximation is used to evaluate the critical temperature for the FM transition in the Ising model on an MN with partial overlap. In order to simplify calculations, it is convenient to assume that the layers of an MN are generated randomly and independently so that the degrees within different layers of nodes belonging to both layers are uncorrelated; the process of generation of an MN with such layers is schematically illustrated in Fig. 1. For brevity, this sort of MNs will be referred to as MNs with independent layers. The simplest way to generate an MN with two independent layers $G^{(A)}$, $G^{(B)}$ and with given distributions of the degrees of nodes within layers $p_{k^{(A)}}$, $p_{k^{(B)}}$ is probably to use the Configuration Model [48] separately and independently for each layer. To generate the first layer $G^{(A)}$, the algorithm starts with assigning to each node $l = 1, 2, \dots N^{(A)}$ in a set of $N^{(A)}$ nodes belonging to this layer a degree, *i.e.*, a random number $k_l^{(A)}$ of ends of edges drawn from a given probability distribution $p_{k^{(A)}}$, with $\tilde{m}^{(A)} < k_l^{(A)} < N^{(A)}$ (the minimum degree of node is $\tilde{m}^{(A)}$, and the maximum one $N^{(A)} - 1$), with the condition that the sum $\sum_l k_l^{(A)}$ is even. The layer is completed by connecting pairs of the ends of edges chosen uniformly at random to make complete edges, respecting the preassigned sequence $k_l^{(A)}$, and under the condition that multiple and self-connections are forbidden. The layer $G^{(B)}$ is generated in a similar way, with the degrees assigned randomly to $N^{(B)}$ nodes from the possibly different probability distribution $p_{k^{(B)}}$. Finally, n nodes randomly chosen from the set of $N^{(A)}$ nodes belonging to the layer $G^{(A)}$ are identified with n nodes randomly chosen from the set of $N^{(B)}$ nodes belonging to the layer $G^{(B)}$ by matching pairs of nodes randomly and uniformly, without repetitions, *i.e.*, it is assumed that a spin located in any of these n nodes interacts both with spins located in the nodes connected to the former node by edges of the layer $G^{(A)}$ or $G^{(B)}$. Thus, the joint probability distribution of the degrees of nodes of the MN with independent layers is

$$p_{k^{(A)},k^{(B)}} = \frac{N^{(A)}-n}{N} p_{k^{(A)}} \delta_{k^{(B)},0} + \frac{N^{(B)}-n}{N} p_{k^{(B)}} \delta_{k^{(A)},0} + \frac{n}{N} p_{k^{(A)}} p_{k^{(B)}} \,. \quad (2)$$

In order to make analytic progress, it is further assumed that the layers $G^{(A)}$, $G^{(B)}$ are uncorrelated networks, *i.e.*, that there are no correlations between the degrees of pairs of nodes connected by edges within each layer. This property concerns separate layers and is not related to the lack of correlations between the degrees within different layers of nodes belonging to both layers, which was assumed above. Analytic calculations are performed for layers in a form of heterogeneous SF networks, with the degree distributions $p_{k^{(A)}} = 0$ for $k^{(A)} < \tilde{m}^{(A)}$, $p_{k^{(A)}} = (\gamma^{(A)} - 1)(\tilde{m}^{(A)})^{\gamma^{(A)}-1}(k^{(A)})^{-\gamma^{(A)}}$ for $k^{(A)} \geq \tilde{m}^{(A)}$ and, similarly, for $p_{k^{(B)}}$. Such SF networks generated from the Configuration Model are uncorrelated for $\gamma^{(A)} > 3$, $\gamma^{(B)} > 3$, and for $\gamma^{(A)} \leq 3$, $\gamma^{(B)} \leq 3$, they are correlated unless artificial constraints are imposed on the maximum degree of nodes [49].

3.2. Heterogeneous mean-field theory

In the heterogeneous MF approximation, the FM transition in the Ising model on an MN with two layers is characterized by two order parameters $\langle S^{(A)} \rangle$, $\langle S^{(B)} \rangle$ which have a form of magnetization weighted by the degrees of nodes within the layers $G^{(A)}$, $G^{(B)}$. Derivation of the MF equations for these order parameters in the case of MNs with partial overlap follows closely that in the case of MNs with full overlap performed in Ref. [35]. As in the latter case, the average value of spin at node i obeys an equation

$$\frac{\mathrm{d}\langle s_i \rangle}{\mathrm{d}t} = -\langle s_i \rangle + \tanh\left(\beta \langle I_i \rangle\right) , \tag{3}$$

where $\beta = 1/T$ and

$$\langle I_i \rangle = J^{(A)} \sum_{\left\{j:(i,j)\in G^{(A)}\right\}} \langle s_j \rangle + J^{(B)} \sum_{\left\{j:(i,j)\in G^{(B)}\right\}} \langle s_j \rangle \tag{4}$$

is the average value of the local field acting at the spin at node i.

The basic assumption of the heterogeneous MF theory for the Ising model on MNs is that the nodes of the network are divided into classes according to their degrees $(k^{(A)}, k^{(B)})$ and that the average values of spins in nodes belonging to the same class $\langle s_{k^{(A)},k^{(B)}} \rangle$ are equal. Taking into account that for uncorrelated layers, the probability that the edge of the layer $G^{(A)}$ attached at one end to the node i is linked at the other end to the node with degrees $(k^{(A)}, k^{(B)})$ is

$$\frac{p_{k^{(A)},k^{(B)}} k^{(A)}}{\sum_{k^{(A)},k^{(B)}} p_{k^{(A)},k^{(B)}} k^{(A)}} = \frac{p_{k^{(A)},k^{(B)}} k^{(A)}}{\left\langle k^{(A)} \right\rangle} \tag{5}$$

and similarly for the layer $G^{(B)}$, and thus that the number of nodes with degrees $(k^{(A)}, k^{(B)})$ connected to the node i by edges of the layer $G^{(A)}$ is

$$k_i^{(A)} \frac{p_{k^{(A)},k^{(B)}} k^{(A)}}{\left\langle k^{(A)} \right\rangle} , \tag{6}$$

and similarly for the layer $G^{(B)}$, and replacing the sums over the indices of nodes by sums over the classes of nodes, Eq. (4) can be written as

$$\begin{aligned}
\langle I_i \rangle &= J^{(A)} k_i^{(A)} \sum_{k^{(A)},k^{(B)}} \frac{p_{k^{(A)},k^{(B)}} k^{(A)}}{\left\langle k^{(A)} \right\rangle} \langle s_{k^{(A)},k^{(B)}} \rangle \\
&\quad + J^{(B)} k_i^{(B)} \sum_{k^{(A)},k^{(B)}} \frac{p_{k^{(A)},k^{(B)}} k^{(B)}}{\left\langle k^{(B)} \right\rangle} \langle s_{k^{(A)},k^{(B)}} \rangle \\
&= J^{(A)} k_i^{(A)} \left\langle S^{(A)} \right\rangle + J^{(B)} k_i^{(B)} \left\langle S^{(B)} \right\rangle ,
\end{aligned} \tag{7}$$

and Eq. (3) as

$$\frac{\mathrm{d}\langle s_i \rangle}{\mathrm{d}t} = -\langle s_i \rangle + \tanh\left[\beta\left(J^{(A)} k_i^{(A)} \left\langle S^{(A)} \right\rangle + J^{(B)} k_i^{(B)} \left\langle S^{(B)} \right\rangle\right)\right] . \tag{8}$$

In the above equations, in a natural way, two order parameters occur

$$\begin{aligned}
\left\langle S^{(A)} \right\rangle &\equiv \frac{1}{N\left\langle k^{(A)} \right\rangle} \sum_{i=1}^{N} k_i^{(A)} \langle s_i \rangle = \sum_{k^{(A)},k^{(B)}} \frac{p_{k^{(A)},k^{(B)}} k^{(A)}}{\left\langle k^{(A)} \right\rangle} \langle s_{k^{(A)},k^{(B)}} \rangle , \\
\left\langle S^{(B)} \right\rangle &\equiv \frac{1}{N\left\langle k^{(B)} \right\rangle} \sum_{i=1}^{N} k_i^{(B)} \langle s_i \rangle = \sum_{k^{(A)},k^{(B)}} \frac{p_{k^{(A)},k^{(B)}} k^{(B)}}{\left\langle k^{(B)} \right\rangle} \langle s_{k^{(A)},k^{(B)}} \rangle .
\end{aligned} \tag{9}$$

Multiplying both sides of Eq. (8) by $\frac{k_i^{(A)}}{N\langle k^{(A)} \rangle}$ $\left(\frac{k_i^{(B)}}{N\langle k^{(B)} \rangle}\right)$, performing the sum over the nodes and replacing it with the sum over the classes of nodes, results in the following system of MF equations for the order parameters:

$$\begin{aligned}
\frac{\mathrm{d}\left\langle S^{(A)} \right\rangle}{\mathrm{d}t} &= -\left\langle S^{(A)} \right\rangle \\
&+ \sum_{k^{(A)},k^{(B)}} \frac{p_{k^{(A)},k^{(B)}} k^{(A)}}{\left\langle k^{(A)} \right\rangle} \tanh\left[\beta\left(J^{(A)} k_i^{(A)} \left\langle S^{(A)} \right\rangle + J^{(B)} k_i^{(B)} \left\langle S^{(B)} \right\rangle\right)\right] , \\
\frac{\mathrm{d}\left\langle S^{(B)} \right\rangle}{\mathrm{d}t} &= -\left\langle S^{(B)} \right\rangle \\
&+ \sum_{k^{(A)},k^{(B)}} \frac{p_{k^{(A)},k^{(B)}} k^{(B)}}{\left\langle k^{(B)} \right\rangle} \tanh\left[\beta\left(J^{(A)} k_i^{(A)} \left\langle S^{(A)} \right\rangle + J^{(B)} k_i^{(B)} \left\langle S^{(B)} \right\rangle\right)\right] .
\end{aligned} \tag{10}$$

3.3. Mean-field critical temperature for the ferromagnetic transition

Equation (10) has a fixed point $(\langle S^{(A)}\rangle, \langle S^{(B)}\rangle) = (0,0)$ corresponding to the PM phase. Expanding Eq. (10) in the vicinity of this fixed point up to linear terms yields

$$\begin{aligned}\frac{\mathrm{d}\left\langle S^{(A)}\right\rangle}{\mathrm{d}t} &= \left(-1+\beta J^{(A)}\frac{\left\langle k^{(A)2}\right\rangle}{\left\langle k^{(A)}\right\rangle}\right)\left\langle S^{(A)}\right\rangle + \beta J^{(B)}\frac{\left\langle k^{(A)}k^{(B)}\right\rangle}{\left\langle k^{(A)}\right\rangle}\left\langle S^{(B)}\right\rangle\,,\\ \frac{\mathrm{d}\left\langle S^{(B)}\right\rangle}{\mathrm{d}t} &= \beta J^{(A)}\frac{\left\langle k^{(A)}k^{(B)}\right\rangle}{\left\langle k^{(B)}\right\rangle}\left\langle S^{(A)}\right\rangle + \left(-1+\beta J^{(B)}\frac{\left\langle k^{(B)2}\right\rangle}{\left\langle k^{(B)}\right\rangle}\right)\left\langle S^{(B)}\right\rangle\,.\end{aligned} \tag{11}$$

Using Eq. (2), it is possible to express the averages with respect to $p_{k^{(A)},k^{(B)}}$ in Eq. (11) by averages with respect to the known degree distributions within layers $p_{k^{(A)}}$, $p_{k^{(B)}}$

$$\begin{aligned}\left\langle k^{(A)}k^{(B)}\right\rangle &= \sum_{k^{(A)},k^{(B)}} k^{(A)}k^{(B)}p_{k^{(A)},k^{(B)}}\\ &= \frac{N^{(A)}-n}{N}\sum_{k^{(A)}}k^{(A)}p_{k^{(A)}}\sum_{k^{(B)}}k^{(B)}\delta_{k^{(B)},0}\\ &\quad+\frac{N^{(B)}-n}{N}\sum_{k^{(A)}}k^{(A)}\delta_{k^{(A)},0}\sum_{k^{(B)}}k^{(B)}p_{k^{(B)}}\\ &\quad+\frac{n}{N}\sum_{k^{(A)}}k^{(A)}p_{k^{(A)}}\sum_{k^{(B)}}k^{(B)}p_{k^{(B)}} = \frac{n}{N}\left\langle k^{(A)}\right\rangle_A\left\langle k^{(B)}\right\rangle_B\,,\end{aligned} \tag{12}$$

$$\left\langle k^{(A)}\right\rangle = \frac{N^{(A)}}{N}\left\langle k^{(A)}\right\rangle_A\,, \qquad \left\langle k^{(B)}\right\rangle = \frac{N^{(B)}}{N}\left\langle k^{(B)}\right\rangle_B\,, \tag{13}$$

$$\left\langle k^{(A)2}\right\rangle = \frac{N^{(A)}}{N}\left\langle k^{(A)2}\right\rangle_A\,, \qquad \left\langle k^{(B)2}\right\rangle = \frac{N^{(B)}}{N}\left\langle k^{(B)2}\right\rangle_B\,. \tag{14}$$

Substituting Eqs. (12)–(14) into Eq. (11) yields the following system of equations for the order parameters:

$$\begin{aligned}\frac{\mathrm{d}\left\langle S^{(A)}\right\rangle}{\mathrm{d}t} &= \left(-1+\beta J^{(A)}\frac{\left\langle k^{(A)2}\right\rangle_A}{\left\langle k^{(A)}\right\rangle_A}\right)\left\langle S^{(A)}\right\rangle + \beta J^{(B)}R^{(A)}\left\langle k^{(B)}\right\rangle_B\left\langle S^{(B)}\right\rangle\,,\\ \frac{\mathrm{d}\left\langle S^{(B)}\right\rangle}{\mathrm{d}t} &= \beta J^{(A)}R^{(B)}\left\langle k^{(A)}\right\rangle_A\left\langle S^{(A)}\right\rangle + \left(-1+\beta J^{(B)}\frac{\left\langle k^{(B)2}\right\rangle_B}{\left\langle k^{(B)}\right\rangle_B}\right)\left\langle S^{(B)}\right\rangle\,.\end{aligned} \tag{15}$$

The paramegnetic fixed point becomes unstable, and the FM phase occurs, if one of the eigenvalues of Eq. (15) crosses zero which takes place if the determinant of the right-hand sides is zero. This, in general, leads to two solutions $T_{c\pm}$. The higher temperature T_{c-} corresponds to the critical temperature for the FM transition from the paramagnetic phase, $T_{c,MF}^{FM} = T_{c-}$. Below $T_{c,MF}^{FM}$, the paramagnetic state is unstable, and the instability at $T = T_{c+} < T_{c,MF}^{FM}$ has no physical meaning. In a simple case with $J^{(A)} = J^{(B)} = J$, $N^{(A)} = N^{(B)}$ and thus $R^{(A)} = R^{(B)} = R$, explicit expressions for $T_{c\pm}$ are

$$T_{c\pm} = 2J \frac{\frac{\left\langle k^{(A)2}\right\rangle_A \left\langle k^{(B)2}\right\rangle_B}{\left\langle k^{(A)}\right\rangle_A \left\langle k^{(B)}\right\rangle_B} - R^2 \left\langle k^{(A)}\right\rangle_A \left\langle k^{(B)}\right\rangle_B}{\frac{\left\langle k^{(A)2}\right\rangle_A}{\left\langle k^{(A)}\right\rangle_A} + \frac{\left\langle k^{(B)2}\right\rangle_B}{\left\langle k^{(B)}\right\rangle_B} \pm \sqrt{\Delta}}\,, \tag{16}$$

where

$$\Delta = \left(\frac{\left\langle k^{(A)2}\right\rangle_A}{\left\langle k^{(A)}\right\rangle_A} - \frac{\left\langle k^{(B)2}\right\rangle_B}{\left\langle k^{(B)}\right\rangle_B} \right)^2 + 4R^2 \left\langle k^{(A)}\right\rangle_A \left\langle k^{(B)}\right\rangle_B \,.$$

In particular, in the case of two layers with identical degree distributions $p_{k^{(B)}} = p_{k^{(A)}}$ and thus with $\langle k^{(B)}\rangle_B = \langle k^{(A)}\rangle_A$, $\langle k^{(B)2}\rangle_B = \langle k^{(A)2}\rangle_A$

$$T_{c\pm} = J \left(\frac{\left\langle k^{(A)2}\right\rangle_A}{\left\langle k^{(A)}\right\rangle_A} \mp R \left\langle k^{(A)}\right\rangle_A \right) . \tag{17}$$

From Eq. (17) follows that the critical temperature for the FM transition increases in general linearly with the size of the overlap R. This is not surprising since then the mean degree of nodes in the aggregate supernetwork also rises which should shift the critical temperature upwards. It is interesting to note that qualitatively, similar dependence of the critical temperature for the FM transition was obtained in the heterogeneous MF approximation for the Ising model on modular networks [7, 8], which are another kind of "networks of networks". This is so though the topology of MNs is different from that of modular networks, which consist of subnetworks such that density of edges connecting nodes belonging to the same subnetwork is significantly higher than that connecting nodes belonging to different subnetworks (loosely corresponding to the size of the overlap R in MNs).

4. Investigation of the ferromagnetic transition using the replica method

4.1. The model for a multiplex network with partial overlap

In order to investigate FM transition in the model under study using the replica method, it is convenient to generate the layers of the MN with desired degree distributions from the static model [50, 51]. Each layer is generated separately and independently (Fig. 1). In order to generate the first layer $G^{(A)}$, the nodes of the MN belonging to this layer are numbered randomly as $l = 1, 2, \dots N^{(A)}$ and weights $v_l^{(A)} > 0$ are assigned to them, while zero weights are assigned to the remaining $N - N^{(A)}$ nodes which do not belong to the layer $G^{(A)}$, $v_i^{(A)} = 0$ for $i \notin G^{(A)}$. The weights are normalized so that $\sum_{i=1}^{N} v_i^{(A)} = \sum_{l=1}^{N^{(A)}} v_l^{(A)} = 1$. Then, nodes are linked with edges in accordance with the prescribed sequence of weights, by selecting a pair of nodes i, j ($i \neq j$) with probablities $v_i^{(A)}$, $v_j^{(A)}$, respectively, linking them with an edge and repeating this process $N\langle k^{(A)}\rangle/2$ times. In this way, a network with N nodes is obtained with the probability that the nodes i, j are linked by an edge $f_{ij} \approx N\langle k^{(A)}\rangle v_i^{(A)} v_j^{(A)}$ and with the mean degree of nodes $\langle k^{(A)}\rangle$. In fact, following this procedure, only $N^{(A)}$ nodes belonging to the layer $G^{(A)}$ can have edges attached, while the remaining $N - N^{(A)}$ edges remain unconnected and do not belong to the layer $G^{(A)}$. In this way, the layer $G^{(A)}$ is generated with $N^{(A)}$ nodes and with the mean degree of nodes $\langle k^{(A)}\rangle_A = N\langle k^{(A)}\rangle/N^{(A)}$. The distribution of the degrees of nodes within the layer $p_{k^{(A)}}$ depends on the choice of weights. In particular, for v_l drawn from the zeta distribution, $v_l = l^{-\mu^{(A)}}/\zeta_{N^{(A)}}(\mu^{(A)})$, where $0 < \mu^{(A)} < 1$ and $\zeta_{N^{(A)}}(\mu^{(A)}) \approx (N^{(A)})^{1-\mu^{(A)}}/(1-\mu^{(A)})$, SF network is obtained with the distribution of the degrees of nodes $p_{k^{(A)}} \propto (k^{(A)})^{-\gamma^{(A)}}$, $\gamma^{(A)} = 1 + 1/\mu^{(A)}$. In an ensemble of layers generated from the static model in the above-mentioned way, the mean degree of a given node i would be $N\langle k^{(A)}\rangle v_i^{(A)} = N^{(A)}\langle k^{(A)}\rangle_A v_i^{(A)}$. The next layer $G^{(B)}$ is generated analogously.

In order to complete the process of generation of an MN, n nodes of the layer $G^{(A)}$ must be identified with n nodes of the layer $G^{(B)}$ (Fig. 1). This identification can be performed in various ways, leading to possible correlation between the degrees of nodes within different layers. The simplest way is to select randomly n out of $N^{(A)}$ nodes belonging to the layer $G^{(A)}$ and n out of $N^{(B)}$ nodes belonging to the layer $G^{(B)}$, and to identify nodes with one another by matching pairs of nodes from these two sets randomly and uniformly, without repetitions. Then the weights, and hence also the degrees of identified nodes within different layers are uncorrelated and for sufficiently large n, their correlation coefficient can be approximated by its expected value

$$\sum_{i=1}^{N} v_i^{(A)} v_i^{(B)} \approx \left\langle \sum_{i=1}^{N} v_i^{(A)} v_i^{(B)} \right\rangle = n \left\langle v_i^{(A)} v_i^{(B)} \right\rangle = \frac{n}{N^{(A)} N^{(B)}} \sum_{l=1}^{N^{(A)}} \sum_{l'=1}^{N^{(B)}} v_l v_{l'}$$
$$= \frac{n}{N^{(A)} N^{(B)}} \left(\sum_{l=1}^{N^{(A)}} v_l \right) \left(\sum_{l'=1}^{N^{(B)}} v_{l'} \right) = \frac{n}{N^{(A)} N^{(B)}} \, . \tag{18}$$

Thus, in this way, an MN with two independent layers is obtained. However, it is also easy to obtain an MN with maximum possible correlation between the weights $v_i^{(A)}$, $v_i^{(B)}$ of nodes belonging to both layers, and thus between their degrees within different layers, by matching nodes l belonging to the layer $G^{(A)}$ with nodes $l' = l$ belonging to the layer $G^{(B)}$ for $l = 1, 2, \ldots n$. An MN with minimum possible correlation between the weights $v_i^{(A)}$, $v_i^{(B)}$ can be easily obtained by matching nodes l belonging to the layer $G^{(A)}$ with nodes $l' = N^{(B)} - l + 1$ belonging to the layer $G^{(B)}$ for $l = 1, 2, \ldots n$ or by matching nodes l' belonging to the layer $G^{(B)}$ with nodes $l = N^{(A)} - l' + 1$ belonging to the layer $G^{(A)}$ for $l' = 1, 2, \ldots n$ (note that for $\mu^{(A)} \neq \mu^{(B)}$ or $N^{(A)} \neq N^{(B)}$, these two ways of matching nodes are not equivalent). For brevity, these two cases are referred to as MNs with maximally and minimally correlated layers. Henceforth in this section, only FM transition in the model on MNs with independent layers with partial overlap is discussed; the effect of correlation between the weights $v_i^{(A)}$, $v_i^{(B)}$ on the critical temperature for the FM transition is briefly discussed in Appendix B.

4.2. The replica symmetric free energy

Investigation of the thermodynamic properties of the Ising model on MNs with partial overlap by means of the replica method follows closely analogous considerations in the case of the Ising model on MNs with full overlap until an explicit form of the weights assigned to the nodes in the static model becomes important; for details, see Refs. [35, 36]. The starting point is to evaluate the free energy averaged over a statistical ensemble of MNs generated from the static model with given weights associated with N nodes and with given overlap, $-\beta F = [\ln Z]_{\mathrm{av}}$, where Z is the partition function for the Ising model on an MN with a particular set of edges (*i.e.*, the two sets of edges in the separately generated layers $G^{(A)}$, $G^{(B)}$), and the average $[\cdot]_{\mathrm{av}}$ is taken over all possible random realizations of a set of edges. In the framework of the replica method, the free energy is formally evaluated as $-\beta F = \lim_{\nu \to 0} \left\{ [Z^{\nu}]_{\mathrm{av}} - 1 \right\} / \nu$. In the case of an MN with separately generated layers, the averages over the realizations of the sets of edges can be evaluated independently for each layer, which yields [35, 36]

$$
\begin{aligned}
&[Z^\nu]_{\mathrm{av}} \\
&= \int \mathrm{d}q_\alpha^{(A)} \int \mathrm{d}q_{\alpha\beta}^{(A)} \ldots \int \mathrm{d}q_\alpha^{(B)} \int \mathrm{d}q_{\alpha\beta}^{(B)} \ldots \mathrm{e}^{-N\nu\beta f\left(q_\alpha^{(A)}, q_{\alpha\beta}^{(A)}, \ldots q_\alpha^{(B)}, q_{\alpha\beta}^{(B)} \ldots\right)} \\
&\equiv \int \mathrm{d}\boldsymbol{q} \exp\left[-N\nu\beta f(\boldsymbol{q})\right] , \qquad (19)
\end{aligned}
$$

with

$$
\begin{aligned}
\nu\beta f(\boldsymbol{q}) \;=\; & \frac{\left\langle k^{(A)}\right\rangle \boldsymbol{T}_1^{(A)}}{2} \sum_\alpha q_\alpha^{(A)2} + \frac{\left\langle k^{(B)}\right\rangle \boldsymbol{T}_1^{(B)}}{2} \sum_\alpha q_\alpha^{(B)2} \\
& + \frac{\left\langle k^{(A)}\right\rangle \boldsymbol{T}_2^{(A)}}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^{(A)2} + \frac{\left\langle k^{(B)}\right\rangle \boldsymbol{T}_2^{(B)}}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^{(B)2} + \ldots \\
& - \frac{1}{N} \sum_{i=1}^{N} \ln \mathrm{Tr}_{\{s_i^\alpha\}} \exp\left(X_i^{(A)} + X_i^{(B)}\right) , \qquad (20)
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{T}_1^{(A)} \;&=\; \cosh^\nu \beta J^{(A)} \tanh \beta J^{(A)} \stackrel{\nu \to 0}{\rightarrow} \tanh \beta J^{(A)} , \\
\boldsymbol{T}_2^{(A)} \;&=\; \cosh^\nu \beta J^{(A)} \tanh^2 \beta J^{(A)} \stackrel{\nu \to 0}{\rightarrow} \tanh^2 \beta J^{(A)} , \qquad (21)
\end{aligned}
$$

and similarly for $\boldsymbol{T}_1^{(B)}$, $\boldsymbol{T}_2^{(B)}$, and $\alpha, \beta, \ldots$ denote different replicas, $\mathrm{Tr}_{\{s_i^\alpha\}}$ is the trace over the replicated spins at node i, and

$$
X_i^{(A)} = N\left\langle k^{(A)}\right\rangle \boldsymbol{T}_1^{(A)} v_i^{(A)} \sum_\alpha q_\alpha^{(A)} s_i^\alpha + N\left\langle k^{(A)}\right\rangle \boldsymbol{T}_2^{(A)} v_i^{(A)} \sum_{\alpha<\beta} q_{\alpha\beta}^{(A)} s_i^\alpha s_i^\beta + \ldots , \qquad (22)
$$

and similarly for $X_i^{(B)}$. The elements of a set $\{\boldsymbol{q}\}$, $q_\alpha^{(A)}, q_{\alpha\beta}^{(A)}, \ldots, q_\alpha^{(B)}, q_{\alpha\beta}^{(B)}, \ldots$ form in a natural way two subsets of the order parameters associated with the two layers of the multiplex network $G^{(A)}$, $G^{(B)}$. The first two order parameters,

$$
q_\alpha^{(A)} = \sum_i v_i^{(A)} \overline{s_i^\alpha} , \qquad q_\alpha^{(B)} = \sum_i v_i^{(B)} \overline{s_i^\alpha} , \qquad (23)
$$

where the averages are evaluated as

$$
\overline{s_i^\alpha} = \frac{\mathrm{Tr}_{\{s_i^\alpha\}} s_i^\alpha \exp\left(X_i^{(A)} + X_i^{(B)}\right)}{\mathrm{Tr}_{\{s_i^\alpha\}} \exp\left(X_i^{(A)} + X_i^{(B)}\right)} ,
$$

are usually called magnetizations; the next two order parameters

$$q_{\alpha\beta}^{(A)} = \sum_i v_i^{(A)} \overline{s_i^\alpha s_i^\beta}\,, \qquad q_{\alpha\beta}^{(B)} = \sum_i v_i^{(B)} \overline{s_i^\alpha s_i^\beta}\,, \tag{24}$$

are called spin glass order parameters, *etc.*

The simplest RS solution for the order parameters is obtained under the assumption that spins with different replica index are indistinguishable. In the case of the Ising model on an MN, this solution has a form of $q_\alpha^{(A)} = m^{(A)}$, $q_{\alpha\beta}^{(A)} = q^{(A)}$, *etc.*, and $q_\alpha^{(B)} = m^{(B)}$, $q_{\alpha\beta}^{(B)} = q^{(B)}$, *etc.*, for $\alpha, \beta = 1, 2 \ldots n$, *etc.*, where, in general, $m^{(A)} \neq m^{(B)}$, $q^{(A)} \neq q^{(B)}$, *etc.* [35, 36]. In the case of the model with purely FM interactions, it is enough to retain in the free energy only terms containg magnetizations $m^{(A)}$, $m^{(B)}$ and truncate in Eq. (20) terms of the order higher than m^2. Assuming the above-mentioned form of the RS solution and taking the limit $\nu \to 0$ yields

$$\begin{aligned}
&\beta f\left(m^{(A)}, m^{(B)}\right) \\
&= \frac{\left\langle k^{(A)}\right\rangle \boldsymbol{T}_1^{(A)}}{2} m^{(A)2} + \frac{\left\langle k^{(B)}\right\rangle \boldsymbol{T}_1^{(B)}}{2} m^{(B)2} - \frac{1}{N}\sum_{i=1}^N \ln\left(2\cosh\eta_i\right)\,,
\end{aligned} \tag{25}$$

where

$$\eta_i = N\left(\left\langle k^{(A)}\right\rangle \boldsymbol{T}_1^{(A)} v_i^{(A)} m^{(A)} + \left\langle k^{(B)}\right\rangle \boldsymbol{T}_1^{(B)} v_i^{(B)} m^{(B)}\right)\,. \tag{26}$$

With $\beta f(\boldsymbol{q})$ given by Eq. (25), the integral in Eq. (19) can be evaluated using the saddle-point method. For this purpose, the minimum of the function $f(m^{(A)}, m^{(B)})$ should be found, and the necessary condition for the existence of extremum leads to the following set of self-consistent equations for the magnetizations $m^{(A)}$, $m^{(B)}$

$$\begin{aligned}
\frac{\partial f}{\partial m^{(A)}} = 0 \;\Leftrightarrow\; m^{(A)} &= \sum_{i=1}^N v_i^{(A)} \tanh\eta_i\,, \\
\frac{\partial f}{\partial m^{(B)}} = 0 \;\Leftrightarrow\; m^{(B)} &= \sum_{i=1}^N v_i^{(B)} \tanh\eta_i\,.
\end{aligned} \tag{27}$$

4.3. Critical temperature for the ferromagnetic transition

After expanding the logarithm in Eq. (25) and retaining only square terms with respect to the order parameters in $f(m^{(A)}, m^{(B)})$, the system of equations in Eq. (27) leads to the following system of linear equations valid for small $m^{(A)}$, $m^{(B)}$:

$$\begin{aligned}
&\left(1 - N\left\langle k^{(A)}\right\rangle \boldsymbol{T}_1^{(A)} \sum_{i=1}^{N} v_i^{(A)2}\right) m^{(A)} \\
&- N\left\langle k^{(B)}\right\rangle \boldsymbol{T}_1^{(B)} \left(\sum_{i=1}^{N} v_i^{(A)} v_i^{(B)}\right) m^{(B)} = 0\,, \\
&- N\left\langle k^{(A)}\right\rangle \boldsymbol{T}_1^{(A)} \left(\sum_{i=1}^{N} v_i^{(A)} v_i^{(B)}\right) m^{(A)} \\
&+ \left(1 - N\left\langle k^{(B)}\right\rangle \boldsymbol{T}_1^{(B)} \sum_{i=1}^{N} v_i^{(B)2}\right) m^{(B)} = 0\,.
\end{aligned} \tag{28}$$

Non-zero solutions of the system of equations (28) exist if the determinant is zero. From this condition, the critical temperature for the FM transition can be evaluated: the corresponding equation for the critical temperature is quadratic with respect to $\tanh\beta J^{(A)}$, $\tanh\beta J^{(B)}$, thus, it has two solutions of which that with a higher value corresponds to $T_{\mathrm{c,RS}}^{\mathrm{FM}}$.

The sums in the diagonal coefficients in the system of equations (28) diverge for $\frac{1}{2} \leq \mu^{(A)} < 1$ $(2 < \gamma^{(A)} \leq 3)$, $\frac{1}{2} \leq \mu^{(B)} < 1$ $(2 < \gamma^{(B)} \leq 3)$, while for $0 < \mu^{(A)} < \frac{1}{2}$ $(\gamma^{(A)} > 3)$, $0 < \mu^{(B)} < \frac{1}{2}$ $(\gamma^{(B)} > 3)$ are [51]

$$N\left\langle k^{(A)}\right\rangle \sum_{i=1}^{N} v_i^{(A)2} = N^{(A)}\left\langle k^{(A)}\right\rangle_A \sum_{l=1}^{N^{(A)}} v_l^{(A)2} \approx \left\langle k^{(A)}\right\rangle_A \frac{\left(1-\mu^{(A)}\right)^2}{1-2\mu^{(A)}}\,, \tag{29}$$

etc. The sums in the off-diagonal coefficients depend on the correlation between the weights $v_i^{(A)}$, $v_i^{(B)}$ of nodes belonging to both layers, and thus between their degrees within different layers. In the case of MN with independent layers for sufficiently large n, they can be approximated by their expected values, as in Eq. (18), which yields

$$\begin{aligned}
N\left\langle k^{(A)}\right\rangle \sum_{i=1}^{N} v_i^{(A)} v_i^{(B)} &\approx R^{(B)}\left\langle k^{(A)}\right\rangle_A\,, \\
N\left\langle k^{(B)}\right\rangle \sum_{i=1}^{N} v_i^{(A)} v_i^{(B)} &\approx R^{(A)}\left\langle k^{(B)}\right\rangle_B\,.
\end{aligned} \tag{30}$$

For simplicity, let us focus on the case of the model of an MN with independent layers such that $J^{(A)} = J^{(B)} = J$, $N^{(A)} = N^{(B)}$ and thus

$R^{(A)} = R^{(B)} = R$. This case was considered in the heterogeneous MF approximation in Sec. 3.3. The critical temperature for the FM transition can be obtained from Eq. (28) using Eqs. (29), (30)

$$T_{\mathrm{c,RS}}^{\mathrm{FM}} = J \,\mathrm{atanh}^{-1} \left\{ \frac{\left\langle k^{(A)} \right\rangle_A \frac{\left(1-\mu^{(A)}\right)^2}{1-2\mu^{(A)}} + \left\langle k^{(B)} \right\rangle_B \frac{\left(1-\mu^{(B)}\right)^2}{1-2\mu^{(B)}} - \sqrt{\Delta}}{2 \left\langle k^{(A)} \right\rangle_A \left\langle k^{(B)} \right\rangle_B \left[\frac{\left(1-\mu^{(A)}\right)^2}{1-2\mu^{(A)}} \frac{\left(1-\mu^{(B)}\right)^2}{1-2\mu^{(B)}} - R^2 \right]} \right\}, \tag{31}$$

where

$$\Delta = \left[\left\langle k^{(A)} \right\rangle_A \frac{\left(1-\mu^{(A)}\right)^2}{1-2\mu^{(A)}} - \left\langle k^{(B)} \right\rangle_B \frac{\left(1-\mu^{(B)}\right)^2}{1-2\mu^{(B)}} \right]^2 + 4R^2 \left\langle k^{(A)} \right\rangle_A \left\langle k^{(B)} \right\rangle_B .$$

In particular, in the case of two layers with identical degree distributions and thus with $\left\langle k^{(B)} \right\rangle_B = \left\langle k^{(A)} \right\rangle_A$, $\mu^{(A)} = \mu^{(B)}$, there is

$$T_{\mathrm{c,RS}}^{\mathrm{FM}} = J \,\mathrm{atanh}^{-1} \left[\left\langle k^{(A)} \right\rangle_A^{-1} \left[\frac{\left(1-\mu^{(A)}\right)^2}{1-2\mu^{(A)}} + R \right]^{-1} \right] . \tag{32}$$

Equations (31), (32) can be written in a more general form taking into account that in Eq. (29), for networks obtained from the static model, the following equality holds [51]:

$$N^{(A)} \sum_{l=1}^{N^{(A)}} v_l^{(A)2} = \frac{\left\langle k^{(A)2} \right\rangle_A - \left\langle k^{(A)} \right\rangle_A}{\left\langle k^{(A)} \right\rangle_A^2} . \tag{33}$$

Then from Eq. (28) using Eqs. (29), (33), the critical temperature for the FM transition can be written as

$$T_{\mathrm{c,RS}}^{\mathrm{FM}} = J \,\mathrm{atanh}^{-1} \left\{ \frac{\frac{\left\langle k^{(A)2} \right\rangle_A}{\left\langle k^{(A)} \right\rangle_A} + \frac{\left\langle k^{(B)2} \right\rangle_B}{\left\langle k^{(B)} \right\rangle_B} - 2 - \sqrt{\Delta}}{2 \left[\left(\frac{\left\langle k^{(A)2} \right\rangle_A}{\left\langle k^{(A)} \right\rangle_A} - 1 \right) \left(\frac{\left\langle k^{(B)2} \right\rangle_B}{\left\langle k^{(B)} \right\rangle_B} - 1 \right) - R^2 \left\langle k^{(A)} \right\rangle_A \left\langle k^{(B)} \right\rangle_B \right]} \right\}, \tag{34}$$

where

$$\Delta = \left(\frac{\left\langle k^{(A)2} \right\rangle_A}{\left\langle k^{(A)} \right\rangle_A} - \frac{\left\langle k^{(B)2} \right\rangle_B}{\left\langle k^{(B)} \right\rangle_B} \right)^2 + 4R^2 \left\langle k^{(A)} \right\rangle_A \left\langle k^{(B)} \right\rangle_B .$$

In particular, in the case of two layers with identical degree distributions and thus with $\langle k^{(B)}\rangle_B = \langle k^{(A)}\rangle_A$, $\langle k^{(B)2}\rangle_B = \langle k^{(A)2}\rangle_A$

$$T_{\mathrm{c,RS}}^{\mathrm{FM}} = J \,\mathrm{atanh}^{-1}\left[\left(\frac{\left\langle k^{(A)2}\right\rangle_A}{\left\langle k^{(A)}\right\rangle_A} + R\left\langle k^{(A)}\right\rangle_A - 1\right)^{-1}\right]. \tag{35}$$

Equations (34) and (35) correspond to Eq. (16) and Eq. (17), respectively, obtained in the heterogeneous MF approximation. Although Eqs. (34), (35) were derived for the static model, it is expected that they can be used for the Ising model on any MN with partial overlap, with degree distributions within heterogeneous layers characterized by the moments $\langle k^{(A)}\rangle_A$, $\langle k^{(A)2}\rangle_A$ for the layer $G^{(A)}$ and similarly for the layer $G^{(B)}$.

5. Comparison with numerical results

The Ising model was investigated numerically on MNs with two independent SF layers with various overlaps R generated from the Configuration Model (Sec. 3.1) with identical degree distributions $p_{k^{(B)}} = p_{k^{(A)}}$, the same numbers of nodes $N^{(A)} = N^{(B)}$ and with $J^{(A)} = J^{(B)} = J = 1$. MC simulations were performed using the Metropolis algorithm and the parallel tempering (replica exchange) method in the form described in Ref. [18]. The numerical value of the critical temperature for the FM transition $T_{\mathrm{c,MC}}^{\mathrm{FM}}$ was determined from the intersection point of the Binder cumulants $U_L^{(M)}$ *vs.* T for different N [52],

$$U_L^{(M)} = \left[1 - \frac{\left\langle M^4\right\rangle_t}{3\left\langle M^2\right\rangle_t^2}\right]_{\mathrm{av}}, \tag{36}$$

where $M = N^{-1}\sum_{i=1}^{N} s_i$ is the usual magnetization, $\langle\cdot\rangle_t$ denotes the time average for the simulation of the Ising model on a particular MN, and $[\cdot]_{\mathrm{av}}$ denotes the average over random realizations of the MN with given parameters.

Exemplary results of MC simulations are shown in Fig. 2. The slow increase of magnetization as well as monotonic increase of the Binder cumulants with decreasing temperature provide evidence for the occurrence of continuous FM transition (Fig. 2 (a)). The critical temperature for this transition obtained from MC simulations increases linearly with the overlap R (Fig. 2 (b)). Dependence of the critical temperature obtained in the MF approximation, Eq. (17), on the size of the overlap reproduces this linear trend but the analytic results slightly overestimate the numerical ones (Fig. 2 (b)). Nevertheless, agreement between theoretical and numerical results is satisfactory. Better quantitative agreement is obtained between theoretical predictions obtained from the RS solution, Eq. (35), which also yields linear dependence of $T^{\rm FM}_{\rm c,RS}$ on R, and results of MC simulations. This is though the SF layers of the MNs under study were generated from the Configuration Model rather than from the static model. This confirms that Eqs. (34), (35) are valid for the Ising model on a broad class of MNs with heterogeneous layers with partial overlap and yield more accurate values of the critical temperature for the FM transition than Eqs. (16), (17) obtained in the MF approximation.

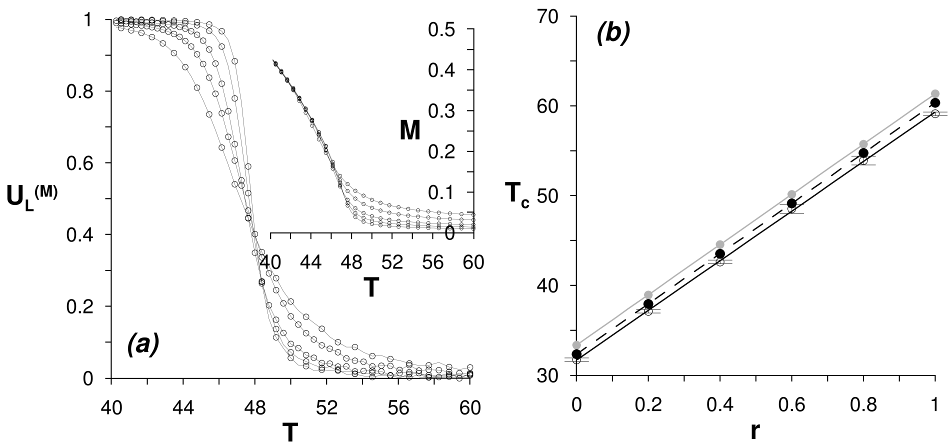

Fig. 2. Results for the Ising model with $J = 1.0$ on an MN with two independent SF layers with identical degree distributions with $\gamma^{(A)} = \gamma^{(B)} = 4.5$, $m^{(A)} = m^{(B)} = 10$. (a) Binder cumulants U_L *vs.* temperature T (inset: magnetization M *vs.* T) obtained from MC simulations of the model with overlap $R = 0.6$ and (from top to bottom for high T) $\tilde{N} = 10^3$, 2×10^3, 5×10^3, 10^4, 2×10^4, solid lines are guides to the eyes. (b) Critical temperature for the FM transition *vs.* overlap R: results of MC simulations (open circles) and least-squares fit $T^{\rm FM}_{\rm c,MC} = 27.61R + 31.67$ (black solid line); results of the MF approximation, Eq. (17) (gray dots) and least-squares fit $T^{\rm FM}_{\rm c,MF} = 28R + 33.33$ (gray solid line); results from the RS solution, Eq. (35) (black dots) and least-squares fit $T^{\rm FM}_{\rm c,RS} = 28R+32.32$ (dashed line).

6. Summary and conclusions

In this paper, the FM transition was investigated in the Ising model on MNs with partial overlap, and detailed study was conducted for MNs with two partly overlapping layers (duplex networks). In such MNs, only a part of nodes belongs to both layers which have, in general, the form of complex, possibly heterogeneous networks. Edges of the layers correspond to FM exchange interactions between spins located in the nodes, so that spins belonging to the overlap provide coupling between layers. Critical temperature for the FM transition was evaluated using the heterogeneous MF approximation as well as the replica method, from the RS solution, and comparable results were obtained from both methods. In particular, in the typical case with no correlations between degrees within different layers of nodes belonging to the overlap, the critical temperature increases linearly with the size of the overlap, and the critical exponent for the magnetization does not depend on the size of the overlap. In the case of maximum or minimum correlation between the above-mentioned degrees for large overlap, the critical temperature is increased or lowered, respectively. The theoretical findings show good quantitative agreement with results of MC simulations of the model on MNs with partly overlapping independent SF layers obtained from the Configuration Model.

This paper extends the theoretical approach to the problem of FM transition in the Ising model on MNs with full overlap [35] to the case of MNs with only partial overlap of layers. Similar extension is possible for other related models, *e.g.*, for the Ising model exhibiting spin glass transition [36], and will be a subject of future research.

Appendix A

Critical exponents for the ferromagnetic transition

For completeness, in the framework of the RS approach let us briefly consider the scaling behavior of the order parameters, *i.e.*, magnetizations $m^{(A)}$, $m^{(B)}$ of the model in the vicinity of the critical temperatures for the FM transition. In the case of MN with SF layers, these temperatures remain finite, and thus the scaling relations for the order parameters are valid, for $\gamma^{(A)} > 3$, $\gamma^{(B)} > 3$. Below the transition point from the MF to the FM phase, the magnetization is expected to increase from zero as ε^{β_m}, where $\varepsilon = (T^{\mathrm{FM}}_{\mathrm{c,RS}} - T)/T^{\mathrm{FM}}_{\mathrm{c,RS}}$. In the case of the Ising model on MNs with full overlap between independent SF layers, the scaling exponent β_m was determined in Ref. [35]. Here, it is briefly argued that its value is not affected by the size of the overlap between layers.

Without losing generality, let us assume in the calculations that $3 < \gamma^{(A)} \leq \gamma^{(B)}$. In order to find the critical exponents for $m^{(A)}$, $m^{(B)}$, the right-hand sides of the equations in system (27) should be expanded with respect to the powers of the order parameters in the vicinity of the respective critical point. Then, *e.g.*, the sum over the indices of nodes in the first equation can be divided in two sums, first over $N^{(A)} - n$ nodes belonging only to the layer $G^{(A)}$ (with $v_i^{(B)} = 0$) and second over n nodes belonging to both layers (for the remaining nodes belonging only to the layer $G^{(B)}$, there is $v_i^{(A)} = 0$). Since n nodes belonging to both layers are selected from the layers $G^{(A)}$, $G^{(B)}$ and identified with one another randomly, the first sum is approximately an $(N^{(A)} - n)/N^{(A)}$ fraction of the respective sum over all nodes belonging to the layer $G^{(A)}$ and the second sum for sufficiently large n can be approximated by its expected value, as in Eq. (18). This yields

$$\begin{aligned}
&\sum_{i=1}^{N} v_i^{(A)} \tanh\left[N\left(\boldsymbol{T}_1^{(A)}\left\langle k^{(A)}\right\rangle v_i^{(A)} m^{(A)} + \boldsymbol{T}_1^{(B)}\left\langle k^{(B)}\right\rangle v_i^{(B)} m^{(B)}\right)\right] \\
&\approx \frac{N^{(A)} - n}{N^{(A)}} \sum_{l=1}^{N^{(A)}} v_l^{(A)} \tanh\left(\boldsymbol{T}_1^{(A)} N^{(A)}\left\langle k^{(A)}\right\rangle_A v_l^{(A)} m^{(A)}\right) \\
&+ \frac{n}{N^{(A)}N^{(B)}} \sum_{l=1}^{N^{(A)}} \sum_{l'=1}^{N^{(B)}} v_l^{(A)} \tanh\left[N^{(A)}\boldsymbol{T}_1^{(A)}\left\langle k^{(A)}\right\rangle_A v_l^{(A)} m^{(A)}\right. \\
&\left. + N^{(B)}\boldsymbol{T}_1^{(B)}\left\langle k^{(B)}\right\rangle_B v_{l'}^{(B)} m^{(B)}\right] . \qquad (37)
\end{aligned}$$

Unfortunately, it is not possible to simply expand $\tanh(\cdot)$ in Eq. (37) with respect to $m^{(A)}$, $m^{(B)}$ due to the occurrence of the terms such as $N^{-1}\sum_{l=1}^{N} v_l^{(A)3}$, *etc.*, which diverge even if the second moments of the distributions of the weights associated with each layer are finite. Nevertheless, the two sums in Eq. (37) can be expanded in the converging series with respect to the powers of the order parameters. This is achieved by first approximating them by integrals

$$\begin{aligned}
&\sum_{i=1}^{N} v_i^{(A)} \tanh\left[N\left(\boldsymbol{T}_1^{(A)}\left\langle k^{(A)}\right\rangle v_i^{(A)} m^{(A)} + \boldsymbol{T}_1^{(B)}\left\langle k^{(B)}\right\rangle v_i^{(B)} m^{(B)}\right)\right] \\
&\approx \frac{N^{(A)} - n}{N^{(A)}} \frac{1-\mu^{(A)}}{N^{(A)}} \int_1^{N^{(A)}} \mathrm{d}y \left(\frac{N^{(A)}}{y}\right)^{\mu^{(A)}} \tanh\left[M^{(A)}\left(\frac{N^{(A)}}{y}\right)^{\mu^{(A)}}\right] \\
&+ \frac{n}{N^{(A)}N^{(B)}} \frac{1-\mu^{(A)}}{N^{(A)}} \int_1^{N^{(A)}} \mathrm{d}y_l \int_1^{N^{(B)}} \mathrm{d}y_{l'} \left(\frac{N^{(A)}}{y_l}\right)^{\mu^{(A)}} \\
&\times \tanh\left[M^{(A)}\left(\frac{N^{(A)}}{y_l}\right)^{\mu^{(A)}} + M^{(B)}\left(\frac{N^{(B)}}{y_{l'}}\right)^{\mu^{(B)}}\right] , \qquad (38)
\end{aligned}$$

where $M^{(A)} = (1-\mu^{(A)})\langle k^{(A)}\rangle_A \boldsymbol{T}_1^{(A)} m^{(A)}$, $M^{(B)} = (1-\mu^{(B)})\langle k^{(B)}\rangle_B \boldsymbol{T}_1^{(B)} m^{(B)}$, then performing changes of variables $u = M^{(A)}(N^{(A)}/y)^{\mu^{(A)}}$ in the first integral and $u_1 = M^{(A)}(N^{(A)}/y_l)^{\mu^{(A)}}$, $u_2 = M^{(B)}(N^{(B)}/y_{l'})^{\mu^{(B)}}$ in the second one as well as taking the limit $N^{(A)}, N^{(B)} \to \infty$, which yields

$$\sum_{i=1}^{N} v_i^{(A)} \tanh\left[N\left(\boldsymbol{T}_1^{(A)}\left\langle k^{(A)}\right\rangle v_i^{(A)} m^{(A)} + \boldsymbol{T}_1^{(B)}\left\langle k^{(B)}\right\rangle v_i^{(B)} m^{(B)}\right)\right]$$
$$\approx \frac{N^{(A)}-n}{N^{(A)}}\left(1-\mu^{(A)}\right)\left(\gamma^{(A)}-1\right)\left(M^{(A)}\right)^{\gamma^{(A)}-2}\int\limits_{M^{(A)}}^{\infty} \mathrm{d}u \frac{u \tanh u}{u^{\gamma^{(A)}}}$$
$$+\frac{n}{N^{(A)}}\left(1-\mu^{(A)}\right)\left(\gamma^{(A)}-1\right)\left(\gamma^{(B)}-1\right)\left(M^{(A)}\right)^{\gamma^{(A)}-2}\left(M^{(B)}\right)^{\gamma^{(B)}-1}$$
$$\times \int\limits_{M^{(A)}}^{\infty} \mathrm{d}u_1 \int\limits_{M^{(B)}}^{\infty} \mathrm{d}u_2 \frac{u_1 \tanh\left(u_1+u_2\right)}{u_1^{\gamma^{(A)}} u_2^{\gamma^{(B)}}}\,, \tag{39}$$

and, finally, by expanding the two integrals in the above equation in the power series with respect to $m^{(A)}$, $m^{(B)}$ using methods described in Ref. [10] and Ref. [35], respectively. Eventually, after some tedious calculations and omitting at the right-hand side the terms which are never dominant for $\gamma^{(B)} > \gamma^{(A)} > 3$ (in particular, those proportional to $(M^{(B)})^{\gamma^{(B)}-1}$), the first equation from system (27) in the vicinity of $T_{\mathrm{c,RS}}^{\mathrm{FM}}$ can be written as

$$\left[\frac{1}{\left\langle k^{(A)}\right\rangle \boldsymbol{T}_1^{(A)}} - \frac{\left(\gamma^{(A)}-2\right)^2}{\left(\gamma^{(A)}-1\right)\left(\gamma^{(A)}-3\right)}\right] M^{(A)} - R^{(A)}\frac{\left(\gamma^{(A)}-2\right)\left(\gamma^{(B)}-1\right)}{\left(\gamma^{(A)}-1\right)\left(\gamma^{(B)}-2\right)} M^{(B)}$$
$$= \frac{\left(\gamma^{(A)}-2\right)^2}{\gamma^{(A)}-1} I\left(\gamma^{(A)}\right)\left(M^{(A)}\right)^{\gamma^{(A)}-2} - \frac{1}{3}\frac{\left(\gamma^{(A)}-2\right)^2 M^{(A)3}}{\left(\gamma^{(A)}-1\right)\left(\gamma^{(A)}-5\right)}$$
$$-R^{(A)}\left[\frac{\left(\gamma^{(A)}-2\right)^2\left(\gamma^{(B)}-1\right) M^{(A)2} M^{(B)}}{\left(\gamma^{(A)}-1\right)\left(\gamma^{(A)}-4\right)\left(\gamma^{(B)}-2\right)}\right.$$
$$\left.+\frac{\left(\gamma^{(A)}-2\right)^2\left(\gamma^{(B)}-1\right) M^{(A)} M^{(B)2}}{\left(\gamma^{(A)}-1\right)\left(\gamma^{(A)}-3\right)\left(\gamma^{(B)}-3\right)} + \frac{\left(\gamma^{(A)}-2\right)\left(\gamma^{(B)}-1\right) M^{(B)3}}{\left(\gamma^{(A)}-1\right)\left(\gamma^{(B)}-4\right)}\right], \tag{40}$$

where

$$I\left(\lambda\right) = \int\limits_0^{\infty} x^{1-\lambda}\left(\tanh x - x\right) \mathrm{d}x \quad \text{for} \quad 3 < \gamma^{(A)} < 5\,.$$

A complementary equation corresponding to the second equation of system (27) can be obtained from Eq. (40) by replacing (A) with (B) and *vice versa*.

The left-hand (linear) part of the system of equations consisting of Eq. (40) and the complementary equation is identical with that of Eq. (28) in the case of independent SF layers. The right-hand (nonlinear) part is almost identical with that in the analogous system of equations for the Ising model on MNs with full overlap between independent SF layers [35]; the only difference is the presence of factors $R^{(A)}$, $R^{(B)}$ which do not affect the scaling behavior of the magnetizations. Hence, it can be immediately concluded that if $3 < \gamma^{(A)} < 5$ and $\gamma^{(A)} < \gamma^{(B)}$, the expected scaling behavior for the magnetization in the vicinity of $T_{\rm c,RS}^{\rm FM}$ is $m^{(A,B)} \propto \varepsilon^{\frac{1}{\gamma^{(A)}-3}}$, and if $\gamma^{(A)} > 5$ and $\gamma^{(A)} < \gamma^{(B)}$, it is $m^{(A,B)} \propto \varepsilon^{1/2}$ [35]. More generally, if $3 < \gamma^{(A)} < 5$ or $3 < \gamma^{(B)} < 5$, the expected scaling behavior for the magnetization in the vicinity of $T_{\rm c,RS}^{\rm FM}$ is $m^{(A,B)} \propto \varepsilon^{\frac{1}{\gamma_{\rm min}-3}}$, where $\gamma_{\rm min} = \min\left\{\gamma^{(A)}, \gamma^{(B)}\right\}$, *i.e.*, it is determined by the more heterogeneous layer, and if $\gamma^{(A)} > 5$, $\gamma^{(B)} > 5$, it is $m^{(A,B)} \propto \varepsilon^{1/2}$ [35]. Thus, the critical exponent β_m for the order parameters of the Ising model on MNs with partial overlap between independent SF layers in the vicinity of the critical temperature for the FM transition from the PM phase is equal to that for the Ising model on MNs with full overlap, and is not affected by the size of the overlap.

Appendix B

Effect of degree correlations between layers on the critical temperature

In the framework of the RS approach, it is relatively easy to study the dependence of the critical temperature for the FM transition on the correlation between degrees within different layers of nodes belonging to the overlapping part of the layers of the MN. As mentioned in Sec. 4.1, different ways of matching nodes from the two layers $G^{(A)}$, $G^{(B)}$ obtained from the static model in order to build the MN with partial overlap lead to different above-mentioned correlations. In particular, using specific ways of matching nodes MNs with extremally, *i.e.*, minimally or maximally correlated layers can be constructed. By matching nodes more or less randomly, MNs with different correlations between weights, and thus between mean degrees within different layers of nodes belonging to the overlap, can be obtained. The correlation coefficient $\sum_{i=1}^{N} v_i^{(A)} v_i^{(B)}$ lies then between the maximum and minimum possible values for the maximally and minimally correlated layers, respectively, and for a typical (most probable) case of totally random matching of nodes, its expected value is given by Eq. (18). As follows from Eq. (28), the critical temperature $T_{\rm c,rS}^{\rm FM}$ is determined by this correlation coefficient, thus below its value is evaluated for the extreme cases of minimally and maximally correlated layers.

Let us assume that the MN with minimally correlated layers is obtained by matching nodes l belonging to the layer $G^{(A)}$ with nodes $l' = N^{(B)} - l + 1$ belonging to the layer $G^{(B)}$ for $l = 1, 2, \ldots n$ (Sec. 4.1). Then,

$$\begin{aligned} N\left\langle k^{(A)}\right\rangle \sum_{i=1}^{N} v_i^{(A)} v_i^{(B)} &= N^{(A)}\left\langle k^{(A)}\right\rangle_A \sum_{l=1}^{n} v_l^{(A)} v_{N^{(B)}-l+1}^{(B)} \\ &= \left(1-\mu^{(A)}\right)\left(1-\mu^{(B)}\right)\left(N^{(A)}\right)^{\mu^{(A)}}\left(N^{(B)}\right)^{\mu^{(A)}-1}\left\langle k^{(A)}\right\rangle_A \\ &\times \sum_{l=1}^{n} l^{-\mu^{(A)}}\left(N^{(B)}-l+1\right)^{-\mu^{(B)}} . \end{aligned} \tag{41}$$

Replacing summation with integration, performing change of variables $y = x/N^{(B)}$ and taking the limit $N^{(B)} \to \infty$ in the integral, it is obtained that

$$\begin{aligned} &N\left\langle k^{(A)}\right\rangle \sum_{i=1}^{N} v_i^{(A)} v_i^{(B)} \\ &= \left(1-\mu^{(A)}\right)\left(1-\mu^{(B)}\right)\left(N^{(A)}\right)^{\mu^{(A)}}\left(N^{(B)}\right)^{\mu^{(A)}-1}\left\langle k^{(A)}\right\rangle_A \\ &\times \int_1^n x^{-\mu^{(A)}}\left(N^{(B)}-x+1\right)^{-\mu^{(B)}} \\ &= \left(1-\mu^{(A)}\right)\left(1-\mu^{(B)}\right)\left(\frac{N^{(A)}}{N^{(B)}}\right)^{\mu^{(B)}}\left\langle k^{(A)}\right\rangle_A \\ &\times \int_{1/N^{(B)}}^{R^{(B)}} y^{-\mu^{(A)}}\left(1-y-\frac{1}{N^{(B)}}\right)^{-\mu^{(B)}} \mathrm{d}y \\ &= \left(1-\mu^{(A)}\right)\left(1-\mu^{(B)}\right)\left(\frac{N^{(A)}}{N^{(B)}}\right)^{\mu^{(B)}}\left\langle k^{(A)}\right\rangle_A B\left(R^{(B)}, 1-\mu^{(A)}, 1-\mu^{(B)}\right), \end{aligned} \tag{42}$$

where B denotes the incomplete Euler beta function. Similarly,

$$N\left\langle k^{(B)}\right\rangle \sum_{i=1}^{N} v_i^{(A)} v_i^{(B)} = N^{(B)}\left\langle k^{(B)}\right\rangle_B \sum_{l=1}^{n} v_l^{(A)} v_{N^{(B)}-l+1}^{(B)}$$
$$= \left(1-\mu^{(A)}\right)\left(1-\mu^{(B)}\right)\left(\frac{N^{(A)}}{N^{(B)}}\right)^{\mu^{(B)}-1}\left\langle k^{(B)}\right\rangle_B B\left(R^{(B)}, 1-\mu^{(A)}, 1-\mu^{(B)}\right) \tag{43}$$

(note partial assymetry between Eqs. (42) and (43); in particular, the overlap coefficient $R^{(B)}$ appears in both equations in the beta function). Hence, in the case of the model on an MN with minimally correlated layers such that $J^{(A)} = J^{(B)} = J$, $N^{(A)} = N^{(B)}$ and thus $R^{(A)} = R^{(B)} = R$, the critical temperature for the FM transition is given by Eq. (31) with the replacement

$$R \to \left(1-\mu^{(A)}\right)\left(1-\mu^{(B)}\right) B\left(R, 1-\mu^{(A)}, 1-\mu^{(B)}\right) .$$

The MN with maximally correlated layers is obtained by matching nodes l belonging to the layer $G^{(A)}$ with nodes $l' = l$ belonging to the layer $G^{(B)}$ for $l = 1, 2, \ldots n$. Again, replacing summation with integration and assuming $n \ll 1$, it is obtained that

$$N\left\langle k^{(A)}\right\rangle \sum_{i=1}^{N} v_i^{(A)} v_i^{(B)}$$
$$\approx \left\langle k^{(A)}\right\rangle_A \frac{\left(1-\mu^{(A)}\right)\left(1-\mu^{(B)}\right)}{1-\left(\mu^{(A)}+\mu^{(B)}\right)} \left(R^{(A)}\right)^{-\mu^{(A)}} \left(R^{(B)}\right)^{1-\mu^{(B)}} ,$$
$$N\left\langle k^{(B)}\right\rangle \sum_{i=1}^{N} v_i^{(A)} v_i^{(B)}$$
$$\approx \left\langle k^{(B)}\right\rangle_B \frac{\left(1-\mu^{(A)}\right)\left(1-\mu^{(B)}\right)}{1-\left(\mu^{(A)}+\mu^{(B)}\right)} \left(R^{(A)}\right)^{1-\mu^{(A)}} \left(R^{(B)}\right)^{-\mu^{(B)}} . \tag{44}$$

Hence, in the case of the model on an MN with minimally correlated layers such that $J^{(A)} = J^{(B)} = J$, $N^{(A)} = N^{(B)}$ and thus $R^{(A)} = R^{(B)} = R$, the critical temperature for the FM transition is again given by Eq. (31) with the replacement

$$R \to \frac{\left(1-\mu^{(A)}\right)\left(1-\mu^{(B)}\right)}{1-\left(\mu^{(A)}+\mu^{(B)}\right)} R^{1-\mu^{(A)}-\mu^{(B)}} .$$

Exemplary dependence of $T_{\mathrm{c,RS}}^{\mathrm{FM}}$ on R for various correlations between weights (and thus mean degrees) within different layers of nodes belonging to the overlapping part of the MN is shown in Fig. 3. In general, for large R, the critical temperature for the FM transition increases with the rise of the above-mentioned correlation and is maximum in the case of maximally correlated layers.

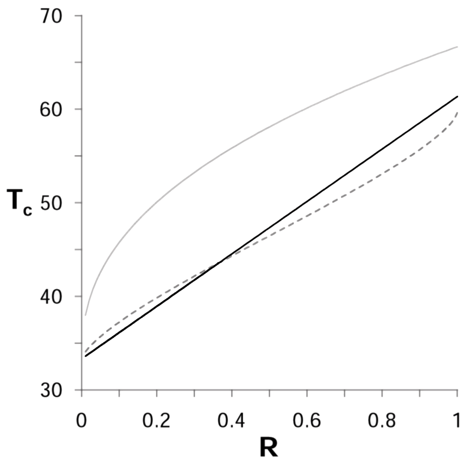

Fig. 3. Critical temperature for the FM transition obtained from the RS solution $T^{\rm FM}_{\rm c,RS}$ *vs.* the overlap R for the Ising model $J = 1.0$ on an MN with two independent SF layers with $N^{(A)} = N^{(B)}$ and identical degree distributions with $\langle k^{(A)}\rangle_A = \langle k^{(B)}\rangle_B = 28$, $\gamma^{(A)} = \gamma^{(B)} = 4.5$ ($\mu^{(A)} = \mu^{(B)} = 2/7$) obtained from the static model. Results of Eq. (31) in the case of uncorrelated layers (black solid line), maximally correlated layers (gray solid line) and minimally correlated layers (dotted line).

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