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SYNCHRONIZATION OF GLOBALLY COUPLED LOZI MAP USING PERIODICALLY VARYING PARAMETER

Baishali Roy[†,‡], Ajanta Bhowal Acharyya[§]

Department of Physics, Lady Brabourne College
P-1/2 Suhrawardy Avenue, Calcutta — 700017, India

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Synchronization in globally coupled Lozi maps has been investigated, when one of the parameters of the Lozi maps, α, is forced to vary periodically. Interestingly, it is observed that due to the variation of the parameter with time, the transition from desynchronized state to synchronized state occurs at lower values of global coupling strength. The synchronized and desynchronized state of CML are distinguished with the help of two statistical quantities: the average fraction of elements belonging to clusters and the standard deviation of the state variables averaged over different realisation of initial conditions. The transition of the system to the synchronised state in globally coupled Lozi map under parametric excitation will be helpful for better way of achieving the synchronised state in the case of various natural and well-known examples of synchronising systems.

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1. Introduction

One of the most complex and intrinsic phenomena in nature is synchronization in which natural dynamical system of multiple objects evolve together in a coupled motion. Some of the well-known instances of synchronizing systems include biological phenomena such as flashing of fireflies in unison, periodically forced circadian rhythms [1], firing neurons [2–4] and physical phenomena such as phase locking in Josephson junction arrays [5], the dynamics of power grids [6, 7] *etc.* Even clapping of audiences in a concert in unison [8], induced oscillation and wobbling of bridges [9], chorusing frogs [10] *etc.* are some daily life experiences of synchronization.

† Present address: Indian Institute of Technology, Gandhinagar, Palaj, Gujarat — 382355, India.

‡ baishi.phy@gmail.com

§ ajanta.bhowal@gmail.com

The presence of synchrony in many natural processes has propelled the extensive study of synchronization in various scientific fields [11]. There are various important applications of synchronisation phenomena, such as in communications, in heartbeat generation, and neural activity. Many mathematical models such as Winfree's coupled oscillator model [12, 13], Kuramoto model [14] as well as discrete-time models, in particular cellular automata [15], coupled map lattices (CMLs) [16, 17] have been studied extensively to understand the various dynamical behaviour exhibited by extended dynamical system. CMLs introduced by Kaneko have received a great deal of attention as different phenomena such as spatio-temporal chaos, synchronization, pattern formation *etc.* can be observed [11] in them.

In CML, each dynamical element (map) is set on a lattice of a given dimension resulting in an extended dynamical system with discrete time ("map"), discrete space ("lattice"), and continuous state. We are mainly interested in the phenomenon of synchronization, exhibited by CML. More specifically, the transition from a chaotic or desynchronized phase in which perturbations grow and each system evolves independently to a synchronized phase in which memory of the initial difference of each system is asymptotically lost and thus all become synchronized, is the subject of discussion. The phenomenon of synchronization, in a system of four/five globally coupled chaotic one-dimensional logistic maps, when the parameter is forced to vary periodically, is investigated [18]. Recently, Cano and Cosenza [19, 20] observed that in a system of large number ($N = 1000$) of globally coupled Lozi maps, the state of the CML may be one of the four states: desynchronized, chimera, cluster and synchronized, depending on the value of the strength of the global coupling parameter. A useful and interesting extension to the study of synchronization on CML can be made by introducing a time-dependent parameter in the Lozi map.

In this study, we investigate the phenomenon of synchronization, in two-dimensional globally coupled Lozi maps [21], when one of the parameters of the Lozi map is forced into periodic variations.

The paper is organized as follows. In Section 2, the behaviour of globally coupled Lozi maps is discussed. The numerical results obtained in globally coupled Lozi maps when the parameter is forced to vary periodically are presented in Section 3. The paper ends with a conclusion in Section 4.

2. Globally coupled Lozi maps

In the communication system, networks of coupled maps showing robust chaotic behaviour are used. We consider here a Lozi map [21] which shows robust chaos and is given by

$$x_{t+1} = 1 - \alpha \mid x_t \mid + y_t \equiv f(x_t, y_t) , \tag{1}$$

$$y_{t+1} = \beta x_t \,, \tag{2}$$

where α and β are real parameters.

The stability diagram of the Lozi map which depends on the values of parameters α and β is described in Ref. [22]. The stable fixed point of the map exists in the region of $\beta > -1$, $\alpha < 1 - \beta$, and $\alpha > \beta - 1$; a stable period-2 orbit occurs in the region of $0 < \beta < 1$, $\alpha < 1+\beta$, and $\alpha > 1-\beta$ (see Ref. [22]). The robust chaos occurs on a bounded region of the parameters α and β. The *global coupling* in the Lozi maps is introduced by the term, ϵh_t, and is described by the equations

$$x^i_{t+1} = (1-\epsilon) f\left(x^i_t, y^i_t\right) + \epsilon h_t \,, \tag{3}$$

$$y^i_{t+1} = \beta x^i_t \,, \tag{4}$$

where

$$h_t = \frac{1}{N} \sum_{j=1}^{N} f\left(x^j_t, y^j_t\right) \,, \tag{5}$$

and where x^i_t and y^i_t are the value of state variables at i^{th} lattice point $(i = 1, \dots N)$ at discrete time t. The function $f(x^i_t, y^i_t)$ is defined by Eq. (1) and the parameter 'ϵ' represents the strength of the global coupling of the maps. Here in Eq. (3), diffusive form of the coupling is assumed.

In the system of globally coupled Lozi maps, four states have been observed by Cano and Cosenza (see Refs. [19, 20]) for different values of the coupling strength 'ϵ'. Though the parameters of the map, α and β, are chosen such that individual mapping is in chaotic range, but depending upon the strength of the global coupling parameter, ϵ, the behaviour of the equilibrium state will be one of the four states: desynchronized, chimera, cluster and synchronized state (in increasing order of coupling strength (ϵ)). In Ref. [19], they have introduced two statistical quantities, σ and p, to characterize the four states of CML.

If the difference between x^i_t and x^j_t, *i.e.*, $d_{ij} = \mid x^i_t - x^j_t \mid$ is less than a threshold value δ, then the pair of elements i and j is considered to belong to a cluster at time t.

The quantity $p(t)$ is defined as the fraction of elements that belongs to the same cluster at time t

$$p(t) = 1 - \frac{1}{N} \sum_{i=1}^{N} \prod_{j=1, j\neq i}^{N} \Theta(d_{ij}(t) - \delta) \,, \tag{6}$$

where $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$.

Asymptotic time average (after discarding a number of transients) of $p(t)$ is defined as p for a given realization of initial conditions. Therefore, a desynchronized state of the CML can be characterized by $p \to 0$, a chimera state by values of p lying within the range, $p_{\min} < p < 1$, where $p_{\min}$ is the minimum cluster size to be taken into consideration, whereas a clustered state of the CML can be characterized by the value $p = 1$. A synchronized state corresponds to the presence of a single cluster of size N and thus in this state, the value of p is 1. Thus, to distinguish between synchronized state and cluster state, another statistical quantity σ is introduced by Cano and Cosenza. The $\sigma(t)$ is defined as the standard deviation of the state variables at time t, *i.e.*, $x_i^t (i = 1, N)$ and is given by

$$\sigma(t) = \left[\frac{1}{N} \sum_{i=1}^{N} (x_t^i - \bar{x}_t)^2 \right]^{1/2} , \tag{7}$$

where

$$\bar{x}_t = \frac{1}{N} \sum_{j=1}^{N} x_t^j . \tag{8}$$

Finally, the asymptotic time average of the standard deviations, $\sigma(t)$, of the state variables is calculated for particular realization of initial conditions x_i^0. Then a synchronized state of the CML is characterized by the values $\sigma = 0$ and $p = 1$, while a cluster state corresponds to $\sigma > 0$ and $p = 1$. A chimera state is given by $p_{\min} < p < 1$ and $\sigma > 0$. An incoherent or desynchronized state corresponds to $p \to 0$ and $\sigma > 0$. The values of the asymptotic time average, σ and p, of the statistical quantities $\sigma(t)$ and $p(t)$ corresponding to each state of coupled map lattice is listed below in Table I.

TABLE I

State of synchronization	σ Time averaged value of $\sigma(t)$	p Time averaged value of $p(t)$
Synchronized	$\sigma = 0$	$p = 1$
Cluster	$\sigma > 0$	$p = 1$
Chimera	$\sigma > 0$	$p_{\min} < p < 1$
Desynchronized	$\sigma > 0$	$p \to 0$

3. Globally coupled Lozi maps with time-dependent parameters: numerical results

In the earlier work, Cano and Cosenza, studied the behaviour of globally coupled Lozi maps when the parameter of Lozi maps remains constant with time. We are interested in observing how the behaviour of globally coupled Lozi maps changes when the time dependence of the parameter is introduced.

The phenomenon of synchronization in globally coupled logistic maps whose parameters are changing periodically with time is studied in Ref. [18]. The authors have studied the synchronization in the case of a small number of coupled maps ($N = 4$ and $N = 5$) and observed five different types of states of CML. However, Cano and Cosenza studied the synchronization phenomena for a large number ($N = 1000$) of globally coupled Lozi maps. They also introduced two statistical quantities, σ and p, to characterize the four different states observed in globally coupled Lozi maps. In this work, we are studying how the states of globally coupled Lozi maps change when the parameter α is time-dependent.

The time dependence of the parameter $\alpha(t)$ is introduced by considering periodically kicked $\alpha(t)$ of time period-T, *i.e.*, α varies with time as follows: $\alpha = \alpha_1$ always except at times, $t = T, 2T, 3T, 4T \ldots$ *etc.*, when α takes the value α_2. Thus, the periodically kicked $\alpha(t)$ of time period (T) follows:

— $\alpha = \alpha_2$, at every "$t = nT$",

— $\alpha = \alpha_1$ otherwise,

where $n \in Z^+$, $T \in Z^+$.

The values of α_1 and α_2 are chosen such that the map is in the chaotic region for α_1 and in the fixed point region for α_2. Thus, the parameter α of the Lozi map is forced to change its value from $\alpha_1 = 1.4$ to $\alpha_2 = 1.0$, after a regular interval of time.

The phase diagram, for a particular value of $\alpha = 1.4$ on the (ϵ, β) parameter space [19], shows that for $\beta < 0$, the state of CML changes from desynchronized to synchronized state as ϵ crosses the critical value, $\epsilon_{\rm c}$, from below, whereas for $\beta > 0$, the state of the CML changes from desynchronized to synchronized through chimera and clustered state.

In this work, we consider $N = 1000$ number of globally coupled Lozi maps with periodically kicked $\alpha(t)$. Here, we study the changes in the behaviour of the state of CML as a result of introducing the time-varying parameter, $\alpha(t)$, only for $\beta < 0$. For $\beta < 0$, when α is fixed at $\alpha_1 = 1.4$, which corresponds to the chaotic region of the Lozi map, the state of the CML changes from desynchronized to synchronized as the value of ϵ crosses $\epsilon_{\rm c}$ from below. Now, we have studied how the value of $\epsilon_{\rm c}$ changes when the parameter α is forced to take the value α_2 from α_1 after every interval of time T.

Figure 1 shows the variation of the two quantities, $\langle\sigma\rangle$ and $\langle p\rangle$, with ϵ for (a) α remaining constant at $\alpha_1 = 1.4$ and (b) periodically kicked $\alpha(t)$ with time period $T = 8$. Here, β is considered as -0.1. We have studied the variation of ϵ_c with the time period (T) of the periodically kicked α for a fixed value of $\beta = -0.1$.

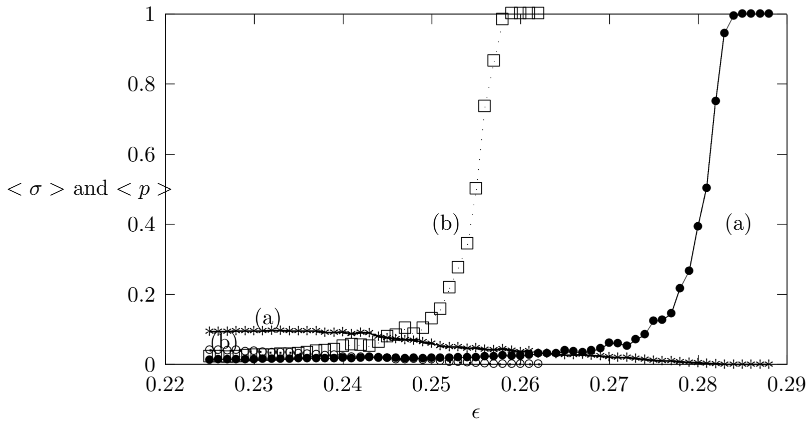

Fig. 1. Plots of the variation of $\langle\sigma\rangle$ and $\langle p\rangle$ with ϵ for a particular value of β, when (a) α remains constant at 1.4 ($\langle\sigma\rangle$ by $(*)$ and $\langle p\rangle$ by $(\bullet)$, (b) periodically varying α with periodicity, $T = 8$ ($\langle\sigma\rangle$ by $(\circ)$ and $\langle p\rangle$ by $(\square)$).

Figure 2 shows the variation of ϵ_c with the time period (T) of periodically kicked $\alpha(t)$, for $\beta = -0.1$, $\alpha_1 = 1.4$ and $\alpha_2 = 1.0$. To characterize the states of CML, we have calculated the time averaged σ and p over 100 units of time after discarding the transient values. The mean values of $\langle\sigma\rangle$ and $\langle p\rangle$ are calculated by averaging over 50 realizations of initial conditions, $x_i(t = 0)$, for $i = 1, N$. For each such realization, x_0^i and y_0^i for $i = 1, N$ can have any value randomly lying between $[-1, 1]$ and distributed uniformly within the range. Figure 2 reveals that the transition from desynchronized to synchronized state occurs at a smaller value of ϵ_c for periodically kicked $\alpha(t)$ compared to the case of constant α.

We have also studied the variation of ϵ_c with the time period (T) of the periodically kicked $\alpha(t)$ for five different values of β, $\beta = -0.1, -0.15, -0.2, -0.25, -0.3$. These variations are plotted in Fig. 3. The figure shows that the nature of variation of ϵ_c with the values of the time period (T) of the periodically kicked α does not change with β.

We have also examined whether the value of the parameter ϵ_c changes with the value of the parameter α_2 for a particular value of α_1. The variation of ϵ_c with the value of α_2 for $\alpha_1 = 1.4$, $\beta = -0.1$ and time period $(T) = 32$ is studied and plotted in Fig. 4. The figure shows that ϵ_c increases very slowly as α_2 increases, *i.e.*, the transition from desynchronized state to synchronized state occurs at a larger value of ϵ_c as α_2 moves towards the boundary of fixed point region and chaotic region.

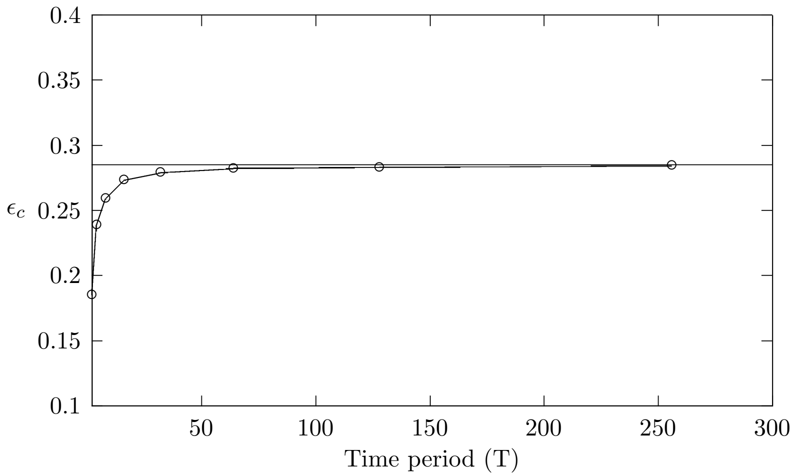

Fig. 2. Plot of the variation of ϵ_c with the time period (T) of periodically kicked $\alpha(t)$ for a particular value of $\beta = -0.1$. The horizontal line represents $\epsilon_c = 0.285$, which is the value of the parameter ϵ, at which the state of CML changes from desynchronized to synchronized when α remains constant at 1.4 ($\beta = -0.1$).

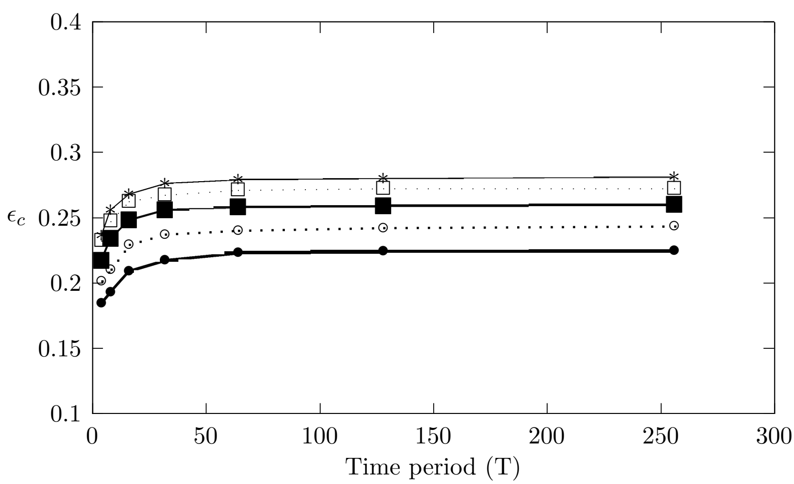

Fig. 3. Plot of the variation of ϵ_c with time period (T) of periodically kicked $\alpha(t)$ for five different values of $\beta = -0.1$ $(*)$, $\beta = -0.15$ $(\square)$, $\beta = -0.2$ $(\blacksquare)$, $\beta = -0.25$ $(\circ)$ and $\beta = -0.3$ $(\bullet)$.

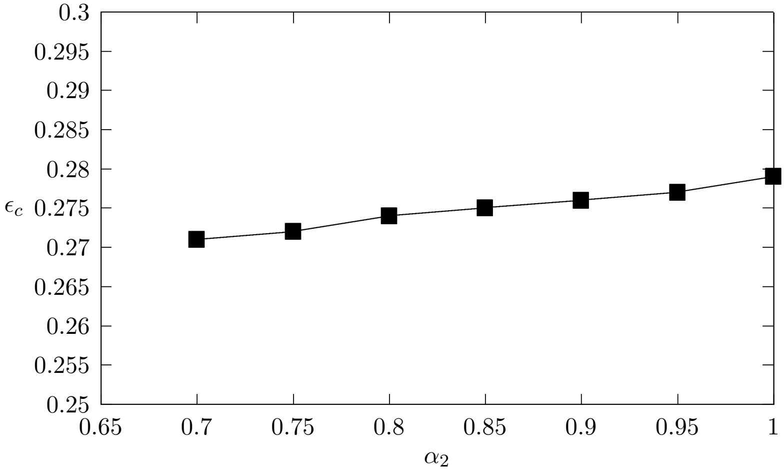

Fig. 4. Plot of the variation of ϵ_c with the values of the parameter α_2 for $\alpha_1 = 1.4$, $\beta = -0.1$ and $T = 32$.

4. Conclusion

The globally coupled Lozi maps with time-dependent parameter have been studied numerically. If the parameters of the Lozi map is considered as $\beta < 0$ and $\alpha = 1.4$, for which the map is chaotic, then the behaviour of the globally coupled Lozi maps is in desynchronized state for lower values of coupling strength ϵ and becomes synchronized as coupling strength increases (say at ϵ_c). In this work, we have introduced time dependence in the parameter $\alpha(t)$ of the Lozi map. We have considered periodically kicked $\alpha(t)$ of time period T. It has been observed that in the globally coupled Lozi maps with such time-dependent parameter, $\alpha(t)$, the transition from desynchronized to synchronized state occurs at lower values of coupling strength, compared to the case of time-independent α. It has also been observed that the value of ϵ_c increases as the time period (T) of the periodically kicked $\alpha(t)$ increases and approaches the value that corresponds to the value of constant $\alpha_1 = 1.4$.

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