STUDIES OF NON-STANDARD PARTICLE MIXINGS THROUGH SINGULAR VALUES*

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Singular values provide a method to study mixing matrices in particle physics. The methods of unitary dilations and the cosine–sine matrix decomposition are discussed in the framework of the Standard Model neutrinos mixing with one non-standard neutrino. We show that the mixings are continuous functions of singular values. It implies that the magnitude of non-standard mixing can be estimated from below and above unambiguously from the experimentally determined interval PMNS mixing matrix.

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1. Introduction

Singular values emerged in mathematics at the end of the 19^{th} century and the name appeared for the first time in 1910 in the context of integral equations [1]. Presently, they are used in many branches of science, *e.g.*, data science [2], image processing [3], mechanics [4]. However, in particle physics, they have not been receiving too much attention so far. We present the survey of a recently developed approach [5] to study properties of interval matrices that emerge in the particle mixing phenomenon with the help of singular values. However, the methodology may equally well be applicable to the quark sector.

Since the discovery of neutrino oscillations, the massive neutrinos have become a fact. However, there are still many unsolved puzzles concerning neutrino physics. Among them is the problem of a number of neutrino states. The existence of neutrinos beyond three standard flavor states is not forbidden. They can be identified at the level of the neutrinos mixing matrix by deviations from unitarity of the so-called PMNS 3-dimensional matrix.

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So far, such signals have been studied with the help of two decompositions known as α and η parametrizations [6–12]. Our approach to this issue involves only singular values and connects in the uniform way the Standard Model (SM) mixing and its extensions.

In the next chapter, we introduce all necessary terminology borrowed from the matrix theory. In the third chapter, "light–heavy" neutrino mixings are connected with singular values in the 3+1 scenario. The work is concluded with a summary and outlook.

2. Terminology and tools

Let A denotes an $n \times n$-dimensional matrix with elements from the field of complex numbers. Singular values (σ_i) are well-defined for general rectangular matrices, however, for our purpose, we consider only square matrices. They are defined as the positive square root of the product of a matrix and its Hermitian conjugation, that is

$$\sigma_i(A) = \sqrt{\lambda\left(AA^\dagger\right)}\,. \tag{1}$$

The convention is that we consider singular values in the decreasing order, *i.e.*, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Another important notion is that of matrix norms. A matrix norm is a function $\|\cdot\|$ from the set of all complex matrices into $\mathbb{R}$ that for any $A, B \in M_{n\times n}$ satisfies the following properties:

$$\begin{aligned} \|A\| &\geq 0 \quad \text{and} \quad \|A\| = 0 \Leftrightarrow A = 0\,, \\ \|\alpha A\| &= |\alpha|\|A\|\,, \qquad \alpha \in \mathbb{C}\,, \\ \|A+B\| &\leq \|A\| + \|B\|\,, \\ \|AB\| &\leq \|A\|\|B\|\,. \end{aligned} \tag{2}$$

In our approach, of a special importance is the operator (or spectral) norm, defined as

$$\|A\| = \max_{\|x\|_2=1} \|Ax\|_2\,, \tag{3}$$

where $x \in \mathbb{C}^n$ and $\|\cdot\|_2$ stands for the Euclidean norm. In this way, we can introduce the notion of contractions, *i.e.*, matrices with the spectral norm less or equal to one

$$\|A\| \leq 1\,. \tag{4}$$

To apply the spectral norm from Eq. (3) properly to our needs, we express it via singular values in the following way

$$\|A\| = \sigma_1(A)\,, \tag{5}$$

so the spectral norm is equal to the largest singular value of a matrix. The next concept which we need is that of a unitary dilation. It allows us to extend matrices in a proper way to a larger unitary matrices. Moreover, only matrices that can be extended is such a way are contractions. Thus, we are interested in the following situation:

$$\|A\| \leq 1 \Rightarrow V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad VV^\dagger = V^\dagger V = I\,. \tag{6}$$

The construction of full unitary matrices from the contraction can be done by the cosine–sine decomposition (CS decomposition) [13].

Theorem 2.1 *Let the unitary matrix* $U \in M_{(n+m)\times(n+m)}$ *be partitioned as*

$$U = \begin{array}{cc} \begin{array}{cc} n & m \end{array} & \\ \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} & \begin{array}{c} n \\ m \end{array} \end{array}\,. \tag{7}$$

If $m \geq n$*, then there are unitary matrices* $W_1, Q_1 \in M_{n\times n}$ *and unitary matrices* $W_2, Q_2 \in M_{m\times m}$ *such that*

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{c|cc} C & -S & 0 \\ \hline S & C & 0 \\ 0 & 0 & I_{m-n} \end{array}\right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix}, \tag{8}$$

where $C \geq 0$ *and* $S \geq 0$ *are diagonal matrices satisfying* $C^2 + S^2 = I_n$*.*

If $n \geq m$, then it is possible to parametrize a unitary dilation of the smallest size

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{cc|c} I_r & 0 & 0 \\ 0 & C & -S \\ \hline 0 & S & C \end{array}\right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix}, \tag{9}$$

where $r = n - m$ is the number of singular values equal to 1 and $C = \mathrm{diag}(\cos\theta_1, \ldots, \cos\theta_m)$ with $|\cos\theta_i| < 1$ for $i = 1, \ldots, m$.

In addition, it is useful to create contractions with a given set of singular values. Such an approach is known as an inverse singular values problem [14]. The method is based on the Horn–Weyl majorization relation between eigenvalues and singular values [15, 16]

$$\prod_{i=1}^{k} |\lambda_i| \leq \prod_{i=1}^{k} \sigma_i\,, \tag{10}$$

with equality when $k = n$, where n is the dimension of the matrix.

3. Application of singular values to the neutrino mixing

3.1. 3-dimensional mixing basis

At the moment, due to imperfect experimental data, the neutrino mixing cannot be decisively concluded by the standard 3×3 unitary PMNS mixing matrix. The issue is how to grasp on the level of 3-dimensional mixing matrices traces of possible BSM scenarios. As mentioned before, all matrices which can be extended to a larger unitary matrix are characterized as contractions (4). For matrices representing physical mixings, the contraction property is necessary but it is not sufficient due to just mentioned restrictions on mixing elements imposed by experimental data. However, a workaround of the problem can be in constructing mixing matrices as the convex combination of unitary matrices which allows to define the space of physically admissible matrices in the following way [5]:

$$\begin{aligned} \Omega \ := \ \mathrm{conv}(U_{\mathrm{PMNS}}) = \Bigg\{ & \sum_{i=1}^{m} \alpha_i U_i \mid U_i \in U(3), \alpha_1, \ldots, \alpha_m \geq 0, \sum_{i=1}^{m} \alpha_i = 1\,, \\ & \theta_{12}, \theta_{13}, \theta_{23} \text{ and } \delta \text{ given by experimental values} \Bigg\}\,. \end{aligned} \tag{11}$$

Moreover, as suggested in Eq. (9), the minimal extension of contraction to a larger unitary matrix is not arbitrary. Dimension of such a minimal unitary dilation is encoded in the number of singular values strictly less than one of a considered contraction. From the physical point of view, this fact can be used to reflect the number of possible additional neutrinos. In the case of 3-dimensional mixing matrices, this gives us three possible minimal unitary extensions, by one, two and three sterile neutrinos. This characteristics provides a structure to the region Ω, *i.e.*, the region can be split into four disjoint subsets, according to the minimal number of additional neutrinos

$$\begin{aligned} V_1, \Sigma \ &= \ \{1, 1, \sigma_3 < 1\} : \text{one additional neutrino}\,, \\ V_2, \Sigma \ &= \ \{1, \sigma_2 < 1, \sigma_3 < 1\} : \text{two additional neutrinos}\,, \\ V_2, \Sigma \ &= \ \{\sigma_3 < 1, \sigma_2 < 1, \sigma_3 < 1\} : \text{three additional neutrinos}\,, \\ V_4, \Sigma \ &= \ \{1, 1, 1\} : \text{only unitary matrices}\,. \end{aligned} \tag{12}$$

The subset V_4 is not interesting from the BSM perspective since it provides only unitary extensions with a decoupled "light–heavy" sector. Recently, we developed a method which allows to construct from the beginning physically admissible mixing matrices with elements within experimental ranges and the prescribed set of singular values. We use an algorithm proposed in [17] by which matrices with a prescribed set of singular values are produced as lower triangular matrices. They can be easily compared with a widely used in global analysis α parametrization. Such an approach allows to control a structure of the 3-dimensional mixing matrix and the corresponding physical properties.

Quantitative results of mixing matrix analysis by singular values can be found in [5, 18, 19]. In these works, the estimation of allowed space for additional neutrinos is discussed, along with statistical analysis of physically admissible mixing matrices within experimental bounds, and the possibility to distinguish sets in (12).

The global analysis allows to estimate the lower and upper bounds for each mixing element which results in an interval matrix data presentation. However, if one would like to consider particular examples of physical mixing matrices, then things get more complicated since in such a representation the correlation between elements is lost. On the other hand, physical matrices are controlled properly by singular values.

3.2. Complete unitary mixing

As singular values control the minimal number of additional neutrinos, 3-dimensional mixing matrices can be enlarged to a complete unitary matrix of some BSM scenario. In order to obtain a complete mixing matrix of the minimal allowed dimension, we need first to construct a contraction from the region Ω and then invoke the CS decomposition in the form of Eq. (9). In this way, the mixing between three known active neutrino states can be maintained. However, if instead of the construction of particular complete mixing matrices, we are only interested in the estimation of the mixing between standard and non-standard neutrinos, it is also possible to do so with the help of singular values and the CS decomposition [19]. In this situation, we are interested in estimation of the top right block U_{12} of the complete mixing matrix. This block is given by

$$U_{12} = W_1(0, -S)^T Q_2^\dagger \,. \tag{13}$$

The matrix W_1 comes from the singular value decomposition of the 3-dimensional mixing matrix. Elements of the matrix S are obtained from the singular values as $s_i = \sqrt{1-\sigma_i^2}$, where σ_i are singular values strictly less than one. Let us consider the case with one additional neutrino. For a review of experimental and global fits in the 3+1 case, see [20]. In such a scenario, the complete matrix is a 4×4 unitary matrix and the 3-dimensional contractions belong to the subset V_1 (12), *i.e.*, only one singular value is strictly less than unity. Thus, the set of singular values is $\Sigma = \{1, 1, \sigma_3\}$

and the CS decomposition in this case takes the form of

$$\begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -s \\ \hline 0 & 0 & s & c \end{array}\right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix} . \tag{14}$$

The "light–heavy" sector is given by

$$U_{12} = W_1 V_{12} Q_2^\dagger \,, \tag{15}$$

where $W_1 \in \mathbb{C}^{3\times 3}$ is unitary, $V_{12} = (0,0,-s)^T$ and $Q_2 = \mathrm{e}^{\mathrm{i}\theta}, \theta \in (0, 2\pi]$. Due to the structure of the CS-matrix for the 3+1 scenario, only the third column of W_1 takes part in U_{12}, so we get

$$U_{12} = -(w_{13}, w_{23}, w_{33})^T s\mathrm{e}^{-\mathrm{i}\theta} \,. \tag{16}$$

As we are interested in the estimation of the absolute values of the "light–heavy" mixing, we obtain

$$\begin{aligned} |U_{e4}| &= |w_{13}s| = \left|w_{13}\sqrt{1-c^2}\right| = \left|w_{13}\sqrt{1-\sigma_3^2}\right| , \\ |U_{\mu 4}| &= \left|w_{23}\sqrt{1-\sigma_3^2}\right| , \\ |U_{\tau 4}| &= \left|w_{33}\sqrt{1-\sigma_3^2}\right| . \end{aligned} \tag{17}$$

Numerical estimation of these bounds for different mass scenario splittings can be found in [19]. The upper bounds have been obtained by looking for the largest absolute value of w_{i3}, $i = 1, 2, 3$, corresponding to allowed singular values for each massive scenario. The behavior of $|w_{13}|$, $\sqrt{1-\sigma_3}$ and $|U_{e4}|$ in the case of light sterile neutrino is presented in Fig. 1. It shows that both $|w_{13}|$ and $\sqrt{1-\sigma_3}$ are continuous functions of σ_3. This implies that $|U_{e4}|$ behaves in a controllable way. As a consequence, numerical estimations of $|U_{i4}|$, $i = e, \mu, \tau$ given in [19] are stable.

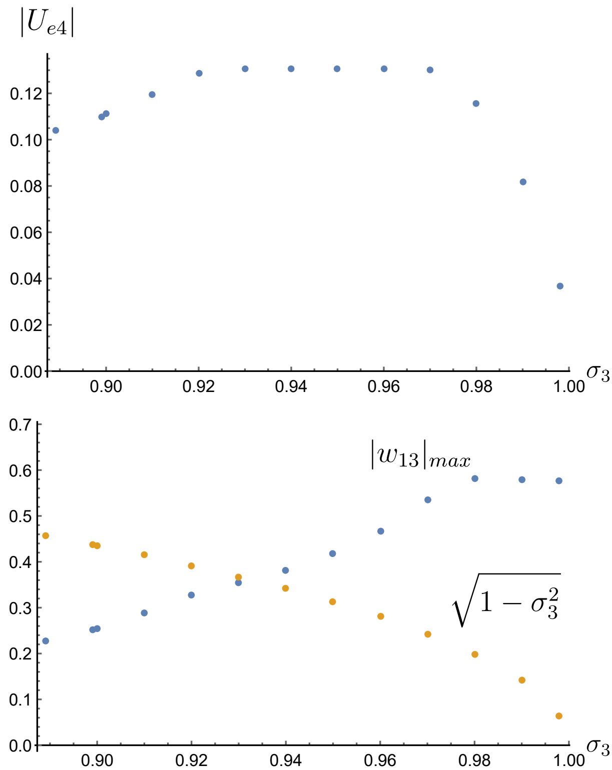

Fig. 1. The top figure presents the behavior of upper bounds of $|U_{e4}|$. We can see that it approaches the maximal value for middle values of σ_3. The bottom figure shows the behavior of particular constituents of $|U_{e4}|$.

4. Summary

In this work, we discussed the application of singular values to the analysis of the three dimensional mixing matrices. Singular values allow to grasp in a uniform way many important physical properties. First of all, they are used to determine physically admissible mixing matrices, *i.e.*, matrices that are unitary or can be extended to a larger unitary mixing matrix. These matrices are classified as contractions. Moreover, singular values encode the minimal number of additional neutrinos. Any contraction can be extended via the unitary dilation procedure to a complete mixing matrix. To do this, we can use the CS decomposition which also allows us to establish bounds for the "light–heavy" sector.

For now, we have done only analysis of the 3+1 scenario. Analysis of scenarios with two and three additional neutrinos is in progress.

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REFERENCES

[1] R.A. Horn, C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991, DOI: 10.1017/9781139020411.

[2] M.E. Wall, A. Rechtsteiner, L.M. Rocha, *Singular Value Decomposition and Principal Component Analysis*, Springer US, Boston, MA, 2003, pp. 91–109, DOI: 10.1007/0-306-47815-3_5.

[3] R.A. Sadek, *Int. J. Adv. Comput. Sci. Appl.* **3**, 26 (2012) [arXiv:1211.7102 [cs.CV]].

[4] J. Meijaard, *Comput. Methods Appl. Mech. Engrg.* **103**, 161 (1993).

[5] K. Bielas, W. Flieger, J. Gluza, M. Gluza, *Phys. Rev. D* **98**, 053001 (2018).

[6] S. Antusch *et al.*, *J. High Energy Phys.* **0610**, 084 (2006) [arXiv:hep-ph/0607020].

[7] E. Fernández-Martínez, M.B. Gavela, J. Lopez-Pavón, O. Yasuda, *Phys. Lett. B* **649**, 427 (2007).

[8] Z.-z. Xing, *Phys. Lett. B* **660**, 515 (2008) [arXiv:0709.2220 [hep-ph]].

[9] Z.-z. Xing, *Phys. Rev. D* **85**, 013008 (2012) [arXiv:1110.0083 [hep-ph]].

[10] F.J. Escrihuela *et al.*, *Phys. Rev. D* **92**, 053009 (2015) [*Erratum ibid.* **93**, 119905 (2016)] [arXiv:1503.08879 [hep-ph]].

[11] F.J. Escrihuela *et al.*, *New J. Phys.* **19**, 093005 (2017) [arXiv:1612.07377 [hep-ph]].

[12] M. Blennow *et al.*, *J. High Energy Phys.* **04**, 153 (2017) [arXiv:1609.08637 [hep-ph]].

[13] A. Allen, D. Arceo, https://apps.dtic.mil/dtic/tr/fulltext/u2/a446226.pdf

[14] M.T. Chu, *SIAM J. Numer. Anal.* **37**, 1004 (2000).

[15] H. Weyl, *Proc. Natl. Acad. Sci. USA* **35**, 408 (1949).

[16] A. Horn, *Proc. Am. Math. Soc.* **5**, 4 (1954).

[17] Chi-Kwong Li, R. Mathias, *BIT Numerical Mathematics* **41**, 115 (2001).

[18] K. Bielas, W. Flieger, *Acta Phys. Pol. B* **48**, 2213 (2017).

[19] W. Flieger, J. Gluza, K. Porwit, arXiv:1910.01233 [hep-ph].

[20] S. Gariazzo, *Acta Phys. Pol. B* **50**, 1719 (2019), this issue.