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PARAMETERIZATION OF AMPLITUDES, FINDING RESONANCES AND UNITARITY, PECULIARITIES AND TRAPS*

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Parameterization of amplitudes for two-body interactions is very common and very important link between the experiment itself and the final results of the analysis — *e.g.* resonance spectrum. There are many methods of parameterization, but only some meet the unitarity condition, which may prove to be crucial in obtaining results, especially when we care about their high precision. It turns out that it is quite easy to ensure that the unitarity condition is fulfilled by amplitude, but amplitudes that break unitarity are very often created and used, especially those for many resonances. Only few conditions must be fulfilled to guarantee unitarity and thus increase the reliability of the obtained results. It is very important presently, when in many data analyses very small, overlapping or broad signals are studied, nonunitary effects can significantly influence results and lead to nonphysical interpretation of obtained parameters.

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1. Introduction

Unitarity — a condition that must satisfy the amplitudes describing the interactions of a given process — can be compared to probability of fulfilling the energy conservation condition. Therefore, unitarity should be the apple of the eye of all those who create such amplitudes — they parameterize them for various interactions and decays. To demonstrate the importance of fulfilling this condition, we concentrate on the simplest case: two-body interactions. The typical and, it would seem, the easiest way to construct amplitudes containing several resonances is to add the appropriate (as small as possible — just enough) number of amplitudes. In the next step, it is of course important to make a good fit to the data, which, however, in itself does not guarantee the correctness of the obtained results. Here, we show

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that amplitudes constructed in such a way may not be unitary and may need additional smooth background to describe the data well enough. Such backgrounds may or may not have physical meaning but, in fact, can be interpreted as an effective influence of all omitted in given parameterization singularities and thresholds at higher energies. These singularities usually lie far from the physical region and can be very model-dependent, therefore their direct interpretation as real resonances can be very doubtful. Of course, resonances which one can find in, for example, Particle Data Group Tables should be model-independent (within their errors). They should also play the leading role in construction of full amplitudes (phase shifts and inelasticities), both elastic and inelastic. Parameters of such resonances (for example mass, width, couplings ...) do, however, depend very much on whether the amplitudes used in analyses were unitary or not.

Square of module of amplitude is proportional to cross section for given process and, of course, varies with energy. Therefore, unitarity (probability) is difficult to apply directly to amplitude (at least not intuitive). A function which behaves like probability is S-matrix whose module is equal to one what strongly constrains analytical structure of unitary amplitude related with S-matrix by

$$A(k) = \frac{S(k) - 1}{2ik}\,, \tag{1}$$

where k is momentum of interacting particles. Generally, the S-matrix can be expressed as a ratio of two Jost functions $D(k)$

$$S(k) = \frac{D(-k)}{D(k)}\,. \tag{2}$$

2. Amplitudes for one-channel scattering

In one-channel case, the S-matrix is just a function of energy and, of course, $S(k) = \mathrm{e}^{2i\delta(k)}$, where $\delta(k)$ is phase shift. Let us construct at the beginning amplitude for the simplest case — for one resonance. To find its position in our S-matrix (or amplitude, see Eq. (1)), the easiest is to assume minimum condition — one zero of the denominator $D(k)$ (*i.e.* pole of $S(k)$) at $k = k_j$ on the 2$^{\mathrm{nd}}$ Riemann sheet and automatically one zero of numerator (zero of $S(k)$) at $k = -k_j$ on 1$^{\mathrm{st}}$ Riemann sheet (see Fig. 1 (a))

$$S(k) = \frac{-k - k_j}{k - k_j}\,. \tag{3}$$

One can, however, easily check that in such a "one pole" case $|S(k)| \neq 1$, therefore, our amplitude $A(k)$ is not unitary. It turns out that it is enough just to add second-symmetric pole and zero (hereafter called "second pole") of $S(k)$ at $k = -k_j^*$ and $k = k_j^*$, respectively (see Fig. 1 (b)). One then gets

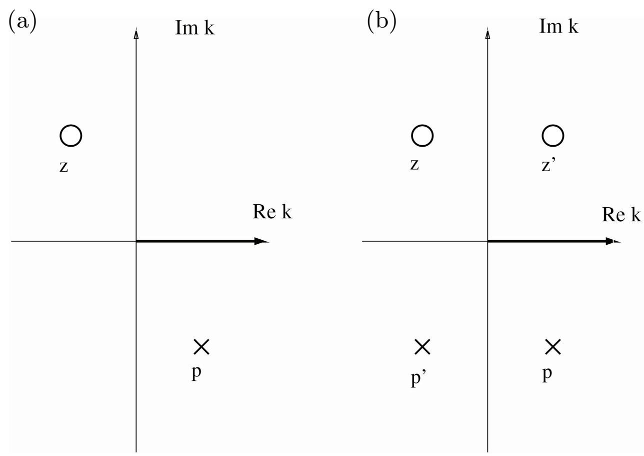

Fig. 1. Schematic positions of poles and zeroes of (a) nonunitary and (b) unitary amplitude for single resonance in the complex momentum plane. Thick black line denotes physical region.

$$S(k) = \frac{(-k - k_j)\left(-k + k_j^*\right)}{(k - k_j)\left(k + k_j^*\right)} \,. \tag{4}$$

One can easily check that now $|S(k)| = 1$ and phase shift $\delta = (-\alpha - \beta + \gamma + \omega)/2$, where α, β, γ and ω are phases of the poles p, p' and zeroes z' and z, respectively, presented in Fig. 2.

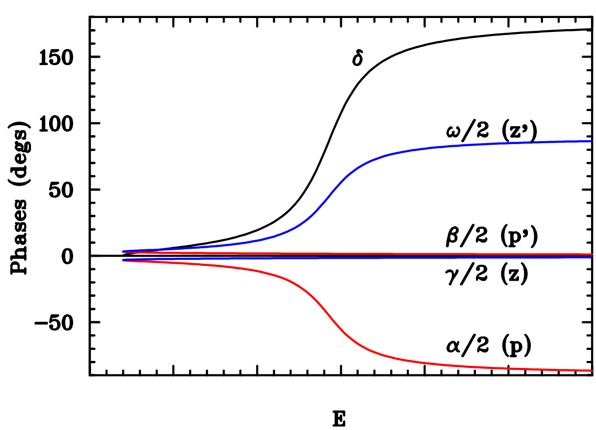

Fig. 2. Phases of all poles and zeroes on Fig. 1 (b) compared with their sum δ — phase shifts of the amplitude.

All those phase components depend on real and imaginary part of k_j by $\mathrm{ArcTan}(\frac{-\mathrm{Im}\,k_j}{k-\mathrm{Re}\,k_j})$ what shows that only pole p and zero z' lying closer to physical region than their mirror pole p', and zero z can together produce increase of the phase shift δ by $\pi/2$ what is characteristic for single resonances. The role of the second pole is smaller and, moreover, decreases with the energy (nowhere is, however, equal to zero). Only at the threshold, *i.e.* right in the middle between p, z' p' and z influence of the all these poles and zeroes on the amplitude is the same.

Hereafter, the method of analysis of resonances using poles and zeroes will be call "pole method".

2.1. Most commonly used amplitudes

The most popular amplitude usually used in experimental analyses is the Breit–Wigner (BW) type amplitude whose nonrelativistic form is

$$\mathrm{BW}(E) = \frac{\Gamma/2k}{M_{\mathrm{BW}} - E - i\Gamma/2}\,, \tag{5}$$

where Γ and M_{BW} are the full width and mass of a resonance, respectively. Using relation (1), one can easily calculate $S_{\mathrm{BW}}(E)$ and check if it is unitary. From definition of $\mathrm{BW}(E)$, one gets $S_{\mathrm{BW}}(E) = \frac{M_{\mathrm{BW}}-E+i\Gamma/2}{M_{\mathrm{BW}}-E-i\Gamma/2}$ what shows that $|S_{\mathrm{BW}}(E)| = 1$ and that single BW approximation is unitary. The Breit-Wigner formula is only an approximation of a real physical amplitude and works well only near the resonance mass, especially for narrow resonances. For example, threshold behavior of such an amplitude (*i.e.* in the limit $E \longrightarrow 2m$) is wrong because $S_{\mathrm{BW}}(E) \longrightarrow \frac{2m-E_j^*}{2m-E_j}$ (where $E_j = M_{\mathrm{BW}} - i\Gamma/2$) and $\delta(E) \longrightarrow \mathrm{ArcTan}(\frac{\Gamma/2}{M_{\mathrm{BW}}-2m}) \neq 0$ (also cross section $\sigma(E) \neq 0$). In the case of unitary S-matrix defined by Jost functions with two poles and zeroes (see Eq. (4)), corresponding limits are correct, *i.e.* $S(k) \longrightarrow \frac{-k_j * k_j}{-k_j * k_j}$, so $\delta(k) \longrightarrow 0$ and $\sigma(k) \longrightarrow 0$.

2.2. Breit–Wigner and pole mass definition and value

Due to different analytical structure of BW amplitude and that given by pole method, masses of a resonance calculated from these two amplitudes are different. In BW approach, mass M_{BW} is defined by energy at which phase shifts cross $\pi/2$. In the pole method, this mass is identified with the real part of the dominant pole. For pole p at $a - ib$, where $a > 0$ and $b > 0$, the phase shift is given by $\delta(k) = \mathrm{ArcTan}(\frac{2bk}{k^2-a^2-b^2}) + \mathrm{ArcTan}(\frac{-2bk}{-k^2-a^2-b^2})$. It is seen that due to nonzero value of b (half of width of a resonance) and presence of the second term, $\delta(k)$ will not cross $\pi/2$ at $k = a$, so $M_{\mathrm{BW}} > 2\sqrt{a^2+m^2}$.

For example, this difference for very well known $\rho(770)$ is few MeV. For wider resonances, it is bigger. It is also worth mentioning the difference between phases of the BW amplitude and that given by the dominant pole even around the resonance mass. The reason is a nonzero "additional" phase produced by second pole p' and zero z (see Fig. 1 (b)).

3. Amplitudes for more channels and more resonances

For one channel amplitudes but with more resonances, very popular is an isobar model being a sum of amplitudes for single resonances. According to Eq. (1), the sum $A_{\mathrm{tot}}(k)$ for two amplitudes $A_1(k)$ and $A_2(k)$ is $\frac{S_1(k)-1}{2ik} + \frac{S_2(k)-1}{2ik}$ what, in elastic region leads to $S_{\mathrm{tot}}(k) = S_1(k)+S_2(k)-1 = \mathrm{e}^{2i\delta_1(k)} + \mathrm{e}^{2i\delta_2(k)} - 1$. Of course $|S_{\mathrm{tot}}(k)| \neq 1$ what means that this sum *i.e.* the isobar model violates unitarity.

Instead of using amplitudes one can use the S-matrices and, for two resonances as above, create product $S_1(k) \times S_2(k)$ which, by definition, fulfils unitarity. For $N > 2$ resonances, the method is the same, $S_{\mathrm{tot}}(k) = \Pi_j^N S_j(k)$. Another effective and popular way of parameterization of multi-resonance amplitudes is to use K-matrix, whose relation with the S-matrix is $S = (1+iK)/(1-iK)$. Sum of two K-matrices does not violate unitarity.

Analytical structure of amplitudes for $n > 1$ channels becomes more complicated. The number of Riemann sheets increases to 2^n and, due to sign ambiguity $k_2 = \pm\sqrt{k_1^2 + m_1^2 - m_2^2}$, each pole splits into 2^{n-1} poles. All these poles lie on various Riemann sheets and are shifted more or less (it depends on strength of coupling between channels) with respect to position of the original pole.

Figure 3 presents Riemann sheets for two channels and schematic positions of poles and zeroes corresponding to one resonance. Names of Riemann sheets are given by signs of imaginary parts of momenta in all channels. For example, in the two-channel case, mark $(-,+)$ means that $\mathrm{Im}(k_1) < 0$ and $\mathrm{Im}(k_2) > 0$.

An example of positions of such shifted S-matrix poles is presented in Table I for three resonances found in two-channel analysis of scalar–isoscalar $\pi\pi$ interactions below 2 GeV (analysis similar to that in [1]). Underlined are poles which play a leading role in the full amplitude, therefore, can be considered as resonances. They were identified checking distances of all found poles from the physical region in complex conformal variable z defined by $z = \frac{k_1+k_2}{\sqrt{m_K^2 - m_\pi^2}}$. The results of such an analysis have been confirmed by analysis of phases and squared modules of amplitudes (proportional to cross section) corresponding to each pole. For example, Figs. (4) and (5) present them for pole poles 1, 1', 2, 2' separately and for corresponding pairs of poles. In both cases, one pole is dominant and the second one plays a minor role.

This is particularly well seen in Fig. 5 where role of the pole 2' is so small that lines for poles 2 and 2+2' are almost indistinguishable.

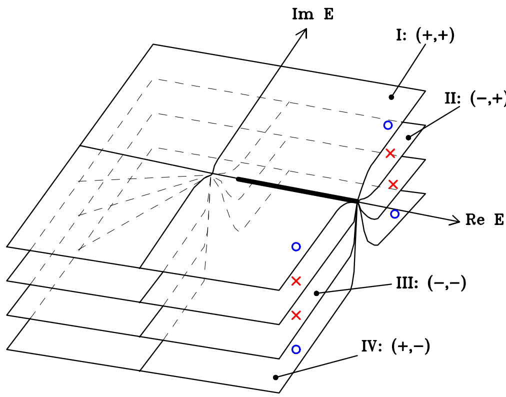

Fig. 3. Riemann sheets for two channels with poles (crosses) and zeroes (circles) for one resonance

TABLE I

Positions of two-channel S-matrix poles found in analysis of scalar–isoscalar $\pi\pi$ interactions below 2 GeV. Underlined are poles related with resonances.

Pole	Re$E_{\rm pole}$ [MeV]	Im$E_{\rm pole}$ [MeV]	Riemann sheet
1	639.6	−323.9	$(-,-)$: III
1'	511.4	−230.6	$(-,+)$: II
2	982.0	−36.9	$(-,+)$: II
2'	432.4	−8.4	$(-,-)$: III
3	1431.7	−79.3	$(-,-)$: III
3'	1394.9	−120.6	$(-,+)$: II

Analysis of amplitudes for more than 2 channels is even more complicated and demanding. One of the reasons is that one cannot define and use similar conformal variable z. The simplest and very effective method of recognition of resonances among set of many S-matrix poles is just an analysis of influence of all found poles on the phase shifts and cross section as was shown in Figs. (4) and (5). Another method relies on presentation of positions of all poles in 3 dimensional combinations of real and/or imaginary parts of complex momenta in all channels. Reasonable choice of axes and careful analysis of distances of these poles from physical region enables to identify the most prominent poles. An example of results of such 3-channel analysis can be found in [2] (Tables 3–7).

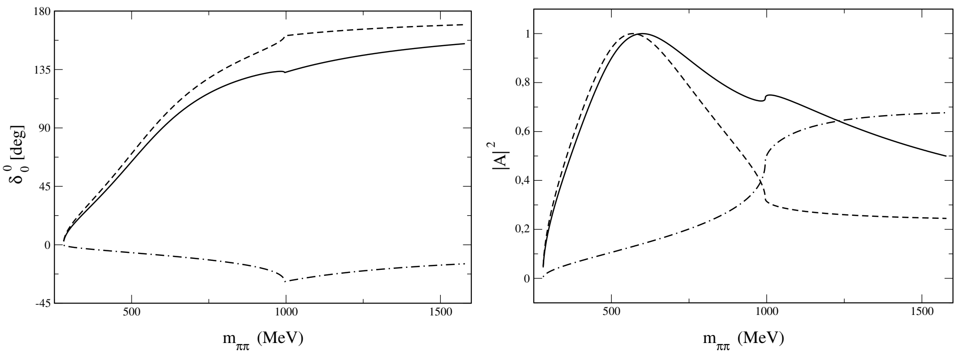

Fig. 4. Phase shifts and squared modules of amplitudes produced by single poles from Table I and their pairs. Dash-dotted lines are for pole 1, dashed for 1' and solid for both these poles together.

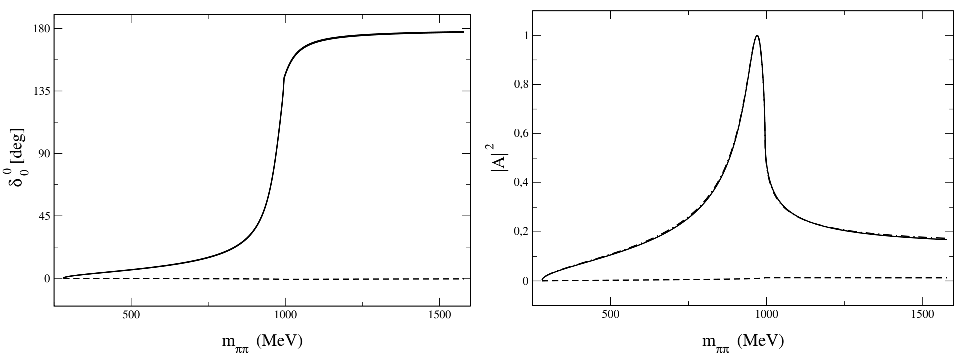

Fig. 5. Phase shifts and squared modules of amplitudes produced by single poles from Table I and their pairs. Dash-dotted lines are for pole 2, dashed for 2' and solid for both these poles together.

Independently of number of analyzed channels and number of resonances crucial is the use of correct *i.e.* unitary amplitude. Recent analysis of pion electromagnetic form factor [3] can serve as an example. Authors present various ways of parameterization of $e^+e^- \to \pi^+\pi^-$ cross section and of vector–isoscalar $\pi\pi$ elastic and inelastic amplitude. In Table II, there are compared parameters of ρ states obtained using the Gounaris–Sakurai approximation and unitary and analytic approach. The latter gives significantly different results than those from PDG Tables and those obtained using the Gounaris–Sakurai model. Mass difference for $\rho(770)$ is about 9 MeV and for ρ' and ρ'' about 170 MeV and 78 MeV respectively. The sign of these differences agrees with what was presented in Section 2. A small phase produced by the second pole (denoted in Section 2 by p') leads to a shift of the main pole p (*i.e.* shift of the mass) to lower energies in comparison with mass determined by the value of phase shift equal to $\pi/2$. In the case of $\rho(770)$, this shift should be few MeV and for wider states, should be bigger what agrees with numbers in Table II.

TABLE II

Parameters of ρ states obtained using the Gounaris–Sakurai model and unitary and analytic approach are compared with values from Particle Data Tables [4].

Parameter	PDG [MeV]	G–S [MeV]	U&A [MeV]
m_ρ	775.26 ± 0.25	774.81 ± 0.01	763.88 ± 0.04
$m_{\rho'}$	1465.00 ± 25.00	1497.70 ± 1.07	1326.35 ± 3.46
$m_{\rho''}$	1720.00 ± 20.00	1848.40 ± 0.09	1770.54 ± 5.49
Γ_ρ	149.10 ± 0.80	149.22 ± 0.01	144.28 ± 0.01
$\Gamma_{\rho'}$	400.00 ± 60.00	442.15 ± 0.54	324.13 ± 12.01
$\Gamma_{\rho''}$	250.00 ± 100.00	322.48 ± 0.69	268.98 ± 11.40
χ^2 pdf		0.98	1.84
		14 param.	11 param.

3.1. Another bond for amplitudes — crossing symmetry

Crossing symmetry is a natural consequence of symmetry between amplitudes “seen” from different channels — for example, s and t channels. Implementation of this condition to amplitudes is quite easy for identical particles and dispersion relations (with two subtractions) for $\pi\pi$ amplitudes were proposed few decades ago by Roy [5] and were later developed and applied in a number of works in the early 2000s *e.g.* [6] and [7]. For nonidentical mesons like π and K, similar analysis was performed recently [8].

It is very advisable to introduce this crossing symmetry to the amplitudes we create because it is a very demanding requirement and makes the amplitude similar to the real one — physical. Introducing this requirement can, therefore, have serious consequences for our amplitudes and significantly change the physical results we draw from them. For example, in the case of scalar–isoscalar $\pi\pi$ amplitudes, crossing symmetry has led to spectacular successes. One of them was to eliminate the long-standing up-down ambiguity in these amplitudes (in favor of the down solution) [9], and the second one to reduce by factor about 6 the uncertainty caused by the significant dispersion of experimental data. This reduction was possible thanks to the application of a newly derived set of the Roy-type dispersion relations GKPY with one subtraction [7]. Particularly spectacular turned out to be the impact of these new dispersion relation on parameters of resonance (pole) $f_0(500)$ (popularly called σ). In Particle Data Tables edited in 2012, its mass has changed from 400–1200 MeV to 400–550 MeV, and the width from 250–500 MeV to 200–350 MeV [10, 11]. This even led to the change of the name from $f_0(600)$ to $f_0(500)$.

4. Conclusions

Construction of amplitudes is a very important element of analyses of results of an experiment. However, it has many peculiarities and it is full of traps that can significantly change the final results. The condition of unitarity is very important and quite easy to introduce when building amplitudes which must meet several conditions, such as for example, the presence of two symmetrical poles and zeroes for each resonance. When drawing conclusions from the analysis of multi-channel amplitudes fitted to the experimental data, it is extremely important to correctly identify the leading poles and link them to the existing (or new) resonances. To avoid ambiguities that appear in the literature in determining the parameters of these resonances, it should always be clearly explained how they were identified and what amplitudes were used.

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