FEW-NUCLEON SYSTEMS WITHOUT PARTIAL WAVE DECOMPOSITION*

Kacper Topolnicki, Jacek Golak, Roman Skibiński
Henryk Witała, Yuriy Volkotrub

M. Smoluchowski Institute of Physics, Jagiellonian University
Łojasiewicza 11, 30-348 Kraków, Poland

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In this contribution, we present an overview of the "three-dimensional" (3D) approach that can be used to describe few-nucleon systems. Instead of relying on the partial wave decomposition of quantum mechanical operators related to a specific problem, the 3D approach works directly with the (three-dimensional) momentum degrees of freedom of the nucleons. Using this approach is, in principle, equivalent to using all partial wave at once and does not require a numerical implementation of heavily oscillating functions. In practice, these beneficial properties are limited by the available computing resources. Nonetheless, some recent results suggest that 3D calculations can be used in situations where traditional calculations are problematic. We briefly describe nucleon–nucleon scattering, and three-nucleon ($3N$) bound state calculations within the 3D approach. We also provide some preliminary results for the ^{3}He bound state with a screened Coulomb interaction.

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1. Introduction

In order to demonstrate the underlying idea of 3D calculations, we will discuss a simple example: a system of two nucleons in the center-of-mass frame. If the individual momenta of the two particles are $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, then it is sufficient to consider relative momentum $\boldsymbol{p} = \frac{1}{2}(\boldsymbol{k}_1 - \boldsymbol{k}_2)$ eigenstates $|\,\boldsymbol{p}\rangle$ of the two-nucleon ($2N$) system. Apart from momentum, the nucleons possess spin and isospin: they are spin-$\frac{1}{2}$ and isospin-$\frac{1}{2}$ particles.

In traditional partial wave (PW) calculations, momentum eigenstates $|\,\boldsymbol{p}\rangle$ are projected onto states with a given orbital angular momentum l, orbital angular momentum projection m_l and relative momentum magnitude

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p: $|\, plm_l\rangle$. The overlap between $|\, \boldsymbol{p}\rangle$ and $|\, plm_l\rangle$ is given by

$$\langle \boldsymbol{p}' \mid plm_l\rangle = \frac{\delta\left(p - |\boldsymbol{p}'|\right)}{p^2} Y_{l\,m_l}\left(\hat{\boldsymbol{p}}'\right)\,, \tag{1}$$

where $Y_{l\,m_l}(\hat{\boldsymbol{p}}')$ is the spherical harmonic. The spin states of the two nucleons can be coupled to a state $|\, sm_s\rangle$ of the total spin s with projection m_s. Analogously, the isospin states of the two particles can be coupled to a total isospin t state $|\, tm_t\rangle$, where m_t is a projection of the total isospin that is related to the charge. Finally, the states $|\, plm_l\rangle$, $|\, sm_s\rangle$, $|\, tm_t\rangle$ can be used to construct PW basis states

$$|\, p(ls)jm\; tm_t\rangle = \sum_{m_l\, m_s} C(lsj, m_l m_s m)\, |\, plm_l\rangle\, |\, sm_s\rangle\, |\, tm_t\rangle\,, \tag{2}$$

where j is the total angular momentum, m is its projection, $C(lsj, m_l m_s m)$ is the Clebsh–Gordan coefficient and the sum runs over all physical values of m_l and m_s. In traditional calculations, all relevant operators are represented in a finite sized PW basis. This is done by computing all the matrix elements of an operator between states (2) that satisfy $j < j_{\max}$, where $j_{\max}$ is some given truncation value for the total angular momentum.

The PW calculations are the mainstream method of performing calculations in the few-nucleon sector. Assuming a reasonable value for $j_{\max}$, the PW calculations can be performed using a moderate amount of computing resources. There are, however, certain disadvantages of using this approach. One is related to the necessity of numerically implementing the heavily oscillating functions in the definition of the spherical harmonics $Y_{lm_l}(\hat{\boldsymbol{p}})$. This can be problematic if the calculations are performed at higher energies or with potentials that have longer ranges like the screened Coulomb potential since many partial wave states have to be included in order for the calculations to converge. Another problem lies in the application of new models of nuclear interactions since each new term in the potential operator creates the necessity to calculate its matrix elements in the PW basis (2). This can be especially tedious for the chiral $3N$ force above N^{2}LO (see *e.g.* [1, 2]). In addition, for systems of three or more particles, there are different ways of coupling the momentum and spin states. This can be confusing since it is not obvious which coupling scheme to use. The 3D approach solves some of these issues by working directly with the (three-dimensional) momentum degrees of freedom of the nucleons. We will demonstrate this approach using the $2N$ transition operator.

The $2N$ transition operator $\check{t}$ can be used to calculate observables in the nucleon–nucleon scattering process (see *e.g.* [3, 4] for more details). It satisfies the Lippmann–Schwinger equation

$$\check{t} = \check{V} + \check{V}\check{G}_0(E)\check{t}\,, \tag{3}$$

where $\check{V}$ is the $2N$ potential and $\check{G}_0(E)$ is the free propagator for the energy E. Using the 3D approach, the transition operator is calculated directly in the momentum eigenstate basis

$$\langle \boldsymbol{p}' \mid \check{t} \mid \boldsymbol{p} \rangle \tag{4}$$

and it is not necessary to calculate its matrix elements in the basis from (2)

$$\langle p' \left(l's'\right) j'm't'm'_t \mid \check{t} \mid p(ls)jmtm_t \rangle \,. \tag{5}$$

Using (4) rather than (5) has several ramifications. Matrix elements of $\check{t}$ in (5) are complex numbers and matrix elements in (4) are operators in the isospin–spin space of the $2N$ system. This means that they are represented by 16×16 matrices, since each of the two nucleons can be in two spin states and in two isospin states. It turns out that the isospin parts can be separated out in (3) and it is sufficient to consider only the spin space of the $2N$ system. This still leaves 4×4 matrices and each element of the matrix is, in general, a function of the three spatial components of $\boldsymbol{p}'$ and three spatial components of $\boldsymbol{p}$. This would mean that calculating the $2N$ transition operator within the 3D approach amounts to calculating $4 \times 4 = 16$ functions of $3 + 3 = 6$ arguments in such a way that they satisfy equation (3) for each isospin case. Even using modern computing resources, this is a challenging numerical task. In order to make the 3D approach more feasible, operator forms of both states and operators are employed in the calculations.

2. Operator form of the $2N$ transition operator

The momentum and isospin space matrix element of the $2N$ transition operator can be written as (see *e.g.* [3])

$$\langle \boldsymbol{p}' \mid \langle t'm'_t \mid \check{t} \mid \boldsymbol{p} \rangle \mid tm_t \rangle = \delta_{t't}\delta_{m'_t m_t} \sum_{i=1}^{6} t_i^{tm_t}\left(p', p, \hat{\boldsymbol{p}}' \cdot \hat{\boldsymbol{p}}\right) \check{w}_i\left(\boldsymbol{p}', \boldsymbol{p}\right) \,, \tag{6}$$

where $\mid t'm'_t \rangle$ and $\mid tm_t \rangle$ are the final and initial isospin states, $\check{w}_i(\boldsymbol{p}', \boldsymbol{p})$ are given (momentum-dependent) spin space operators, listed *e.g.* in [4], and $t_i^{tm_t}(p', p, \hat{\boldsymbol{p}}' \cdot \hat{\boldsymbol{p}})$ are scalar functions of the initial and final momenta (they depend only on the magnitudes of the momenta and the angle between them). These scalar functions effectively define the transition operator and are the central object of the 3D calculations.

The operator form (6) is inserted into the Lippmann–Schwinger equation (3) together with the corresponding operator form of the $2N$ potential. Next, the spin dependencies are removed as described in [4], and (3) is transformed into a set of coupled linear equations for the scalar functions

$t_i^{tm_t}(p', p, \hat{\boldsymbol{p}}' \cdot \hat{\boldsymbol{p}})$ that define the transition operator. At this point, instead of having to calculate 16 functions of 6 arguments for each isospin case, it is only necessary is to work out (for each isospin case) the 6 scalar functions of only 3 real arguments. This makes 3D calculations of the transition operator feasible. Results of these computations can be found in [4, 5]. A direct link between the scalar functions and observables is given in [3].

A similar method of employing operator forms in order to reduce the numerical cost of the 3D calculations was used for other nuclear systems beyond two nucleons. Recent work was focused on calculating the ^{3}He bound state with a screened Coulomb interaction and in the next section, we show the operator form of the $3N$ bound state together with some preliminary results related to an upcoming paper [6].

3. Operator form of the $3N$ bound state and preliminary results for ^{3}He

$3N$ bound state calculations are carried out within the Faddeev formalism

$$| \psi\rangle = \check{G}_0(E)\check{V}\left(1+\check{P}\right) | \psi\rangle + \check{G}_0(E)\check{V}^{(1)}\left(1+\check{P}\right) | \psi\rangle \,. \tag{7}$$

In (7), $\check{G}_0(E)$ is the $3N$ free propagator for energy E, $| \psi\rangle$ is a Faddeev component of the bound state $| \Psi\rangle$ and $\check{V}$, $\check{V}^{(1)}$ are the $2N$ potential of the $(2,3)$ subsystem and the part of the $3N$ potential that is symmetric with respect to the exchange of particles $2,3$ respectively. Finally, the permutation operator $\check{P} = \check{P}_{12}\check{P}_{23} + \check{P}_{13}\check{P}_{23}$ is built from $\check{P}_{ij}$ operators that interchange particles i and j. The full bound state of the $3N$ system $| \Psi\rangle$ can be obtained by applying $1+\check{P}$ to the Faddeev component $| \psi\rangle$. Note that we use a version of the Faddeev equation without the $2N$ transition operator. This saves us from having to perform the difficult, additional step of calculating this operator.

Using the operator form of the $3N$ state from [7], (7) can be transformed into a set of coupled linear equations for scalar functions that define the Faddeev component. Details on this procedure can be found in [8]. The relation between the scalar functions and Faddeev component is given by [7]

$$\left\langle \boldsymbol{pq}; \left(t\tfrac{1}{2}\right) TM_T \mid \psi \right\rangle = \sum_{i=1}^{8} \phi_{tT}^{(i)}\left(p, q, \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{q}}\right) \check{O}_i\left(\boldsymbol{p}, \boldsymbol{q}\right) \,, \tag{8}$$

where $\check{O}_i(\boldsymbol{p}, \boldsymbol{q})$ are spin operators listed in [8], $\boldsymbol{p}$ and $\boldsymbol{q}$ are Jacobi momenta of the $3N$ system, t is the total isospin of the $2N$ subsystem, T is the total isospin, and M_T is the projection of the total isospin that is directly related

to the charge. Finally, $\phi_{tT}^{(i)}(p, q, \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{q}})$ are scalar functions that define the Faddeev component of the bound state that are the central object of the 3D calculations.

The newest work [6] extends the $3N$ bound state calculations to ^{3}He with a screened Coulomb interaction from [9]. In Fig. 1, we show preliminary results for selected scalar functions that define the Faddeev component. The disappearance of the $T = \frac{3}{2}$ component can be observed when going from ^{3}He to ^{3}H. Small differences in the scalar functions lead to substantial differences in the expectation values of different observables, more details will be given in [6].

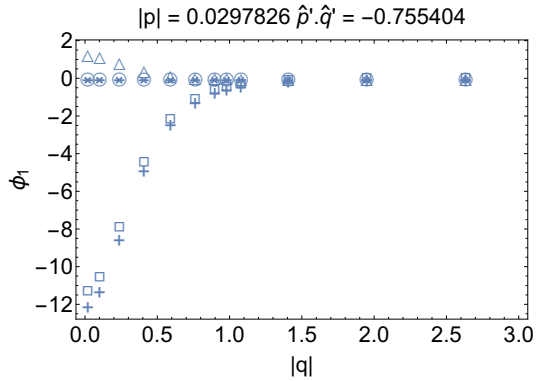

Fig. 1. Comparison of the dominant scalar function $\phi^{(1)}(|\boldsymbol{p}|, |\boldsymbol{q}|, \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{q}})$ from (8) calculated for ^{3}H ($\times$; $+$; $-$ correspond to $t = 0, T = \frac{1}{2}$; $t = 1, T = \frac{1}{2}$; $t = 1, T = \frac{3}{2}$) and ^{3}He (circles; squares; triangles correspond to $t = 0, T = \frac{1}{2}$; $t = 1, T = \frac{1}{2}$; $t = 1, T = \frac{3}{2}$). ^{3}He calculations use the screened Coulomb interaction form [9] with the screening radius 10 fm.

4. Summary and outlook

Results obtained for the ^{3}He bound state suggest that the "three-dimensional" approach can be successfully applied to problems that utilize not only short-range nuclear potentials but also longer-range screened Coulomb interactions. A detailed description of these calculations together with the results for the expectation values of selected operators and matrix elements of nuclear currents related to the beta decay of the triton will be published in a separate paper.

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