HIGH-RANK SYMMETRIES IN NUCLEI: CHALLENGES FOR PREDICTION CAPACITIES OF THE NUCLEAR MEAN-FIELD THEORIES*

J. Dudek[a,b], I. Dedes[b], J. Yang[b], A. Baran[b], D. Curien[a]
T. Dickel[c,d], A. Góźdź[b], D. Rouvel[a,e], H.L. Wang[a,f]

[a]Université de Strasbourg, IPHC
23, rue du Loess, 67037 Strasbourg, France and
CNRS, UMR7178, 67037 Strasbourg, France
[b]Institute of Physics, Maria Curie-Skłodowska University
pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland
[c]II. Physikalisches Institut, Justus-Liebig-Universität Gießen
35392 Gießen, Germany
[d]GSI Helmholtzzentrum für Schwerionenforschung GmbH
64291 Darmstadt, Germany
[e]Lycée Pascal, 74 rue du Logelbach, 68025 Colmar, France
[f]School of Physics and Engineering, Zhengzhou University
Zhengzhou 450001, China

(Received November 30, 2018)

We report on the recent progress in the application of the group-theory criteria combined with the nuclear mean-field theory methods in the context of the experimental verification of the presence of the tetrahedral and octahedral (sometimes referred to as high-rank) symmetries in sub-atomic physics. In this article, we focus on the possible coexistence of the two classes of shapes representing the two symmetries in nuclei simultaneously as well as we discuss the possible spontaneous breaking of the octahedral $\mathrm{O_h}$-group symmetry by its tetrahedral $\mathrm{T_d}$-subgroup symmetry partner. Experimental methods which are envisaged for the identification of the discussed symmetries, the former based on the mass spectrometry and isomer detection techniques are briefly discussed.

DOI:10.5506/APhysPolB.50.685

1. Introduction

The possible existence of nuclear states manifesting tetrahedral and octahedral shape-symmetries, the latter also referred to as high-rank, or exotic

* Presented at the Zakopane Conference on Nuclear Physics "Extremes of the Nuclear Landscape", Zakopane, Poland, August 26–September 2, 2018.

symmetries, attracted increasing attention of theorists and experimentalists during the recent years. Among the first predictions obtained using the methods similar to those employed in this article, let us quote Ref. [1] following the early pilot project in Ref. [2], followed by Ref. [3], together with an overview article [4] discussing selected theory aspects (*cf.* also Refs. [5–8] and references therein). An analogous overview addressing the experimental issues can be found in Ref. [9]. The reasons for the growing interest are numerous and are related to the specific unprecedented quantum properties of the nuclear states with such symmetries. Let us mention a few of them.

Firstly, at the exact symmetry limit, certain nucleonic states in such nuclei appear as 4-fold degenerate, an unprecedented mechanism, leading among others to the existence of the low-lying 8-fold, and 16-fold degenerate particle–hole excited states with distinct (orthogonal) wave functions.

Secondly, since the high-rank symmetric nuclei are non-spherical, their orientation in space can be defined and, as any other deformed nucleus, they generate collective rotational bands. At the same time, their collective dipole and quadrupole moments vanish due to the strongly restrictive symmetry properties, Ref. [3]. This eliminates the possibility of the usually strong electromagnetic decay via collective E1 or E2 transitions. Both these conditions lead to the presence of $E_I \propto I(I+1)$ sequences, which energy-wise look like rotational bands, except that no collective (neither E2 nor E1) transitions can depopulate these states. This implies the presence of the collective rotational bands without collective E2 transitions.

Thirdly, one can demonstrate with the help of the elementary methods of representations of group-theory (see below), that rotational bands associated with tetrahedral and/or octahedral symmetry states are *not* like the other rotational bands whose dozens of thousands are encountered in the literature: The corresponding spin-parity (I^π) rotational sequences do *not* follow the rules either of the type $\Delta I \equiv I_{\rm out} - I_{\rm in} = 2$, or of $\pi_{\rm in}\pi_{\rm out} = +1$. Very characteristically and very importantly, those sequences contain doublets, triplets, *etc.*, of degenerate rotational state energies, and in the case of tetrahedral symmetry, they may mix parities in a very characteristic, unique manner allowing to identify these symmetries in subatomic physics. To our knowledge, the first example of identification of such structures in sub-atomic physics has been presented in Ref. [10].

Fourthly, the strict absence of the collective E2 and E1 transitions at the exact symmetry limit makes out of the new type of rotational states excellent candidates for the long lived, possibly very long lived isomers. This suggests that the most efficient methods of experimental verifications of the presence of the high-rank symmetries in nature should involve mass spectrometry techniques rather than γ-detection techniques, see below.

Let us also emphasise that theory predicts the existence of the whole families of nuclei over the Periodic Table presenting the symmetries in question, Ref. [1], centred around doubly-magic tetrahedral nuclei with the Z_t/N_t combinations given by: 32, 40, 56, 64, 70, 90, 136, ... This leads to the prediction of existence of the following doubly-magic tetrahedral nuclei of interest in the context (some of them very exotic):

$$^{64,72,88}\mathrm{Ge}\,,^{80,96,104,110}\,\mathrm{Zr}\,,^{112,120,126,146}\,\mathrm{Ba}\,,^{134,154,174}\,\mathrm{Gd}\,,^{160,180}\,\mathrm{Yb}\,,^{226}\,\mathrm{Th}\,. \tag{1}$$

Therefore, the states of interest are expected to appear in many regions of the Periodic Table, possibly as shape coexisting configurations.

Last but not least, predicted existence of the new class of isomers may open the new ways to studying the very exotic nuclei whose ground states might live much shorter than the isomeric high-rank symmetry states. By the same token, the isomers of this new category may lead to the new families of the waiting-point nuclear configurations in the stellar nucleosynthesis processes and may become, more generally, of a broad interest for astrophysics.

The above discussion shows that the field of high rank symmetry research involves numerous unprecedented quantum features/mechanisms and thus whose studying can be seen as self-motivated. When fully confirmed, the high-rank symmetries will open new fields of research of, especially, exotic and, possibly, superheavy nuclei.

We proceed to presenting the summary-description of the methods needed to help the identification of the new cases after the first discovery of Ref. [10].

2. Towards identification criteria of tetrahedral and octahedral symmetries in subatomic physics

In order to be able to identify the point group high-rank symmetries, it is necessary to be able to construct the nuclear Hamiltonian capable of describing the sought symmetries in nuclei and to construct the unique identification criteria suitable for detecting the presence of the symmetries in question in the available experimental data.

In this article, we employ the nuclear mean-field approach with the standard phenomenological "universal" Woods–Saxon Hamiltonian of Ref. [11]. In what follows, we present briefly, in Sect. 2.1, the main lines of our technique of generating Hamiltonians with a predefined point group symmetry. The identification criteria suitable for detecting the presence of tetrahedral and octahedral symmetries in nuclei in terms of the experimental data are summarised in Sect. 2.2.

2.1. Generating mean-field Hamiltonian with a predefined point-group symmetry

To describe nuclear shapes in a phenomenological mean-field approach, we employ the basis of spherical harmonics, $\{Y_{\lambda\mu}(\vartheta,\varphi)\}$, with the help of what we describe any nuclear surface Σ employing deformation parameters whose full set is denoted $\{\alpha_{\lambda,\mu}\}$ as

$$R(\vartheta,\varphi) \sim \left[1+\sum_{\lambda}\sum_{\mu}\alpha^{*}_{\lambda\mu}Y_{\lambda\mu}(\vartheta,\varphi)\right] . \tag{2}$$

Consider a given symmetry point group $\mathcal{G}$ composed of elements $\hat{g}\in\mathcal{G}$. Our goal is to construct a surface, say Σ, invariant under the action of all the symmetry elements of this group

$$\forall\hat{g}\in\mathcal{G}:\ \Sigma \xrightarrow{\hat{g}} \Sigma' = \Sigma \Rightarrow \sum_{\lambda\mu}\alpha^{*}_{\lambda\mu}\left[\hat{g}\,Y_{\lambda\mu}(\vartheta,\varphi)\right] = \sum_{\lambda\mu}\alpha^{*}_{\lambda\mu}Y_{\lambda\mu}(\vartheta,\varphi) . \tag{3}$$

One may demonstrate that Eq. (3) represents a system of as many relations as the number of the group elements and that it is equivalent to a system of linear equations for $\{\alpha_{\lambda\mu}\}$, separately at each given order λ. The equations in question imply, as it was shown in Ref. [12], that tetrahedral symmetry surfaces can be generated with the help of specific odd ($\lambda \geq 3$) spherical harmonics. We find

$$\lambda = 3:\ \ \alpha_{3,\pm2}\equiv t_3 , \tag{4}$$

the order $\lambda = 5$ is excluded by the symmetry, and the next solutions are

$$\lambda\ =\ 7:\ \ \alpha_{7,\pm2}\equiv t_7 ; \qquad \alpha_{7,\pm6}\equiv -\sqrt{11/13}\times t_7 , \tag{5}$$

$$\lambda\ =\ 9:\ \ \alpha_{9,\pm2}\equiv t_9 ; \qquad \alpha_{9,\pm6}\equiv +\sqrt{28/198}\times t_9 ,\ \ldots \tag{6}$$

Similarly, for the octahedral symmetry surfaces, one obtains valid solutions exclusively for selected spherical harmonics of even order $\lambda \geq 4$ satisfying

$$\lambda\ =\ 4:\ \ \alpha_{40}\equiv o_4;\ \ \alpha_{4,\pm4}\equiv \pm\sqrt{5/14}\times o_4 , \tag{7}$$

$$\lambda\ =\ 6:\ \ \alpha_{60}\equiv o_6;\ \ \alpha_{6,\pm4}\equiv -\sqrt{7/2}\times o_6 , \tag{8}$$

$$\lambda\ =\ 8:\ \ \alpha_{80}\equiv o_8;\ \ \alpha_{8,\pm4}\ \ \sqrt{28/198}\times o_8;\ \ \ \alpha_{8,\pm8}\ \ \sqrt{65/198}\times o_8 ,\ \ldots \tag{9}$$

Since the deformed Woods–Saxon Hamiltonian is based on potentials with the structure

$$V_{\mathrm{WS}}(\vec{r}\,) = \frac{V_o}{1+\exp\left[\mathrm{dist}_{\Sigma}(\vec{r};R_o)/a_o\right]} , \tag{10}$$

in which $\mathrm{dist}_\Sigma(\vec{r};R_o)$ represents the distance of a point $\vec{r}$ from the nuclear surface Σ, and where V_o, R_o and a_o are parameters, it follows that for the surfaces invariant under the symmetry operations $\hat{g}$ of the group $\mathcal{G}$, the whole Hamiltonian becomes invariant under the same symmetry operations. This property has been used to generate the potential energy surfaces as functions of deformations illustrated below — in particular for the cases of tetrahedral and octahedral symmetries.

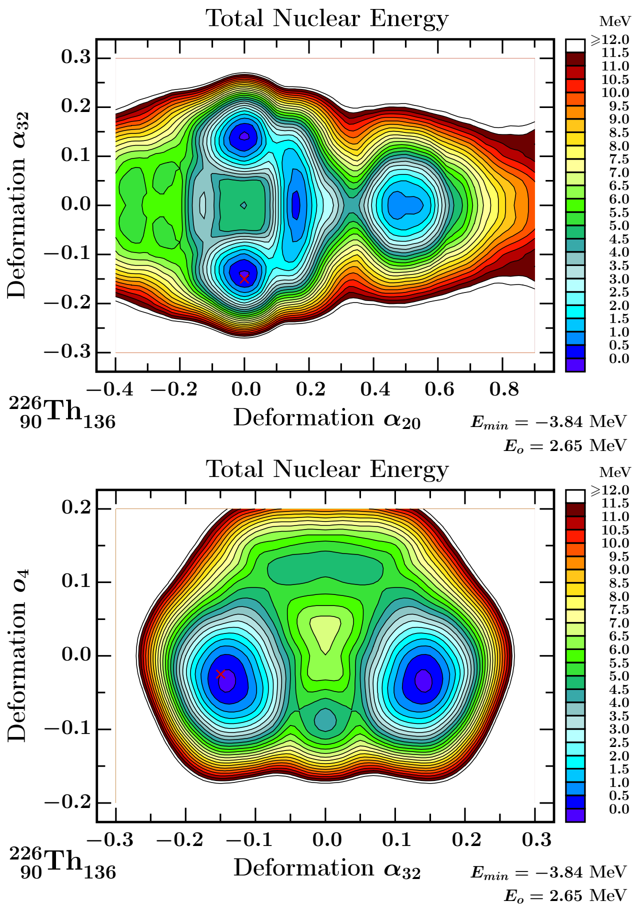

Fig. 1. Top: Total energy surfaces for ^{226}Th showing the presence of tetrahedral symmetry minima at $\alpha_{20} = 0$ in competition with quadrupole symmetry minima. At each point, the energy was minimised over octahedral symmetry deformations o_4 and o_6, *cf.* Eqs. (7) and (8). Bottom: Projection over t_3-tetrahedral *vs.* o_4-octahedral symmetry deformations minimised over o_6 showing combined tetrahedral–octahedral symmetry minima at $o_4 \approx -0.04$. By convention we normalise the macroscopic energy to zero at zero deformation; E_o is the corresponding shell-energy, whereas $E_{\min}$ denotes the energy at the absolute minimum.

In the following, we use the standard Strutinsky method with the single particle spectra generated by the Universal Woods–Saxon Hamiltonian of Ref. [11] together with the folded-Yukawa macroscopic energy formula of Ref. [13]. In this article, we limit ourselves to presenting the *coexistence of the tetrahedral and octahedral symmetry* degrees of freedom in the tetrahedral doubly magic nucleus ^{226}Th, which is representative for a number of neighbouring heavy (actinide) nuclei. The main purpose of this article is to encourage the experimental efforts towards identifying the coexistence of these important symmetries in heavy nuclei using mass-spectrometry methods knowing that the tetrahedral symmetry group $\mathrm{T_d}$ is a subgroup of the octahedral symmetry group $\mathrm{O_h}$. Some specific criteria facilitating such an identification as well as the short description of the envisaged experimental approach are presented below.

Figure 1, top, shows the potential energy of the ^{226}Th nucleus projected on the quadrupole (elongation α_{20}) and the first order tetrahedral (t_3) deformation plane, where at each deformation point, the energy was minimised over the octahedral first-, and the second order deformations, o_4 and o_6, respectively. Illustration shows clearly the presence of two strong tetrahedral symmetry minima which, in the discussed deformation space, appear as the lowest ones. Figure 1, bottom, shows that the pure tetrahedral symmetry minima gain about 1 MeV when the minimisation over the octahedral symmetry is allowed and thus that the predicted exotic-symmetry configurations combine both the tetrahedral and octahedral symmetry shape components. This has interesting and measurable consequences as shown below.

2.2. Rotation properties of tetrahedral and octahedral symmetry systems: identification criteria based on group theory

The properties of rotational states which arise within the mean-field theory have been studied recently using the Gogny–Hartree–Fock–Bogolyubov approach with the spin, parity and particle number projection techniques, Refs. [14, 15]. It has been demonstrated by explicit calculations that the rotational $E_I \propto I(I+1)$ sequences arise naturally within the projected mean-field theory, but the spin-parity combinations of the tetrahedral-symmetry rotational states are not like those known from dozens of thousands of rotational bands described in the literature. Instead, they follow the structures predicted by the group theory methods applicable to the quantum rotor Hamiltonians (see below). The corresponding technique will be summarised in what follows; interested reader may consult Refs. [14, 15] for details.

Let $\mathcal{G}$ be the symmetry group of the quantum rotor Hamiltonian. Let $\{D_i,\, i=1,2,\ldots M\}$ be irreducible representations of $\mathcal{G}$. Representations $D^{(I\pi)}$ of the group of rotation characterised by the definite spin-parity I^π-combination can be decomposed in terms of D_i as follows:

$$D^{(I\pi)} = \sum_{i=1}^{M} a_i^{(I\pi)} D_i \,, \tag{11}$$

where multiplicity factors $a_i^{(I\pi)}$ are given (*cf.* Ref. [16]) by

$$a_i^{(I\pi)} = \frac{1}{N_G} \sum_{R \in G} \chi_{I\pi}(R) \chi_i(R) = \frac{1}{N_G} \sum_{\alpha=1}^{M} g_\alpha \chi_{I\pi}(R_\alpha) \chi_i(R_\alpha) \,. \tag{12}$$

Above, N_G denotes the order of the group $\mathcal{G}$, $\{\chi_{I\pi}, \chi_i\}$ are characters associated with $\{D^{(I\pi)}, D_i\}$, R denotes the group elements and g_α is the number of elements in the class α, whose representative element is R_α. Tetrahedral group has 5 irreducible representations, here denoted A_1, A_2, E, F_1 and F_2 and 5 classes. Using this information and the known tables of group characters one may demonstrate that the collective band built on the $I^\pi = 0^+$ *tetrahedral ground-state* is composed of the following spin-parity combination spanned by the irreducible representation A_1, *cf.* *e.g.* Refs. [14, 16]:

$$\underbrace{\mathrm{A}_1: \quad 0^+, 3^-, 4^+, \underbrace{(6^+, 6^-)}_{\text{doublet}}, 7^-, 8^+, \underbrace{(9^+, 9^-)}_{\text{doublet}}, \underbrace{(10^+, 10^-)}_{\text{doublet}}, 11^-, \underbrace{2 \times 12^+, 12^-}_{\text{triplet}}, \cdots}_{\textbf{Forming a common parabola}} \tag{13}$$

If instead the tetrahedral symmetry mean-field is perturbed by the octahedral shape components, one should expect two branches, one with the positive

$$\underbrace{A_{1g}: \; 0^+, 4^+, 6^+, 8^+, 9^+, 10^+, \underbrace{(12^+, 12^+)}_{\text{doublet}} \; \ldots, \quad I^\pi = I^+}_{\textbf{Forming a common parabola}} \tag{14}$$

and one with the negative parity

$$\underbrace{A_{2u}: \; 3^-, 6^-, 7^-, 9^-, 10^-, 11^-, 12^-, \ldots, \; I^\pi = I^-}_{\textbf{Forming another (common) parabola}} . \tag{15}$$

The discussed situation is illustrated qualitatively in Fig. 2, left hand-side. The positive and negative parity branches lie symmetrically with respect to a parabola corresponding to Eq. (13). The discussed symmetry properties have been used in Ref. [10] to identify the combination of octahedral and tetrahedral symmetries using existing experimental data on ^{152}Sm nucleus.

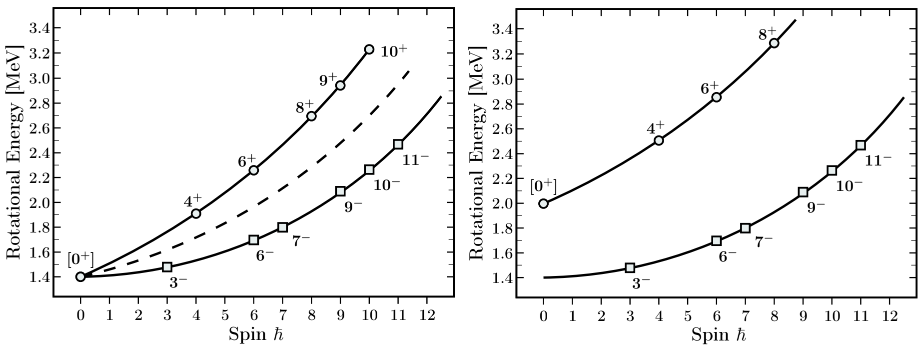

Fig. 2. Qualitative. Left: The characteristic configuration of two parabolic energy *vs.* spin sequences of opposite parities aiming at a common $I^\pi = 0^+$ lowest energy point (tetrahedral symmetry 'ground-state'). In the case of pure tetrahedral symmetry, the two branches are expected to form a single one coinciding with the dashed line (for details *cf.* Ref. [10]). Right: Hypothetical pure octahedral symmetry configuration. The two auxiliary curves are expected to be approximately parallel and they are not expected to join the common $I^\pi = 0^+$ point at the bottom. This feature can be used to distinguish between the scenarios of single or coexisting symmetry configurations.

On the basis of the calculation results in Fig. 1, indicating the presence of the strong tetrahedral minima perturbed by the presence of the octahedral symmetry deformation components, we should expect that the results of the future experiments should resemble the pattern in the left-hand side of Fig. 2 rather than that in the right-hand side.

3. Search of high-rank symmetries — experimental perspectives

Nuclear-decay spectroscopy relays on coincidence techniques in order to identify the studied states and suppress the background. This becomes challenging for states which neither decay via γ-transitions, nor α-, or β-transitions — like the high-rank symmetry states discussed in this article — or manifest coincidence times above one millisecond. To study long-life isomeric states, high-resolution mass spectrometry is the method of choice. States are identified without necessity of detecting their decay but rather via identification of their mass/excitation energy. Mapping or searching in unknown regions always requires measurement methods that are: *(i)* non-scanning, thus cover the whole region of interest in a single measurement, *(ii)* highly sensitive, to detect also the rarest events, *(iii)* large dynamic range ($\gg 10$), to detected also weakly populated states, *(iv)* fast, to measure short-lived states and *(v)* high resolving power, to be able to unambiguously identify and measure also close-lying states, Ref. [17].

The mass spectrometry methods which fulfil these requirements are, for half-lives longer than a few seconds, the Schottky mass spectrometry at the storage rings, Ref. [18], and for all half-lives down to $\sim 10\,\mathrm{ms}$, the multiple-refection time-of-flight mass spectrometer (MR-TOF-MS) of the FRS Ion Catcher at GSI, Darmstadt, Refs. [19, 20], see Fig. 3. For this system, even the spatial separation of excited and ground state has been demonstrated in a measurement time of about 8 milliseconds, Ref. [21].

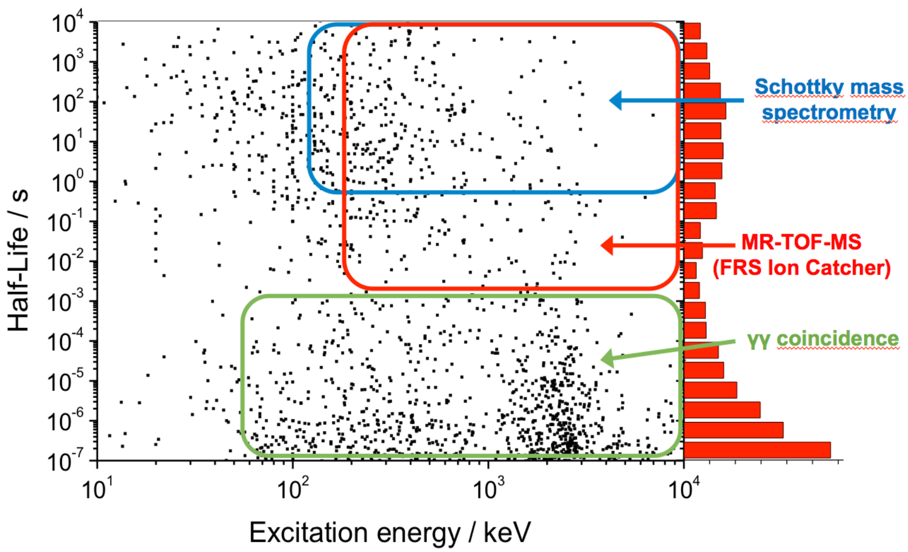

Fig. 3. In this scatter plot, the excitation energy and half-live of all known isomeric states according to the NUBASE2012, Ref. [22], are shown. The accessible half-lives and excitation energy of the different measurement methods (see the text) are indicated. On the right-hand side, a histogram of the half-lives of the known isomers is shown. This shows a clear minimum in the millisecond region. The reason for this is that this region becomes accessible only now with the MR-TOF-MS of the FRS Ion Catcher experiments.

It is expected that the intensive future measurements using these and similar techniques will become the methods of choice for the identification of the high-rank symmetry nuclear configurations whose γ-decay probabilities vanish at the exact symmetry limit.

4. Summary and conclusions

In this article, we briefly summarised the theoretical mean-field approach allowing to study the so-called high-rank symmetries, tetrahedral ($\mathrm{T_d}$) and octahedral ($\mathrm{O_h}$) ones, and their mutual spontaneous symmetry breaking leading to the coexistence of two approximate realisations of the symmetries. We illustrated the theory predictions on the example of a heavy nucleus ^{226}Th which, after the first discovery of the presence of both these symmetries in nature in Ref. [10], could become a favourable next experimental test-case.

One of the authors, J.D., performed part of this work within the EMPIR Project 15SIB10 MetroBeta. This project has received funding from the EMPIR programme co-financed by the Participating States and from the European Union Horizon 2020 research and innovation programme. This work was partially supported by the National Science Centre, Poland (NCN) under contracts No. 2013/08/M/ST2/00257 and 2016/21/B/ST2/01227 and Polish–French COPIN-IN2P3 collaboration agreement under project numbers 04-113 and 05-119.

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