ON THE POSSIBILITY OF THE FORMATION OF PARTICLES WITH A NONZERO REST MASS IN THE "GAS" OF NULL-STRINGS

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A solution of the Einstein equations for a multi-string system having a layered structure is found. It is shown that the influence of the gravitational field of such a multi-string system can lead to stable in time oscillations of the test null-string in the vicinity of fixed point of space. This situation can be interpreted as a particle localized in space with an effective nonzero rest mass.

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1. Introduction

One of directions of the string theory is to study a role of one-dimensional extended objects in cosmology. Gauge Grand Unified Theories (GUT) predict a possibility of a formation of one-dimensional topological defects in a process of phase transitions in the early Universe. These defects are called cosmic strings [1–8]. It is not excluded that cosmic strings can be preserved until modern era and can be observable [9, 10].

The radius of the cross section of a cosmic string is estimated as $r_{\rm s} \approx 10^{-31}$ m. For a description of a string motion in the case when a string cross-section radius $r_{\rm s}$ is much less than a radius of string curvature, the following approximation is used. A position of a string is determined by a line in D-dimensional space-time. Then, a trajectory of a string is a two-dimensional world surface which is mathematically defined by functions $x^m(\tau,\sigma)$, where τ and σ are parameters on a world surface of a string.

Null-strings realize a limit case of zero tension for cosmic strings [7, 11, 15]. They describe a limit case in which points of a string can interact only with a surrounding (external) gravitational field, but not with each other. It is assumed that null-strings realize a high-temperature phase of a string theory [15], *i.e.* they could have been formed in early stages of Universe evolution. Thus, it is possible that they were taking part in processes of formation of the observable Universe structure.

For example, in [15], the possibility of a null-string inflation mechanism for the case of D-dimensional Friedmann–Robertson–Walker spaces (FRW) described by a homogeneous and isotropic metric was considered. In the cosmic time $t = x^0$, it has the form of

$$\mathrm{d}S^2 = \left(\mathrm{d}x^0\right)^2 - R^2\left(x^0\right)\mathrm{d}x^i\delta_{ij}\mathrm{d}x^j\,,$$

where $i,\, j = 1, \ldots, D-1$.

The work notes the possibility of the existence of a phase of an ideal null-string gas (contracting or expanding) described by the exact equation of state

$$\rho = P(D-1)\,.$$

Considering this phase of null-string gas as the dominant source of gravity in FRW spaces, possible scaling factors $R(t)$ were calculated

$$\begin{aligned} R_{\mathrm{I}}(t) &= \left[q\cdot(t_{\mathrm{c}}-t)\right]^{2/D}\,, \qquad t<t_{\mathrm{c}}\,,\\ R_{\mathrm{II}}(t) &= \left[q\cdot(t-t_{\mathrm{c}})\right]^{2/D}\,, \qquad t>t_{\mathrm{c}}\,, \end{aligned}$$

where $q = \left(4\pi G_D A/(D-1)(D-2)\right)^{1/2}$; G_D, A, t_{c} are constants. The solution $R_{\mathrm{I}}(t)$ describes the regime of accelerated contraction of the D-dimensional Universe ($\mathrm{d}R/\mathrm{d}t < 0$, $\mathrm{d}^2R/\mathrm{d}t^2 < 0$) with collapse at the moment of time $t = t_{\mathrm{c}}$. The second solution $R_{\mathrm{II}}(t)$ describes the delayed expansion of the Universe ($\mathrm{d}R/\mathrm{d}t > 0$, $\mathrm{d}^2R/\mathrm{d}t^2 < 0$) from the superconstricted state with zero volume.

We note that in this paper, the transition to an ideal null-string gas was accomplished by means of the $(D-1)$-dimensional spatial averaging of the energy-momentum tensor of one null-string over a multi-string ensemble.

This example operates with the concept of a gas (net) of null-strings, however the properties of this gas are still not clear. One of the possible directions in understanding that the properties of null-string gas can be the study of the influence of the gravitational field of various time-stable null-string configurations on the dynamics of a test null-string.

The study of the test null-string motion carried out in [17–21] suggests that a number of interesting properties of null-string gas can exist. For example, the formation of stable structures in a space filled with null-string gas [17]. Namely, the possibility of the existence of a state (phase) of a gas of null-strings, in which closed null-strings are arranged in parallel planes and, without changing their initial form (not interacting with each other), move in one direction.

The first part of the proposed work is devoted to finding a solution that describes the gravitational field of the simplest realization of such a state of a null-string gas (multi-string system). Namely, the states in which $m \times n$ (m, n are constants) of closed null-strings having the shape of a circle move in one direction.

In the second part of the work, the motion of a test null-string in the gravitational field of a multi-string system under study is considered for the case in which the test null-string moves "towards" the multi-string system.

In this work, a unit system is chosen in which the speed of light $c = 1$.

2. The Einstein equations

In the cylindrical coordinate system $x^0 = t$, $x^1 = \rho$, $x^2 = \theta$, $x^3 = z$, the functions $x^\alpha(\tau, \sigma)$, defining the trajectories of motion (world surfaces) of closed null-strings forming a multi-string system that moves along the negative direction of the z axis, have the form of

$$t = \tau\,, \qquad \rho = R_i\,, \qquad \theta = \sigma\,, \qquad z = z_j^0 - \tau\,, \tag{1}$$

where: τ and σ are parameters on a world surface of a null-string; $\tau \in (-\infty, +\infty)$, $\sigma \in [0; 2\pi]$; $R_i =$const, $i = 1 \dots n$; $z_j^0 =$const, $j = 1 \dots m$; moreover $R_{i+1} > R_i$, and $z_{j+1}^0 > z_j^0$.

It can be noted that trajectory (1) describes a case of a multi-string system motion with a layered structure. Specifically, the system has m layers (surfaces) the distances between which are determined by constants z_j^0. There are n closed coaxial null-strings of different, but constant in time, radiuses R_i on each layer. Moreover, the position of a closed null-string on each such a layer is the same.

The energy-momentum tensor for a secluded null-string has the form [5] of

$$T^{\alpha\beta}\sqrt{-g} = \varsigma \int \mathrm{d}\tau \mathrm{d}\sigma \; x^\alpha_{,\tau} x^\beta_{,\tau} \delta^4 \left(x^\omega - x^\omega\left(\tau, \sigma\right)\right)\,, \tag{2}$$

where indexes α, β, ω take values $0, 1, 2, 3$; functions $x^\omega(\tau, \sigma)$ define a trajectory of a null-string motion (world surface); $x^\alpha_{,\tau} = \partial x^\alpha / \partial \tau$; $g = |g_{\alpha\beta}|$; $g_{\alpha\beta}$ is the metric tensor of external space-time; $\varsigma =$ const.

Since an interaction in the considered multi-string system is only gravitational, then generalizing (2) on a considered in the work case, we can write

$$T^{\alpha\beta}_{\text{tot}} = \sum_{j=1}^{m} \sum_{i=1}^{n} \left(T^{\alpha\beta}\right)_{ij}\,, \tag{3}$$

where $T_{\rm tot}^{\alpha\beta}$ and $\left(T^{\alpha\beta}\right)_{ij}$ are, respectively, energy-momentum tensor of multi-string system, and secluded null-string numerated in this system by indexes i, j (*i.e.* the null-string with a radius R_i and in each fixed moment of time $t = t_0$ located in the plane $z = z_j^0 - t_0$).

Considering (2), nonzero components of tensor (3) for trajectories (1) have the form of

$$T_{\rm tot}^{00} = T_{\rm tot}^{33} = -T_{\rm tot}^{03} = \frac{\varsigma}{\sqrt{-g}} \sum_{j=1}^{m} \sum_{i=1}^{n} \delta\left(q - z_j^0\right) \delta\left(\rho - R_i\right) , \tag{4}$$

where $q = t + z$.

Since for conserving trajectories of a motion of multi-string systems (1) all directions on hyper-surfaces $z =$const are equivalent, then the metric functions are $g_{\alpha\beta} = g_{\alpha\beta}(t, \rho, z)$. Thus, using an invariance of a quadratic form relatively to an inversion of θ on $-\theta$, we obtain $g_{02} = g_{12} = g_{32} = 0$.

It can be also noticed that for the considered multi-string systems, the quadratic form must be invariant relatively to a simultaneous inversion $t \to -t$; $z \to -z$. Then

$$g_{\alpha\beta}(t, \rho, z) = g_{\alpha\beta}(-t, \rho, -z)\,. \tag{5}$$

The result of (5) is $g_{01} = g_{31} = 0$. Finally, using a freedom of choice of a coordinate system in GTR, we partially fix it by choosing $g_{03} = 0$.

Thus, a quadratic form for the task can be represented in the form of

$$\mathrm{d}S^2 = e^{2\nu}(\mathrm{d}t)^2 - A(\mathrm{d}\rho)^2 - B(\mathrm{d}\theta)^2 - e^{2\mu}(\mathrm{d}z)^2\,, \tag{6}$$

where ν, μ, A, B are functions of variables t, ρ, z satisfying conditions (5).

A motion of a null-string in pseudo-Riemannian manifold is determined by a system of equations [15]

$$x^{\alpha}_{,\tau\tau} + \mathit{\Gamma}^{\alpha}_{pq} x^{p}_{,\tau} x^{q}_{,\tau} \;=\; 0\,, \tag{7}$$

$$g_{\alpha\beta} x^{\alpha}_{,\tau} x^{\beta}_{,\tau} \;=\; 0\,, \qquad g_{\alpha\beta} x^{\alpha}_{,\tau} x^{\beta}_{,\sigma} = 0\,, \tag{8}$$

where $\mathit{\Gamma}^{\alpha}_{pq}$ are the Christoffel symbols. Trajectories of a null-strings motion, forming multi-string system, must be a particular solution of motion equations. Thus, an analysis of these equations could give additional limitations on functions of the quadratic form (6). Considering equations of motion of a null-string (7), (8) for (6), it can be shown directly that for trajectories (1), equations (8) lead to an equation

$$e^{2\nu} - e^{2\mu} = 0\,, \tag{9}$$

from which

$$\nu = \mu\,. \tag{10}$$

Equation (7), considering equality (10), leads to an equation

$$\nu_{,t} - \nu_{,z} = 0\,, \tag{11}$$

from which

$$\nu = \nu(q,\rho)\,, \tag{12}$$

where

$$q = t + z\,. \tag{13}$$

The Einstein equations system built for multi-string formations (1) allows extending a definition of a dependence of quadratic form functions (6) (considering (10), (12))

$$A = A(q,\rho)\,, \qquad B = B(q,\rho)\,. \tag{14}$$

It can be represented in the form of

$$\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right)_{,q} - 2\nu_{,q}\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right) + \frac{1}{2}\left(\left(\frac{A_{,q}}{A}\right)^2+\left(\frac{B_{,q}}{B}\right)^2\right) = -2\chi T_{00}\,, \tag{15}$$

$$\left(\frac{B_{,\rho}}{B}+2\nu_{,\rho}\right)_{,q} - \nu_{,\rho}\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right) - \frac{1}{2}\frac{B_{,\rho}}{B}\left(\frac{A_{,q}}{A}-\frac{B_{,q}}{B}\right) = 0\,, \tag{16}$$

$$\left(\frac{B_{,\rho}}{B}\right)_{,\rho} + \frac{1}{2}\left(\frac{B_{,\rho}}{B}\right)^2 + 2\nu_{,\rho}\frac{B_{,\rho}}{B} - \frac{1}{2}\frac{A_{,\rho}}{A}\frac{B_{,\rho}}{B} = 0\,, \tag{17}$$

$$\nu_{,\rho\rho} + 2(\nu_{,\rho})^2 + \frac{\nu_{,\rho}}{2}\left(\frac{B_{,\rho}}{B}-\frac{A_{,\rho}}{A}\right) = 0\,, \tag{18}$$

$$(\nu_{,\rho})^2 + \nu_{,\rho}\frac{B_{,\rho}}{B} = 0\,, \tag{19}$$

where

$$T_{00} = \varsigma\frac{e^{2\nu}}{\sqrt{AB}}\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\delta\left(q-z_j^0\right)\delta(\rho-R_i)\right)\,,$$

$\chi = 8\pi G$, G is the gravitational constant.

Integrating system of equations (15)–(19), let us use the algorithm suggested in work [16]. Since a cross-section area radius of a cosmic string is small ($\sim 10^{-31}$ m) but still finite, then a model of a null-string in the form of a thin tube is physically more justified. Therefore, let us consider components of string energy-momentum tensor (4) as a limit of some "smeared" distribution. As such, it is convenient to choose a real massless scalar field (since we consider a system of scalar null objects). Then, let us constrict this "smeared" distribution into the considered multi-string system. We demand that components of the energy-momentum tensor of a scalar field in a limit of contraction would asymptotically coincide with components of the null-string energy-momentum tensor. In this approach, we, in fact, deny one-dimensionality of null-strings, forming a multi-string system and move to a physically justified model of a null-string in the form of thin tube ("smeared" null-string).

Components of the energy-momentum tensor for a real massless scalar field have the form [2] of

$$T_{\alpha\beta} = \varphi_{,\alpha}\varphi_{,\beta} - \tfrac{1}{2} g_{\alpha\beta} L \,, \tag{20}$$

where $L = g^{\omega\lambda}\varphi_{,\omega}\varphi_{,\lambda}$; $\varphi_{,\alpha} = \partial\varphi/\partial x^{\alpha}$; φ is a distribution function of a scalar field, and indices $\alpha, \beta, \omega, \lambda$ take values $0, 1, 2, 3$. In order to provide self-consistency of the Einstein equations for (6), (10), (12), (14), (20), we demand

$$T_{\alpha\beta} = T_{\alpha\beta}\left(q,\rho\right) \Rightarrow \varphi = \varphi\left(q,\rho\right) . \tag{21}$$

The system of Einstein equations for (6), (10), (12), (14), (20), (21) may be represented in the form of

$$\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right)_{,q} - 2\nu_{,q}\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right) + \frac{1}{2}\left(\left(\frac{A_{,q}}{A}\right)^{2}+\left(\frac{B_{,q}}{B}\right)^{2}\right) = -2\chi\left(\varphi_{,q}\right)^{2} , \tag{22}$$

$$\left(\frac{B_{,\rho}}{B}+2\nu_{,\rho}\right)_{,q} - \nu_{,\rho}\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right) - \frac{1}{2}\frac{B_{,\rho}}{B}\left(\frac{A_{,q}}{A}-\frac{B_{,q}}{B}\right) = -2\chi\varphi_{,q}\varphi_{,\rho} \,, \tag{23}$$

$$\left(\frac{B_{,\rho}}{B}\right)_{,\rho} + \frac{1}{2}\left(\frac{B_{,\rho}}{B}\right)^{2} + 2\nu_{,\rho}\frac{B_{,\rho}}{B} - \frac{1}{2}\frac{A_{,\rho}}{A}\frac{B_{,\rho}}{B} = 0 \,, \tag{24}$$

$$\nu_{,\rho\rho} + 2(\nu_{,\rho})^{2} + \frac{\nu_{,\rho}}{2}\left(\frac{B_{,\rho}}{B}-\frac{A_{,\rho}}{A}\right) = 0 \,, \tag{25}$$

$$(\nu_{,\rho})^{2} + \nu_{,\rho}\frac{B_{,\rho}}{B} = \frac{\chi}{2}\left(\varphi_{,\rho}\right)^{2} . \tag{26}$$

Comparing the Einstein equations system (15)–(19) with the system (22)–(26), one can see that during a constriction of a scalar field in a multi-string system, the following should be valid:

$$\begin{aligned} \left(\varphi_{,\rho}\right)^{2}\Big|_{q\to z_j^0,\rho\to R_i} &\to 0 \,, \qquad \left(\varphi_{,q}\right)^{2}\Big|_{q\to z_j^0,\rho\to R_i} \to \infty \,, \\ \left(\varphi_{,q}\varphi_{,\rho}\right)\Big|_{q\to z_j^0,\rho\to R_i} &\to 0 \,, \end{aligned} \tag{27}$$

and outside a region where a scalar field is concentrated (*i.e.* at $q \neq z_j^0$, $\rho \neq R_i$)

$$\varphi \to 0\,, \qquad \varphi_{,q} \to 0\,, \qquad \varphi_{,\rho} \to 0\,. \tag{28}$$

A generalization of a distribution function of a scalar field suggested in work [16] on the considered case of a multi-string system is

$$\varphi(q,\rho) = \ln\left(\frac{1}{\alpha(q)+\lambda(q)f(\rho)}\right)\,, \tag{29}$$

where

1. functions $\lambda(q)$ and $\alpha(q)$ are connected by the relation

$$\lambda(q) = (1-\alpha(q))/f_0\,, \qquad f_0 = \mathrm{const}\,, \tag{30}$$

2. functions $\alpha(q)$ and $f(\rho)$ are limited and for all $q \in (-\infty,+\infty)$ and $\rho \in [0,+\infty)$ take values in interval of

$$0 < \alpha(q) < 1\,, \qquad 0 < f(\rho) < f_0\,. \tag{31}$$

Moreover,

$$\alpha(q)|_{q\notin\left(z_j^0-\Delta q_j;z_j^0+\Delta q_j\right)} \to 1\,, \qquad \alpha(q)|_{q\to z_j^0} \to 0\,, \tag{32}$$

$$f(\rho)|_{\rho\notin(R_i-\Delta\rho_i;R_i+\Delta\rho_i)} \to f_0\,, \qquad f(\rho)|_{\rho\to R_i} \to 0\,, \tag{33}$$

where Δq_j and $\Delta\rho_i$ are small positive constants, defining a "thickness" of a ring ("smeared" null-string) numerated by indexes i and j.

What is more, in a limit of a constriction of a scalar field into a multi-string system, conditions must be implemented (at $\Delta q_j \to 0$, $\Delta\rho_i \to 0$):

$$\left|\frac{\alpha_{,q}}{\alpha(q)}\right|_{q\to z_j^0} \to \infty\,, \qquad \left.\frac{f_{,\rho}}{f(\rho)}\right|_{\rho\to R_i} \to 0\,, \tag{34}$$

$$\left.\frac{\alpha_{,q}}{\alpha(q)} \times \frac{f_{,\rho}}{f(\rho)}\right|_{q\to z_j^0,\rho\to R_i} \to 0\,. \tag{35}$$

One of possible examples of functions $\alpha(q)$ and $f(\rho)$, satisfying conditions (31)–(35), is

$$\alpha(q) = \exp\left(\sum_{j=1}^{m}\frac{-1}{\left(\xi_j\left(q-z_j^0+(\epsilon_q/\xi_j)\right)\right)^2}\right)\,, \tag{36}$$

$$f(\rho) = f_0\exp\left(-\gamma\left(1-\exp\left(\sum_{i=1}^{n}\frac{-1}{\left(\zeta_i(\rho-R_i+(\epsilon_\rho/\zeta_i))\right)^2}\right)\right)\right)\,. \tag{37}$$

Constants ξ_j and ζ_i define a size ("thickness") of a ring ("smeared" null-string) numerated by indexes i and j inside which a scalar field in variables q and ρ, respectively, is concentrated. Specifically, as it follows from (36) and (37), at $\Delta q_j \to 0$ $\Delta\rho_i \to 0$,

$$\xi_j \to \infty\,, \qquad \zeta_i \to \infty\,, \tag{38}$$

and positive constants ϵ_q, ϵ_ρ and γ provide a validation of conditions (32)–(35) at $\Delta\rho_i \to 0$, $\Delta q_j \to 0$, $\rho \to R_i$, $q \to z_j^0$. Specifically, at $\Delta q_j \ll 1$, $\Delta\rho_i \ll 1$

$$\epsilon_q \ll 1\,, \qquad \epsilon_\rho \ll 1\,, \qquad \gamma \gg 1\,, \tag{39}$$

and during a following constriction of a scalar field into a multi-string system, *i.e.* at $\Delta\rho_i \to 0$, $\Delta q_j \to 0$

$$\epsilon_q \to 0\,, \qquad \epsilon_\rho \to 0\,, \qquad \gamma \to \infty\,. \tag{40}$$

Indeed, for (36), (37), (38), (40)

$$\left|\frac{\alpha_{,q}}{\alpha(q)}\right|_{q\to z_j^0} = \frac{2\xi_j}{\epsilon_q^3} \to \infty\,, \tag{41}$$

$$\left.\frac{f_{,\rho}}{f(\rho)}\right|_{\rho\to R_i} = \gamma\exp\left(-\frac{1}{(\epsilon_\rho)^2}\right)\frac{2\zeta_i}{\epsilon_\rho^3} \to 0\,. \tag{42}$$

Then condition (35) in a limit of contraction of a scalar field into a multi-string system is fulfilled for all points of "smeared" null-string numerated by indexes i and j and located inside the region $\left|q - z_j^0\right| < \epsilon_q/\xi_j$, $|\rho - R_i| < \epsilon_\rho/\zeta_i$.

Using (30), (36), (37) for (29), we obtain an expression of one of possible distributions of a massless scalar field. During constriction, its energy-momentum tensor components asymptotically coincide with energy-momentum tensor components of the considered multi-string system. We note that the form of the distribution function (29), (30), (36), (37) is not general. This choice should be considered as one of the possible ways of "smearing" of null-strings forming the multi-string system. The form of the distribution function of the scalar field must influence gravitational properties of the string model in the form of a tube of a scalar field. However, since null-strings correspond to the case in which the scalar field contracts to a one-dimensional objects, the "smearing" method in the limit cases (38)–(40) cannot be significant.

3. Solution of the Einstein equations

Let us complete the system of Einstein equations (22)–(26) by the scalar field equation which for tensor (20) is

$$\left(g^{\alpha\beta}\varphi_{,\alpha}\right)_{;\beta}=0\,, \tag{43}$$

where the semicolon denotes the covariant derivative. For (10), (12), (14), (21), equation (43) takes the form of

$$\left(\varphi_{,\rho}A^{-1}\right)_{,\rho}+\left(\varphi_{,\rho}A^{-1}\right)\left(2\nu_{,\rho}+\frac{1}{2}\left(\frac{A_{,\rho}}{A}+\frac{B_{,\rho}}{B}\right)\right)=0\,. \tag{44}$$

Integrating equations (44), (25) and (24), one can find the connection between metric functions and the distribution function of the scalar field $\varphi(q,\rho)$, specifically

$$A(q,\rho) \;=\; \frac{\beta(q)}{(c_1)^2}\left(\varphi_{,\rho}\right)^2\exp\left(\varphi\left(\tilde{c}_3+4\tilde{c}_2\right)+4\nu_0(q)\right)\,, \tag{45}$$

$$\nu(q,\rho) \;=\; \tilde{c}_2\varphi+\nu_0(q)\,,\qquad B(q,\rho)=\beta(q)\exp\left(\tilde{c}_3\varphi\right)\,, \tag{46}$$

where $\tilde{c}_2=c_2/c_1$, $\tilde{c}_3=c_3/c_1$, $c_1=c_1(q)$, $c_2=c_2(q)$, $c_3=c_3(q)$, $\nu_0(q)$ and $\beta(q)$ are integration “constants”.

The remaining three equations of the system: (22), (23) and (26) considered for (45) and (46) define the conditions, connecting functions (integration “constants”): $c_1(q)$, $c_2(q)$, $c_3(q)$, $\nu_0(q)$ and $\beta(q)$, and its derivatives. Thus, equations (23) and (26) for functions (45), (46) are, respectively,

$$c_{3,q}+2c_{2,q}-2c_2(q)\frac{\beta_{,q}}{\beta(q)}-2\nu_{0,q}\left(c_3(q)+2c_2(q)\right)=0\,, \tag{47}$$

and

$$\left(\tilde{c}_2\right)^2+\left(\tilde{c}_2\right)\left(\tilde{c}_3\right)=\frac{\chi}{2}\,. \tag{48}$$

Wherein it is convenient to write equation (22) for functions (45), (46) in the form of

$$\Psi_0+\Psi_1\varphi+\Psi_2\varphi^2+\Psi_3\varphi\varphi_{,q}+\Psi_4\varphi_{,q}+\Psi_5\left(\varphi_{,q}\right)^2+\varphi_{,qq}=0\,, \tag{49}$$

where Ψ_i, $i=0,1,\ldots,5$ are functions containing the integration “constants”, their derivatives, and also functions determining the distribution of the scalar field, for example,

$$
\begin{aligned}
\Psi_0 &= \Psi^{-1}\left(2\Gamma_{,qq} - 4\nu_{0,q}\Gamma_{,q} + \frac{1}{2}\left(\left(\frac{\beta_{,q}}{\beta(q)}\right)^2 + \left(\tilde{\Gamma}_{,q}\right)^2\right)\right), \\
\Psi &= 2\left(\tilde{c}_3 + 2\tilde{c}_2 + 1\right), \\
\Gamma &= \ln\left\{\frac{\beta(q)\lambda(q)}{c_1(q)}e^{2\nu_0(q)}\right\}, \qquad \tilde{\Gamma} = \ln\left\{\frac{\beta(q)\left(\lambda(q)\right)^2}{\left(c_1(q)\right)^2}e^{4\nu_0(q)}\right\}.
\end{aligned}
$$

According to (29) and (30), the function $\varphi(q,\rho)$, determining the distribution of the scalar field, satisfies the equation

$$
\varphi_{,qq} - \frac{\lambda_{,qq}}{\lambda_{,q}}\varphi_{,q} - \left(\varphi_{,q}\right)^2 = 0\,. \tag{50}
$$

Equating the coefficients at the same powers of the function φ and its derivatives in equations (49) and (50), we find conditions on the functions Ψ_i, $i = 0, \ldots, 5$ of equation (49)

$$
\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0\,, \tag{51}
$$

$$
\Psi_4 = -\frac{\lambda_{,qq}}{\lambda_{,q}}, \qquad \Psi_5 = -1\,. \tag{52}
$$

An analysis of equations (47), (48), (51), (52) allows to determine the explicit form of the functions $c_i(q)$, $i = 1, 2, 3$, $\nu_0(q)$, and $\beta(q)$, specifically

$$
c_1(q) = c_0\alpha_0\left|\lambda_{,q}\right|\left(c_6\lambda(q) + c_7\right)^{(1+\tilde{c}_2)/c_4}, \tag{53}
$$

$$
\tilde{c}_3 = \frac{c_3(q)}{c_1(q)} = -\sqrt{4 - 2\chi}\,, \tag{54}
$$

$$
\tilde{c}_2 = \frac{c_2(q)}{c_1(q)} = -1 + \frac{1}{2}\sqrt{4 - 2\chi}\,, \tag{55}
$$

$$
\beta(q) = \left(c_6\lambda(q) + c_7\right)^{1/c_4}, \tag{56}
$$

$$
e^{2\nu_0(q)} = \alpha_0\left|\lambda_{,q}\right|\left(c_6\lambda(q) + c_7\right)^{(1+2\tilde{c}_2)/c_4}, \tag{57}
$$

where

$$
c_4 = -\frac{2(\tilde{c}_2)^2 + 4\tilde{c}_2 + 1}{2(1 + \tilde{c}_2)}, \qquad c_6 = -c_4 \cdot c_5\,, \tag{58}
$$

α_0, c_0 c_5 and c_7 are the integration constants, wherein $c_0 \neq 0$.

The found expressions for functions (53)–(57) fix the form of the sought metric functions (45), (46) up to the values of the integration constants (c_0, α_0, c_5 and c_7). To determine the values of these constants, it is convenient to study the expression for the scalar curvature $K = g^{\alpha\beta}g^{\mu\nu}R_{\alpha\mu\beta\nu}$, where $R_{\alpha\mu\beta\nu}$ is the Riemann–Christoffel tensor which for the found functions has the form of

$$K = -\chi(\varphi_{,\rho})^2 A^{-1}(q,\rho) = -\chi\frac{(c_0)^2\left(c_6\lambda(q)+c_7\right)^{-(1+2\tilde{c}_2)/c_4}}{(\alpha(q)+\lambda(q)f(\rho))^{-(\tilde{c}_3+4\tilde{c}_2)}}\,. \tag{59}$$

Outside the region where the scalar field is concentrated, according to (30) and (32), $\alpha(q)\to 1$, $\lambda(q)\to 0$. Then, equality (59) outside the region where the scalar field is concentrated takes the form of

$$K = -\chi(c_0)^2\left(c_7\right)^{-(1+2\tilde{c}_2)/c_4}\,. \tag{60}$$

We note that for the values of the constants $\tilde{c}_2$ and $\tilde{c}_3$, determined by (54) and (55), the constants $-(1+2\tilde{c}_2)/c_4$ and $-(\tilde{c}_3+4\tilde{c}_2)$ in equality (59) are positive

$$-\frac{(1+2\tilde{c}_2)}{c_4} = 2+\left(\frac{2-\sqrt{4-2\chi}}{1-\chi}\right) > 0\,, \tag{61}$$

$$-(\tilde{c}_3+4\tilde{c}_2) = 4-\sqrt{4-2\chi} > 0\,. \tag{62}$$

Then, in order to ensure that the scalar curvature K tends to zero outside the region where the scalar field is concentrated, it is necessary to require

$$c_7 = 0\,. \tag{63}$$

When the scalar field is contracted into a multi-string system, that is, at $\Delta q_j\to 0$, $\Delta\rho_i\to 0$, the following must be satisfied:

$$K|_{q\to z_j^0,\rho\to R_i} = -\chi(c_0)^2F\left(\frac{c_6}{f_0}\right)^{-(1+2\tilde{c}_2)/c_4}\to 0\,, \tag{64}$$

where, applying (30), (36), (37),

$$F = \frac{\left(1-e^{-1/(\epsilon_q)^2}\right)^{-(1+2\tilde{c}_2)/c_4}}{\left(e^{-1/(\epsilon_q)^2}+\left(1-e^{-1/(\epsilon_q)^2}\right)e^{-\gamma\left(1-e^{-1/(\epsilon_\rho)^2}\right)}\right)^{-(\tilde{c}_3+4\tilde{c}_2)}}\,. \tag{65}$$

We note that according to (40), (61), (62) at the contraction of the scalar field into a multi-string system, that is at $\Delta\rho_i\to 0$, $\Delta q_j\to 0$,

$$F|_{\epsilon_\rho\to 0;\epsilon_q\to 0;\gamma\to\infty}\to\infty\,. \tag{66}$$

Then fixing in (58), for example,

$$c_5 = -\frac{1}{c_4} F^{2c_4/(1+2\tilde{c}_2)}\,, \tag{67}$$

to which

$$c_6 = F^{2c_4/(1+2\tilde{c}_2)} \tag{68}$$

corresponds, and taking into account (66), for (64), we obtain (at $\Delta\rho_i \to 0$, $\Delta q_j \to 0$)

$$K|_{q\to z_j^0,\rho\to R_i} = -\chi\frac{(c_0)^2}{F}\,(f_0)^{(1+2\tilde{c}_2)/c_4} \to 0\,. \tag{69}$$

Applying (53)–(58), (63), (67), (68) for equalities (45) and (46), we find the required solution of the system of equations (22)–(26)

$$e^{2\nu(q,\rho)} = \acute{\alpha}\frac{|\lambda_{,q}|}{(\lambda(q))^2}\left(\frac{\alpha(q)+\lambda(q)f(\rho)}{(\lambda(q))^{1/(1-\chi)}}\right)^{2-\sqrt{4-2\chi}}\,, \tag{70}$$

$$B(q,\rho) = \acute{\beta}\left(\frac{\alpha(q)+\lambda(q)f(\rho)}{(\lambda(q))^{1/(1-\chi)}}\right)^{\sqrt{4-2\chi}}\,, \tag{71}$$

$$A(q,\rho) = \acute{\gamma}(f_{,\rho})^2\left(\frac{\alpha(q)+\lambda(q)f(\rho)}{(\lambda(q))^{1/(1-\chi)}}\right)^{2-\sqrt{4-2\chi}}\,, \tag{72}$$

where the functions $\alpha(q)$ and $f(\rho)$ satisfy conditions (31)–(35) and can be represented in the form of (36), (37), with the constants $\acute{\alpha} = \alpha_0 F^2$, $\acute{\beta} = F^{2/(\sqrt{4-2\chi}-1)}$, $\acute{\gamma} = (c_0)^{-2}F^2$. The constants α_0 and c_0 in solution (70)–(72) can be considered as scale factors and equal to 1.

4. Motion characteristics of a test null-string in a gravitational field of a multi-string system

According to (69) and (65), the curvature of the space in which the multi-string system is located depends on the value of the constants determining the "thickness" of the null-strings forming the multi-string system: ϵ_q, ϵ_ρ, ξ_j, γ, ζ_i, f_0. Moreover, in the limit of contraction of a scalar field into a multi-string system, the curvature of the space tends to zero. Then, according to (38)–(40), the conditions under which the influence of the gravitational field of the test null-strings can be neglected (the condition of applicability of the concept of a test null-string) are

$$(\tilde{\epsilon_q},\tilde{\epsilon_\rho}) < (\epsilon_q,\epsilon_\rho)\,, \qquad \left(\tilde{\xi},\tilde{\gamma},\tilde{\zeta},\tilde{f_0}\right) > (\xi_j,\gamma,\zeta_i,f_0)\,, \tag{73}$$

where $\tilde{\epsilon_q}$, $\tilde{\epsilon_\rho}$, $\tilde{\xi}$, $\tilde{\gamma}$, $\tilde{\zeta}$, $\tilde{f_0}$ are the constants that determine the "thickness" of the test null-string.

For a quadratic form (6), considering (10), (12), (14), equations of a test null-string motion (7) and (8) can be represented in the form of

$$q_{,\tau\tau} + 2\nu_{,\tau} q_{,\tau} = 0\,, \tag{74}$$

$$\eta_{,\tau\tau} + 2\nu_{,\rho}\eta_{,\tau}\rho_{,\tau} + e^{-2\nu}\left(A_{,q}(\rho_{,\tau})^2 + B_{,q}(\theta_{,\tau})^2\right) = 0\,, \tag{75}$$

$$\rho_{,\tau\tau} + \frac{1}{A}\left\{e^{2(\nu)}\nu_{,\rho}q_{,\tau}\eta_{,\tau} + A_{,q}q_{,\tau}\rho_{,\tau} + \frac{1}{2}\left(A_{,\rho}(\rho_{,\tau})^2 - B_{,\rho}(\theta_{,\tau})^2\right)\right\} = 0\,, \tag{76}$$

$$\theta_{,\tau\tau} + \frac{B_{,\tau}}{B}\theta_{,\tau} = 0\,, \tag{77}$$

$$e^{2\nu}\eta_{,\tau}q_{,\tau} - A(\rho_{,\tau})^2 - B(\theta_{,\tau})^2 = 0\,, \tag{78}$$

$$\frac{1}{2}e^{2\nu}\left(\eta_{,\tau}q_{,\sigma} + \eta_{,\sigma}q_{,\tau}\right) - A\rho_{,\tau}\rho_{,\sigma} - B\theta_{,\tau}\theta_{,\sigma} = 0\,, \tag{79}$$

where

$$\eta = t - z\,. \tag{80}$$

In this work, an analysis of possible trajectories of a test null-string motion will be provided for a particular case

$$q = q(\tau)\,, \quad \eta = \eta(\tau)\,, \quad \rho = \rho(\tau)\,, \quad q_{,\tau} > 0\,, \quad \eta_{,\tau} > 0\,, \quad \theta = \theta(\sigma)\,. \tag{81}$$

For (81), a test null-string at each instant of its "counter" motion with respect to the multi-string system has a shape of a circle. Its radius is time-dependent and it is always located in a plane parallel to a plane of the multi-string system.

The considered case of movement of a test null-string "towards" a multi-string system ($q_{,\tau} > 0$) includes situations when:

- The test null-string moves in the positive direction of the z axis.
- The test null-string moves in a plane perpendicular to the z axis (for example, the test null-string expands radially or collapses radially).
- The test null-string moves in the same direction as the multi-string system, but the projection of the velocity of the test null-string points on the z axis is less than the speed of light (for example, it moves in the negative direction of the z axis while changing its size (radius)).

For the case of (81), equations (77) and (79) hold identically and first integrals of equations (74)–(76), considering (70)–(72) and (78), respectively take the form of

$$q_{,\tau} \;=\; P_1 e^{-2\nu}\,, \tag{82}$$

$$\eta_{,\tau} = \left(\frac{P_2}{P_1}\right)^2 \cdot \frac{|\lambda_{,q}|}{(\lambda(q))^2} q_{,\tau} \,, \tag{83}$$

$$|f_{,\rho}\rho_{,\tau}| = \frac{P_2}{P_1} \cdot \frac{|\lambda_{,q}|}{(\lambda(q))^2} q_{,\tau} \,, \tag{84}$$

where P_1 and P_2 are integration constants, moreover,

$$P_1 > 0\,, \qquad P_2 > 0\,. \tag{85}$$

Constant P_2 defines velocities of test null-string points in variable ρ in the initial moment of time.

From equations (30), (32), (33), it follows that for equations (83) and (84), the whole range of the variables q and ρ ($q \in (-\infty, +\infty)$, $\rho \in [0, +\infty)$) splits into regions, depending on the signs of the derivatives of $\lambda(q)$ and $f(\rho)$. A quantity of regions is defined by the amount of null-strings in the multi-string system.

Consider the simplest example of a multi-string system in which two closed null-strings of the radii R_1 and R_2 ($R_1 < R_2$) are located in the $q = 0$ plane at each moment of time and move in the negative direction of the z axis. For this configuration, the range of the variables q and ρ, depending on signs of the derivatives of the functions $\lambda(q)$ and $f(\rho)$, splits into the following eight regions:

Regions I and V:

$$\rho \in (R_2, +\infty) \quad \text{for I} \quad \text{and} \quad \rho \in (R_1, R) \quad \text{for V}\,, \qquad q \in (-\infty, 0)\,, \tag{86}$$

in which

$$f_{,\rho} > 0\,, \qquad \lambda_{,q} > 0\,. \tag{87}$$

Regions II and VI:

$$\rho \in (R, R_2) \quad \text{for II} \quad \text{and} \quad \rho \in (0, R_1) \quad \text{for VI}\,, \qquad q \in (-\infty, 0)\,, \tag{88}$$

in which

$$f_{,\rho} < 0\,, \qquad \lambda_{,q} > 0\,. \tag{89}$$

Regions III and VII:

$$\rho \in (R_2, +\infty) \quad \text{for III} \quad \text{and} \quad \rho \in (R_1, R) \quad \text{for VII}\,, \qquad q \in (0, +\infty)\,, \tag{90}$$

in which

$$f_{,\rho} > 0\,, \qquad \lambda_{,q} < 0\,. \tag{91}$$

Regions IV and VIII:

$$\rho \in (R, R_2) \quad \text{for IV} \quad \text{and} \quad \rho \in (0, R_1) \quad \text{for VIII}\,, \qquad q \in (0, +\infty)\,, \quad (92)$$

in which

$$f_{,\rho} < 0\,, \qquad \lambda_{,q} < 0\,, \quad (93)$$

where $R = (R_1 + R_2)/2$.

Figure 1 shows an arrangement of regions I–VIII in respect to the surfaces $\rho = R_1$, $\rho = R$, $\rho = R_2$, $q = 0$.

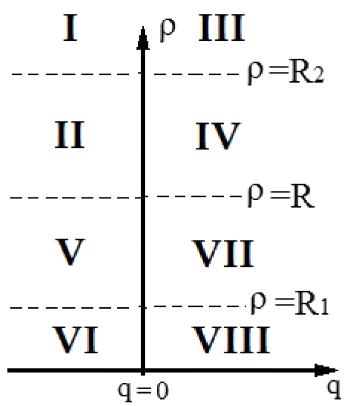

Fig. 1. An illustration of the range of the variables ρ and q.

According to (30), (32), (33), the functions $f(\rho)$ and $\lambda(q)$ at boundaries between regions I–VIII (*i.e.*, at $q = 0$, $\rho = R_1$, $\rho = R_2$, $\rho = R$) have extrema

$$f_{,\rho}|_{\rho=R_1;R_2;R} = 0\,, \qquad \lambda_{,q}|_{q=0} = 0\,, \quad (94)$$

moreover, there is no null-string at the boundary $\rho = R$, unlike $\rho = R_1$, and $\rho = R_2$.

In work [17], it has been shown that for a test null-string, there is always only a "narrow" region ("interaction zone"), getting in which a test null-string can interact with a source null-string. Width of this "zone" is defined by values of constants f_0 and P_2, having the same physical interpretation. Moreover, the bigger the value of the constant f_0 and the less the value of the constant P_2, the wider is the "interaction zone". It is obvious that if distances between null-strings in the multi-string system are bigger than sizes of corresponding "interaction zones" ("interaction zones" do not overlap), then a motion of a test null-string in a gravitational field of such a source is defined only by its interaction with a specific null-string. It is not dependent on an influence of surroundings. In this case, possible trajectories of a test null-string motion do not differ from the trajectories provided in work [17]. In the proposed paper, we will be interested in the case in which the "interaction zones" for null-strings forming the multi-string system overlap.

Integrating equation (84) in regions I–VIII, considering that in each of them two cases $\rho_{,\tau} > 0$ and $\rho_{,\tau} < 0$ can be realized, we find

$$f_L^k = U_L^k + \varGamma_L^k \frac{P_2}{P_1} (\lambda(q))^{-1} , \tag{95}$$

where U_L^k are the integration constants. The index L takes values I–VIII and denotes the number of a region in which the solution found is realized. The index k takes values $0, 1$ and denotes at $k = 0$ the case $\rho_{,\tau} > 0$ and at $k = 1$ the case $\rho_{,\tau} < 0$. The constants $\varGamma_L^k$ take values

$$\begin{aligned} \varGamma_{\mathrm{I}}^0 &= \varGamma_{\mathrm{V}}^0 = \varGamma_{\mathrm{IV}}^0 = \varGamma_{\mathrm{VIII}}^0 = \varGamma_{\mathrm{II}}^1 = \varGamma_{\mathrm{VI}}^1 = \varGamma_{\mathrm{III}}^1 = \varGamma_{\mathrm{VII}}^1 = -1 , \\ \varGamma_{\mathrm{II}}^0 &= \varGamma_{\mathrm{VI}}^0 = \varGamma_{\mathrm{III}}^0 = \varGamma_{\mathrm{VII}}^0 = \varGamma_{\mathrm{I}}^1 = \varGamma_{\mathrm{V}}^1 = \varGamma_{\mathrm{IV}}^1 = \varGamma_{\mathrm{VIII}}^1 = 1 . \end{aligned} \tag{96}$$

The integration constants U_L^k are fixed by the value of the functions $f(\rho)$ and $\lambda(q)$ at the boundaries of the regions. From equalities (84) and (94), it follows [17] that during approach of the test null-string to the boundaries $q = 0$, depending on initial conditions, the following situations can be realized:

— A test null-string on the boundaries $\rho = R_1$ and $\rho = R_2$ collides with a null-string forming a multi-string system.

— At $\rho = R$, the test null-string crosses the boundary $q = 0$.

— In the regions I, III, near the boundary $q = 0$, the test null-string has infinitely large radius.

— In regions VI, VIII, near the boundary $q = 0$, the test null-string has an infinitesimal radius.

For these conditions, the constants U_L^k have the form of

$$\begin{aligned} U_{\mathrm{I}}^0 &= U_{\mathrm{VI}}^1 = U_{\mathrm{III}}^1 = U_{\mathrm{VIII}}^0 = f_0 \left(1 + \frac{P_2}{P_1} \right) , \\ U_{\mathrm{I}}^1 &= U_{\mathrm{IV}}^1 = U_{\mathrm{V}}^1 = U_{\mathrm{VIII}}^1 = U_{\mathrm{II}}^0 = U_{\mathrm{III}}^0 = U_{\mathrm{VI}}^0 = U_{\mathrm{VII}}^0 = -f_0 \frac{P_2}{P_1} , \\ U_{\mathrm{V}}^0 &= U_{\mathrm{VII}}^1 = U_{\mathrm{II}}^1 = U_{\mathrm{IV}}^0 = f(R) + f_0 \frac{P_2}{P_1} . \end{aligned} \tag{97}$$

It follows from solution (95)–(97) that after crossing the boundary $q = 0$, at $\rho = R$, both the case $\rho_{,\tau} < 0$ and the case $\rho_{,\tau} > 0$ can be realized. For example, using explicit form of constants (96) and (97), it can be seen that equality (95) realized in region II in the case $\rho_{,\tau} < 0$, completely coincides with the solution in region IV for $\rho_{,\tau} > 0$, and also with the solution $\rho_{,\tau} < 0$ which is realized in region VII.

It can be shown that the solution of equations (82), (83) (that is, the dependence variables t and z on the time-like parameter τ) in the I–VIII regions is:

— in regions I, II, V, VI ($q < 0$)

$$t = \frac{1}{2}\left(\frac{P_2^2}{P_1}\tau - \frac{\epsilon_q}{\xi} - \frac{1}{\xi}\sqrt{\ln^{-1}\left(F_-(\tau)\right)^{-1}}\right), \tag{98}$$

$$z = -\frac{1}{2}\left(\frac{P_2^2}{P_1}\tau + \frac{\epsilon_q}{\xi} + \frac{1}{\xi}\sqrt{\ln^{-1}\left(F_-(\tau)\right)^{-1}}\right). \tag{99}$$

— in regions III, IV, VII, VIII ($q > 0$)

$$t = \frac{1}{2}\left(\frac{P_2^2}{P_1}\tau - \frac{\epsilon_q}{\xi} + \frac{1}{\xi}\sqrt{\ln^{-1}\left(F_+(\tau)\right)^{-1}}\right), \tag{100}$$

$$z = \frac{1}{2}\left(-\frac{P_2^2}{P_1}\tau - \frac{\epsilon_q}{\xi} + \frac{1}{\xi}\sqrt{\ln^{-1}\left(F_+(\tau)\right)^{-1}}\right), \tag{101}$$

where

$$F_{\mp}(\tau) = 1 - \frac{f_0}{f_0\left(1 - e^{-1/(\epsilon_q)^2}\right)^{-1} \mp P_1\tau}. \tag{102}$$

Let us note that integration constants in the solution of equations (82) and (83) were fixed by the following conditions:

— at the moment of time $t = 0$, the test null-string and the considered multi-string system are located in the same plane $z = 0$ ($q = 0$, $\eta = 0$),

— at the boundary $q = 0$, the value of the parameter $\tau = 0$ (*i.e.* for the value $\tau = 0$ during the "counter" motion the test null-string and the multi-string system appear in one plane $z = 0$), moreover,

in regions I, II, V, VI ($q < 0$)

$$\text{at} \quad q \in (-\infty, 0), \quad \text{the value of the parameter} \quad \tau \in (-\infty, 0), \tag{103}$$

in regions III, IV, VII, VIII ($q > 0$)

$$\text{at} \quad q \in (0, +\infty), \quad \text{the value of the parameter} \quad \tau \in (0, +\infty). \tag{104}$$

Consider an influence of an initial momentum of test null-string points in the variable ρ (the constant P_2) on a motion of the test null-string. For this purpose, it is convenient to consider the value of constant P_2 relatively to P_1, *i.e.* consider that

$$P_2 = \alpha P_1 \,, \tag{105}$$

where the proportionality coefficient $\alpha > 0$.

Let us find the value of the variables t and z at boundaries of an interaction zone which, according to [21], are reached at

$$|\tau| = \frac{f_0}{P_2} \,. \tag{106}$$

Applying (105) and (106) for equalities (98)–(102), neglecting small summand at $\epsilon_q \ll 1$ respectively, we find:

— in the region $q < 0$ (the left boundary of the "interaction zone": $\tau = -f_0/P_2$),

$$t = -\frac{1}{2}\left(\alpha f_0 + \frac{1}{\xi}\sqrt{\ln^{-1}(1+\alpha)}\right) , \tag{107}$$

$$z = \frac{1}{2}\left(\alpha f_0 - \frac{1}{\xi}\sqrt{\ln^{-1}(1+\alpha)}\right) , \tag{108}$$

— in the region $q > 0$ (the right boundary of the "interaction zone": $\tau = f_0/P_2$),

$$t = \frac{1}{2}\left(\alpha f_0 + \frac{1}{\xi}\sqrt{\ln^{-1}(1+\alpha)}\right) , \tag{109}$$

$$z = \frac{1}{2}\left(-\alpha f_0 + \frac{1}{\xi}\sqrt{\ln^{-1}(1+\alpha)}\right) . \tag{110}$$

Notice that at $\alpha \to 0$ (the initial momenta in variable ρ are small), for the left boundary of the interaction zone (equalities (107), (108)), we have $t < 0$, $z < 0$ and for the right boundary of the interaction zone (equalities (109), (110)), the values $t > 0$, $z > 0$. Thus, a motion of a test null-string counter (in positive direction of the z axis) to the multi-string system corresponds to the case $\alpha \to 0$.

At $\alpha \gg 1$ (the initial momenta in variable ρ are very big), for the left boundary of the interaction zone, we have $t < 0$, $z > 0$ and for the right boundary of the interaction zone, the values are $t > 0$, $z < 0$ (*i.e.*, on the z axis, the left boundary is located "more to the right" and the right one is located "more to the left"). From which it follows that in the case $\alpha \gg 1$, a test null-string is moving in the same direction as the multi-string system (in negative direction of the z axis), but the projection of a velocity of test null-string points on the z axis is less than the speed of light, because of a fast change of the radius.

It can be noticed that for equalities (107)–(110)), one more very interesting case is possible. Specifically, when the value of the constant α is the solution of equation

$$\alpha f_0 - \frac{1}{\xi}\sqrt{\ln^{-1}(1+\alpha)} = 0\,,$$

or which is the same

$$(\alpha+1)^{\alpha^2} = e^{(f_0\xi)^{-2}}\,. \tag{111}$$

It is easy to see that in this case, for the left boundary of the interaction zone (*i.e.* at $\tau = -f_0/P_2$), we have $t < 0$, $z = 0$ for the right boundary of the interaction zone (*i.e.* at $\tau = f_0/P_2$), the values $t > 0$, $z = 0$. Furthermore, as was spoken before, at $\tau = 0$ which the value of the variable $t = 0$ corresponds to, a test null-string and the multi-string system are also located in the plane $z = 0$. From the spoken, it follows that for the value of the constant α which is the root of equation (111), an influence of a gravitational field of the multi-string system leads to an appearance of test null-string oscillations in the variable z in a vicinity of the surface $z = 0$.

Note that since the test null-string can cross the boundary $q = 0$ only at $\rho = R = (R_1 + R_2)/2$ for the considered multi-string system, then the oscillation regime of the test null-string is possible only in regions II, IV, V, VII.

Figures 2 and 3 provide the graphs of the functions $z(\tau)$ and $t(\tau)$ which corresponds to equalities (98)–(101) in the case of an oscillatory motion of a test null-string for the values of the constants $f_0 = P_1 = 50$, $P_2 = 3$, $\xi = 1.4$, $\epsilon_q = 10^{-3}$, $\tau \in [-16; 16]$. From Fig. 2, it is seen that at $\tau = -16$ ($\tau = -f_0/P_2 = 50/3 \approx 16$), the test null-string gets at the left boundary of the "interaction zone" (point **A** on the graph of the function $z(\tau)$) and in this moment of time is located in plane $z = 0$. Then it moves in the negative direction of the z axis, *i.e.* moves in the same direction as the multi-string system (region **A**–**B**). At point **B**, it changes a direction of its motion and in regions **B**–**C**–**D** moves towards the multi-string system, wherein, in point **C**, the test null-string and the multi-string system are located in one plane (plane $z = 0$). At point **D**, it changes its motion direction again and in region **D**–**F** approaches the plane $z = 0$. At point **F** ($\tau = 16$), the test null-string leaves the "interaction zone", being in the plane $z = 0$. Wherein the time function over the entire interval $\tau \in [-16; 16]$ is increasing (Fig. 3).

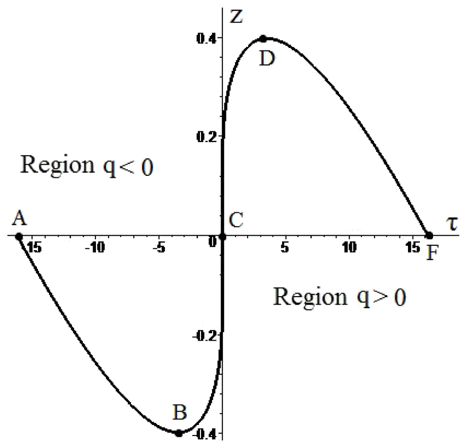

Fig. 2. The figure shows the graph of the functions $z(\tau)$, in the case of oscillatory motion of the test null-string.

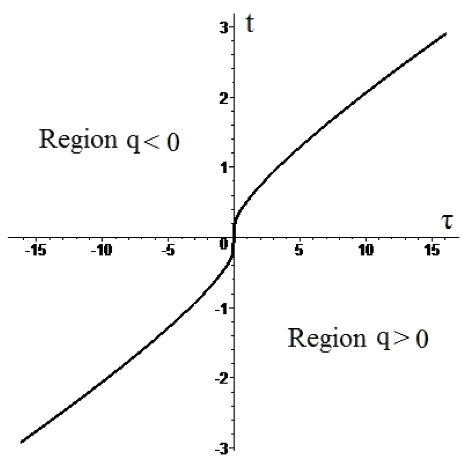

Fig. 3. The figure shows the graph of the functions $t(\tau)$, in the case of oscillatory motion of the test null-string.

Figures 4 and 5 show schematically a change of a test null-string radius in the variables z, ρ in the case of an oscillation regime which is realized during its motion respectively in regions II, IV, and II, VII. On the provided figures, points **A**–**F** correspond to the points of graph of the function $z(\tau)$ (Fig. 2) and stand for the moments of the change of a test null-string motion direction.

For the case of motion shown in Fig. 4, the test null-string enters the gravitational field of a single-layer multi-string system in region II (point **A** on the graph) at $\rho = R_2$, $\rho_{,\tau} < 0$. Then, it intersects the boundary $z = 0$

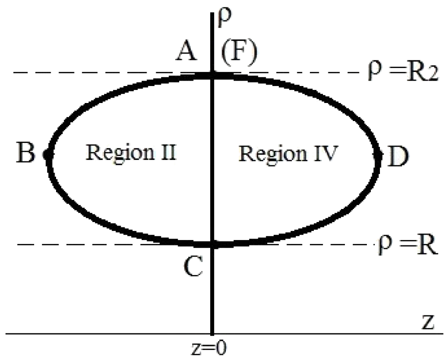

Fig. 4. The figure shows schematically a change of a radius of the test null-string depending on a position on the z axis, in the case of an oscillatory motion which is realized during its motion in regions II and IV.

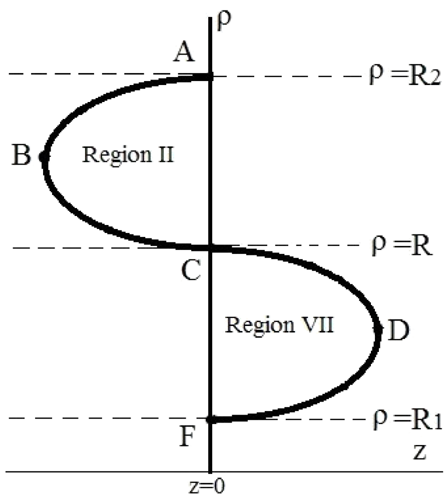

Fig. 5. The figure shows schematically a change of a radius of the test null-string depending on a position on the z axis, axis in the case of an oscillatory motion which is realized during its motion in regions II and VII.

at $\rho = R$ (point **C** on the graph). Then, moving in region IV, it increases its size ($\rho_{,\tau} > 0$) and at $\rho = R_2$, it appears on the right border of region IV (point **F** on the graph).

For the case of motion shown in Fig. 5, the test null-string enters the gravitational field of single-layer multi-string system in region II (point **A** on the graph) at $\rho = R_2$, $\rho_{,\tau} < 0$. Then, it intersects the boundary $z = 0$ at $\rho = R$ (point **C** on the graph). Then, moving in region IV, it decreases its size ($\rho_{,\tau} < 0$) and at $\rho = R_2$, it appears on the right border of region IV (point **F** on the graph).

The trajectories of the test null-string motion in the field of the multi-string system having a layered structure (*i.e.*, according to (1), there are m layers and on each layer there are n closed coaxial null-strings) can be obtained by combining possible trajectories of the test null-string motion in the field of a single-layer multi-string system.

Thus, for example, Fig. 6 schematically provides one of possible motion trajectories of the test null-string for a multi-string system consisted of two layers in the variables q and ρ. The first layer of the multi-sting system is located at $q = 0$ and the second one at $q = 2z_0$. There are two null-strings of radii R_1 and R_2, $R = (R_1 + R_2)/2$ on each layer. It can be noted that provided motion trajectory of the test null-string (point **A** is an initial point of the trajectory, point **F'** is an end point of the trajectory) consists of two similar regions. The region of trajectory **A** $\rightarrow$ **C** $\rightarrow$ **F** describes motion of the test null-string in the gravitational field of the first layer of the multi-string system ($q = 0$). The region of trajectory $\mathbf{A}' \rightarrow \mathbf{C}' \rightarrow \mathbf{F}'$ describes motion of the test null-string in the gravitational field of the second layer of the multi-string system ($q = 2z_0$). It is seen that both provided regions are completely identical. Let us note that the motion trajectory **A** $\rightarrow$ **C** $\rightarrow$ **F** provided in Fig. 6 in the variables q and ρ coincides with the analogical points for the trajectory in the variables z and ρ provided in Fig. 4. Thus, for the provided example, the test null-string undergoes two complete oscillations in the region limited both in variable z and variable ρ. It can be noted that the motion trajectory of the test null-string provided in Fig. 6 is not the only one possible.

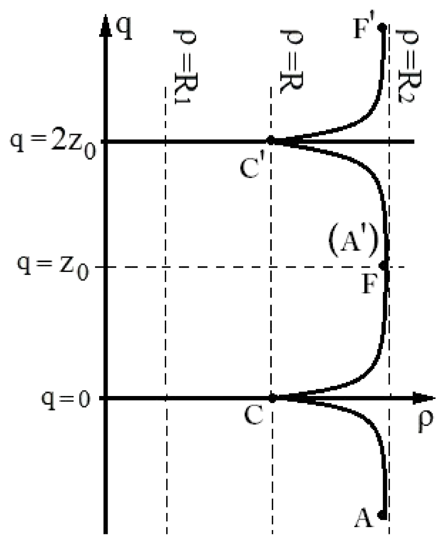

Fig. 6. Schematic illustration of one of possible motion trajectories of the test null-string.

Thus, Fig. 7 provides schematically another possible motion trajectory of the test null-string for the multi-string system considered above and consisted of two layers (point **A** is an initial point of the trajectory, point **F'** is an end point of the trajectory). This trajectory also consists of two regions. The region of trajectory $\mathbf{A} \to \mathbf{C} \to \mathbf{F}$ describes motion of the test null-string in the gravitational field of the first layer of the multi-string system ($q = 0$) and for this region, $\rho_{,\tau} < 0$. The region of trajectory $\mathbf{A}' \to \mathbf{C}' \to \mathbf{F}'$ describes motion of the test null-string in the gravitational field of the first layer of the multi-string system ($q = 2z_0$) and for this region, $\rho_{,\tau} > 0$. The motion trajectory $\mathbf{A} \to \mathbf{C} \to \mathbf{F}$ provided in Fig. 7 in the variables q and ρ coincides with the analogical points for the trajectory in the variables z and ρ provided in Fig. 5. The motion trajectory $\mathbf{A}' \to \mathbf{C}' \to \mathbf{F}'$ is qualitatively analogical to trajectory $\mathbf{A} \to \mathbf{C} \to \mathbf{F}$, but it describes the motion of the test null-string with an increasing radius. For the provided example the test null-string also undergoes oscillations in the region limited both in variable z and variable ρ. However, both sizes and motion trajectory of the test null-string in this region are considerably different.

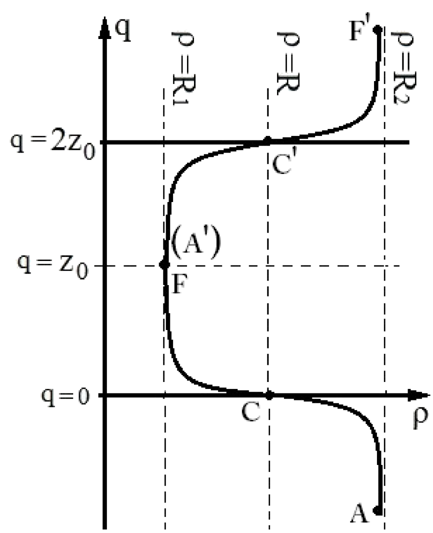

Fig. 7. Schematic illustration of one of possible motion trajectories of the test null-string.

Thus, it can be said that, depending on the value of the initial momenta of the test null-string points with respect to the variable ρ (the constant P_2), the influence of the gravitational field of a layered multi-string system can lead to time-stable oscillations of a test null-string in a vicinity of a fixed point of space (the repeating trajectories are limited both in the variable z and in the variable ρ).

Figures 8 and 9 provide schematically two-dimensional world surfaces of a test null-string which are realized during its oscillating motion in a gravitational field of the multi-string system in the variables t, z, ρ, respectively, for the trajectories for the cases illustrated in Figs. 6 and 7.

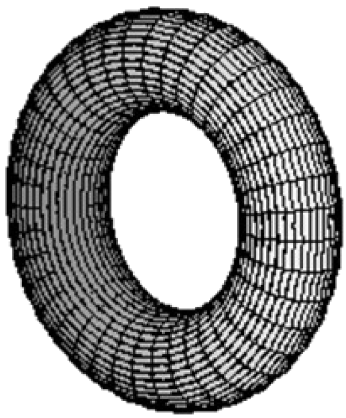

Fig. 8. The figure provides a two-dimensional world surface of a test null-string in a case of an oscillating motion which corresponds to Fig. 4.

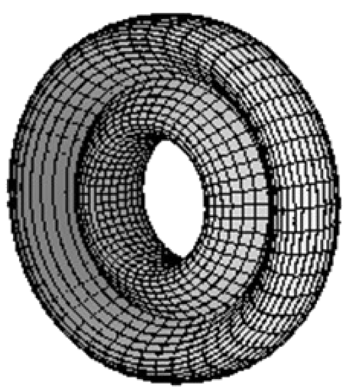

Fig. 9. The figure provides a two-dimensional world surface of a test null-string in a case of an oscillating motion which corresponds to Fig. 5.

It can be considered especially interesting that a stable in time and limited in space regions inside which a null-string oscillations occur (in a vicinity of a fixed point) can be considered as a localized in space particles with effective nonzero rest mass. For such particles, a concept of "lifetime" can be introduced. It, obviously, will depend on the amount of layers in the multi-string system. A concept "mass" (energy) of the particle can be introduced. It should be connected with the size of the region in which oscillations of a null-string occur and with a motion trajectory of a null-string in this region.

5. Conclusions

1. A solution of the Einstein equations for a multi-string system moving along the z axis and having a layered structure is found. Specifically, there are m layers in the system, and there are n closed co-axial null-strings of different constant in time radii on each layer. The arrangement of closed null-strings on each such a layer is the same and orthogonal to the direction of motion.

2. A motion of a test null-string in the gravitational field of the investigated multi-string system is considered for the case in which the "interaction zones" of null-strings forming a multi-string system overlap. An analysis of possible trajectories of the test null-string was carried out under the condition that, when moving towards the multi-string system, the test null-string has always the form of a circle. Its radius changes with time and is completely located in the plane parallel to the plane of the null-strings forming the multi-string system

3. It was shown that depending on the value of the initial momenta of the test null-string points with respect to the variable ρ (constant P_2), the influence of the gravitational field of a layered multi-string system can lead either to pulsating (limited in the variable ρ and not limited in the variable z) motion of the test null-string, or to time-stable oscillations of a test null-string in a vicinity of a fixed point of space (the repeated trajectories are limited both in the variable z and in the variable ρ).

4. Aa an interesting result of the proposed work can be considered the fact that time-stable and space-limited regions within which null-string oscillations occur can be considered as particles localized in space with an effective nonzero rest mass.

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