# $N$-SOLITON SOLUTIONS FOR A NONLINEAR WAVE EQUATION VIA RIEMANN–HILBERT APPROACH*

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A nonlinear wave equation is investigated via the Riemann–Hilbert approach. Based on the spectral analysis for the Lax pair, a Riemann–Hilbert problem of the nonlinear wave equation is established. For the reflectionless cases, we obtain $N$-soliton solutions of the nonlinear wave equation and discuss the dynamic behavior of its soliton solutions.

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## 1. Introduction

The study of explicit solutions for various soliton equations has been very important in modern mathematics with ramifications to several areas of mathematics, physics and other sciences. Several systematic methods have been developed to solve soliton equations, for example, the inverse scattering transformation [1, 2], the Hirota bilinear method [3], the Bäcklund transformation [4], the Darboux transformation [5, 6], the algebra-geometric method [7], the Painlevé analysis and so on [8–20]. Among these methods, the Riemann–Hilbert (RH) approach is proved to be a powerful tool for solving nonlinear evolution equations [21]. This approach cannot only solve the initial value problem of soliton equation, but also study the initial-boundary value problem of soliton equation [22–30].

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The main aim of this paper is to study the following nonlinear wave equation [31]:

$$\begin{aligned} u_{xt} + 2uw - i(uw)_x &= 0\,, \\ w_x + \left(|u|^2\right)_t &= 0 \end{aligned} \tag{1}$$

via the Riemann–Hilbert approach, from which we obtain $N$-soliton solutions of the nonlinear wave equation and discuss the dynamic behavior of its soliton solutions.

The present paper is organized as follows. In Section 2, we introduce a Lax pair of the nonlinear wave equation (1) and study the spectral analysis for the Jost solutions. In Section 3, the Riemann–Hilbert problem of the nonlinear wave equation (1) is constructed by using the inverse scattering transformation. In Section 4, we study the symmetric relations of the potential matrix and the scattering data. In Section 5, $N$-soliton solutions of the nonlinear wave equation are obtained on the basis of solving the Riemann–Hilbert problem for the reflectionless cases. Moreover, the dynamic behaviors of soliton solutions are discussed.

## 2. Lax pairs

In this section, we shall investigate the direct scattering transformation by formulating a Riemann–Hilbert problem for the nonlinear wave equation (1). We first consider the Lax pair of equation (1)

$$Y_x = UY\,, \qquad Y_t = VY \tag{2}$$

with

$$\begin{aligned} U &= \begin{pmatrix} -\frac{ik^2}{1+k^2} & \frac{1}{1+ik}u \\ \frac{1}{1-ik}\bar{u} & \frac{ik^2}{1+k^2} \end{pmatrix}, \\ V &= \frac{1}{2}\begin{pmatrix} -\frac{i}{k^2}w & -\frac{1-ik}{k^2}(uw + iu_t) \\ -\frac{1+ik}{k^2}(\bar{u}w - i\bar{u}_t) & \frac{i}{k^2}w \end{pmatrix}, \end{aligned}$$

where $Y = Y(x,t;k)$ is a matrix function and $k$ is a complex spectral parameter with $k \neq 0$, $k \neq \pm i$, $w$ is a real-valued function, $u$ is a complex valued function and $\bar{u}$ is the complex conjugate quantity of $u$.

Throughout this work, we assume that $u \to 0$, $w \to 1$ as $x \to \pm\infty$. When $u = 0$, $w = 1$, Eq. (2) turns into

$$E_x = -\frac{ik^2}{1+k^2}\sigma_3 E\,, \qquad E_t = -\frac{i}{2k^2}\sigma_3 E\,, \tag{3}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Equation (2) has a special solution

$$E = e^{-\frac{ik^2}{1+k^2}\sigma_3 x - \frac{i}{2k^2}\sigma_3 t}.$$

For the sake of convenience, we define a new matrix spectral function $J = J(x,t;k)$ by

$$Y = JE\,. \tag{4}$$

Under transformation (4), the Lax pair (2) can be rewritten as

$$\begin{aligned} J_x &= -\frac{ik^2}{1+k^2}[\sigma_3, J] + \widetilde{U}J\,, \\ J_t &= -\frac{i}{2k^2}[\sigma_3, J] + \widetilde{V}J\,, \end{aligned} \tag{5}$$

where $[\sigma_3, J] = \sigma_3 J - J\sigma_3$ is the commutator and

$$\begin{aligned} \widetilde{U} &= \begin{pmatrix} 0 & \frac{1}{1+ik}u \\ \frac{1}{1-ik}\bar{u} & 0 \end{pmatrix}, \\ \widetilde{V} &= \frac{1}{2}\begin{pmatrix} -\frac{i}{k^2}(w-1) & -\frac{1-ik}{k^2}(uw+iu_t) \\ -\frac{1+ik}{k^2}(\bar{u}w - i\bar{u}_t) & \frac{i}{k^2}(w-1) \end{pmatrix}. \end{aligned}$$

In the following, we concentrate on studying the direct scattering by using the space part of the Lax pair (5), where the time variable $t$ is viewed as a dummy variable and is hided. Now, we introduce matrix Jost solutions $J_\pm = J_\pm(x,k)$ for the space part of the Lax pair (5)

$$J_- = ([J_-]_1, [J_-]_2)\,, \qquad J_+ = ([J_+]_1, [J_+]_2)\,, \tag{6}$$

with the asymptotic conditions

$$J_\pm \to I\,, \qquad x \to \pm\infty\,, \tag{7}$$

where each $[J_\pm]_l$ $(l = 1, 2)$ denotes the $l^{\text{th}}$ column of $J_\pm$, respectively, and the symbol $I$ is the $2\times 2$ identity matrix. Using the large-$x$ asymptotic condition (7), we can turn the $x$-part of (5) into the Volterra integral equations

$$J_\pm(x,k) = I + \int_{\pm\infty}^{x} e^{\frac{ik^2}{1+k^2}\hat{\sigma}_3(y-x)}\widetilde{U}(y)J_\pm(y,k)\mathrm{d}y\,, \tag{8}$$

where $\hat{\sigma}_3$ acts on a $2\times 2$ matrix $X$ by $\hat{\sigma}_3 X = [\sigma_3, X]$, the $e^{\hat{\sigma}_3}X = e^{\sigma_3}Xe^{-\sigma_3}$.

By performing the standard procedures on the Volterra integral equations (8), one can prove the existence and uniqueness of the Jost solutions $J_\pm$. Moreover, it is important that $[J_-]_1$, $[J_+]_2$ can be analytically extended into $D_+$, and $[J_+]_1$, $[J_-]_2$ into $D_-$, where the regions $D_\pm$ are defined by

$$\begin{aligned} D_+ &= \{k \in \mathbb{C} |\arg k \in (0, \pi/2) \cup (\pi, 3\pi/2)\}\,, \\ D_- &= \{k \in \mathbb{C} |\arg k \in (\pi/2, \pi) \cup (3\pi/2, 2\pi)\}\,, \end{aligned}$$

and $\partial D = \{\mathbb{R} \cup i\mathbb{R}\}$.

Now, we investigate the properties of $J_\pm$. Indeed, the fact that the potential matrix $\widetilde{U}$ is zero-trace implies that $\det J_\pm$ are independent of the variable $x$. In particular, by evaluating $\det J_+$ at $x = +\infty$ and $\det J_-$ at $x = -\infty$, respectively, we have

$$\det J_\pm = 1\,, \qquad k \in \partial D\,. \tag{9}$$

Since $J_- E_1$ and $J_+ E_1$ are both fundamental solutions of the $x$-part of Eq. (2) for $k \in \partial D$ with $E_1 = e^{-\frac{ik^2}{1+k^2}\sigma_3 x}$, they are linearly related. That is, there exists a scattering matrix $S(k)$ such that

$$J_- E_1 = J_+ E_1 S(k)\,, \qquad k \in \partial D\,, \tag{10}$$

where $\det S(k) = 1$ and

$$S(k) = \begin{pmatrix} a(k) & -\widetilde{b}(k) \\ b(k) & \widetilde{a}(k) \end{pmatrix}.$$

Furthermore, we find from the scattering relation (10) that

$$\begin{aligned} a(k) &= W([J_-]_1, [J_+]_2)\,, \qquad b(k) = e^{-\frac{2ik^2}{1+k^2}x} W([J_+]_1, [J_-]_1)\,, \\ \widetilde{a}(k) &= W([J_+]_1, [J_-]_2)\,, \qquad \widetilde{b}(k) = e^{\frac{2ik^2}{1+k^2}x} W([J_+]_2, [J_-]_2)\,, \end{aligned} \tag{11}$$

where $W(\cdot,\cdot)$ denotes the Wronski determinant. Then it follows from the analytic property of $J_\pm$ that $a(k)$ can be analytically extended to $D_+$, $\widetilde{a}(k)$ allows analytic extensions to $D_-$.

## 3. Riemann–Hilbert problem

To construct the Riemann–Hilbert problem on $\partial D$ by using the analytic properties of the Jost solutions $J_\pm$, it is important to introduce a matrix function $P_1 = P_1(x, k)$ which is analytic in $D_+$

$$P_1 = ([J_-]_1, [J_+]_2)\,, \tag{12}$$

which solves the linear spectral problem (2). By (11) and (12), we have

$$\det P_1 = a(k)\,, \qquad k \in D_+\,. \tag{13}$$

On the other hand, in order to construct an analytic matrix function $P_2$ in $D_-$, we write the inverse of $J_\pm$ as

$$J_-^{-1} = \begin{pmatrix} [J_-^{-1}]^1 \\ [J_-^{-1}]^2 \end{pmatrix}, \qquad J_+^{-1} = \begin{pmatrix} [J_+^{-1}]^1 \\ [J_+^{-1}]^2 \end{pmatrix}, \tag{14}$$

where each $[J_\pm^{-1}]^l$ $(l = 1, 2)$ denotes the $l^{\text{th}}$ row of $J_\pm^{-1}$, respectively. It is easy to verify that $J_\pm^{-1}$ satisfy the equation of

$$\begin{aligned} K_x &= -\frac{ik^2}{1+k^2}[\sigma_3, K] - K\widetilde{U}\,, \\ K_t &= -\frac{i}{2k^2}[\sigma_3, K] - K\widetilde{V}\,. \end{aligned} \tag{15}$$

Resorting to the spectral analysis of (15), we can define a matrix function $P_2 = P_2(x, k)$ which is analytic for $k$ in $D_-$

$$P_2 = \begin{pmatrix} [J_-^{-1}]^1 \\ [J_+^{-1}]^2 \end{pmatrix}. \tag{16}$$

By (11) and (16), we have

$$\det P_2 = \widetilde{a}(k)\,, \qquad k \in D_-\,. \tag{17}$$

Let us consider the asymptotic expansion of $P_1$

$$P_1(k) = I + k^{-1}P_1^{(1)} + k^{-2}P_1^{(2)} + \ldots\,, \qquad k \to \infty\,, \tag{18}$$

and substitute this expansion into (5). Then we find that the potentials $u$ and $w$ can be reconstructed as

$$\begin{aligned} u &= \partial_x \left(P_1^{(1)}\right)_{12} + 2i\left(P_1^{(1)}\right)_{12}\,, \\ w &= 1 - 2i\left(P_1^{(1)}\right)_{21}\partial_t\left(P_1^{(1)}\right)_{12} + 2i\partial_t\left(P_1^{(2)}\right)_{11}\,, \end{aligned} \tag{19}$$

where $(P_1^{(l)})_{jk}$ is the $(j, k)$-entry of $P_1^{(l)}$.

Summarizing the above results, we have constructed two matrix functions $P_1$ and $P_2$, which are analytic in $D_+$ and $D_-$, respectively. Now, we denote the limit of $P_1$ when $k$ approaches $\partial D$ inside $D_+$ as $P^+$, and the limit of $P_2$ when $k$ approaches $\partial D$ inside $D_-$ as $P^-$, respectively. Consequently, we can formulate the RH problem. In fact, $P^+$ and $P^-$ satisfy the jump condition on the curve $\partial D$

$$P^-(x,k)P^+(x,k) = G(x,k)\,, \qquad k \in \partial D\,, \tag{20}$$

where

$$G(x,k) = \begin{pmatrix} 1 & \widetilde{b}(k)e^{-\frac{2ik^2}{1+k^2}x} \\ b(k)e^{\frac{2ik^2}{1+k^2}x} & 1 \end{pmatrix}.$$

In order to ensure the uniqueness of the solution of (20), we consider the large-$k$ asymptotic behavior of $P_1$ and $P_2$, then we have for $\forall k \in D_+$

$$\begin{aligned} P_1(x,k) &\to I\,, \qquad k \in D_+ \to \infty\,, \\ P_2(x,k) &\to I\,, \qquad k \in D_- \to \infty\,. \end{aligned} \tag{21}$$

(21) is the canonical normalization condition. Equation (20) is the RH problem of the nonlinear wave equation (1).

In the process of solving the RH problem, the index of RH problem is a very important factor. For the RH problem: $P^-P^+ = G$, the index is usually defined as

$$\operatorname{ind} G(k) = \frac{1}{2\pi i}[\ln \det G(k)]|_{\partial D}\,.$$

From (13), (17) and (20), we have

$$\det G(k) = 1 - b(k)\,\widetilde{b}(k) = a(k)\,\widetilde{a}(k) = \det P_1\ \det P_2\,.$$

$a(k)$ and $\widetilde{a}(k)$ could only have zeros, because $a(k)$ and $\widetilde{a}(k)$ are analytic functions in $D_+$ and $D_-$, respectively. So the zeros of $\det G(k)$ are the same as the zeros of $a(k)$ and $\widetilde{a}(k)$. We also refer the zeros of $\det G(k)$ to the zeros of the RH problem. The RH problem with nonzero index is named as an irregular RH problem. In this paper, we solve the RH problem (20) by the technique of regularization.

## 4. Symmetric relations

To establish the Riemman–Hilbert problem for Eq. (1), we notice that there are symmetry relations for the matrix $U$ and $V$

$$\sigma_3 \widehat{U}^\dagger\left(\bar{k}\right)\sigma_3 = -\widehat{U}(k)\,, \qquad \sigma_3 \widehat{V}^\dagger\left(\bar{k}\right)\sigma_3 = -\widehat{V}(k)\,, \tag{22}$$

$$\sigma \widehat{U}(-k)\sigma^{-1} = \widehat{U}(k)\,, \qquad \sigma \widehat{V}(-k)\sigma^{-1} = \widehat{V}(k)\,, \tag{23}$$

where † means the Hermitian conjugate, $\bar{k}$ is the complex conjugate quantity of $k$ and

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+ik}{1-ik} \end{pmatrix}.$$

Since $Y^{\dagger}(\bar{k})$ and $Y^{-1}(k)$ satisfy the same differential equation and boundary value problem, we obtain

$$\sigma_3 \widehat{Y}^{\dagger}\left(\bar{k}\right)\sigma_3 = \widehat{Y}^{-1}(k)\,. \tag{24}$$

Similarly, we have

$$\sigma \widehat{Y}(-k)\sigma^{-1} = \widehat{Y}(k)\,. \tag{25}$$

From (4), we know

$$\sigma_3 J^{\dagger}\left(\bar{k}\right)\sigma_3 = J^{-1}(k)\,, \tag{26}$$
$$\sigma J(-k)\sigma^{-1} = J(k)\,. \tag{27}$$

Due to (10), we obtain

$$\sigma_3 S^{\dagger}\left(\bar{k}\right)\sigma_3 = S^{-1}(k)\,, \tag{28}$$
$$\sigma S(-k)\sigma^{-1} = S(k)\,. \tag{29}$$

Based on (28) and (29), we find

$$a(-k) = a(k)\,, \qquad k \in D_+\,, \tag{30}$$
$$\widetilde{a}(-k) = \widetilde{a}(k)\,, \qquad k \in D_-\,, \tag{31}$$
$$\widetilde{a}(k) = \bar{a}\left(\bar{k}\right)\,, \qquad k \in \partial D\,. \tag{32}$$

Owing to the definitions of $P_1$, $P_2$, we observe

$$\sigma_3 P_1^{\dagger}\left(\bar{k}\right)\sigma_3 = P_2(k)\,, \tag{33}$$
$$\sigma P_j(-k)\sigma^{-1} = P_j(k)\,, \qquad j = 1, 2\,. \tag{34}$$

Then from the symmetric relations of above and (13), (17), we can show the zeros of $\det P_1$ and $\det P_2$ as: if $k$ is a zero of $\det P_1$, then $-k$ is also a zero of $\det P_1$ and $\hat{k} = \bar{k}$ is a zero of $\det P_2$. So, we can assume that $\det P_1$ have $2N$ single zeros $\{k_j\}_1^{2N}$ in $D_+$, where $k_{N+j} = -k_j (1 \le j \le N)$. $\det P_2$ have $2N$ single zeros $\{\hat{k}_j\}_1^{2N}$ in $D_-$, satisfying $\hat{k}_j = \bar{k}_j, (1 \le j \le 2N)$. So, we have

$$\det P_1(x,t,k_j) = 0\,, \qquad \det P_2\left(x,t,\hat{k}_j\right) = 0\,. \tag{35}$$

There exist nonzero feature vectors $v_j$, $\hat{v}_j$, satisfying

$$\begin{aligned} P_1(k_j)v_j &= 0\,, \qquad 1 \le j \le 2N\,, &(36)\\ \hat{v}_j P_2\left(\hat{k}_j\right) &= 0\,, \qquad 1 \le j \le 2N\,. &(37)\end{aligned}$$

Then, from (33), (34) and (36), we have

$$\begin{aligned} \hat{v}_j &= v_j^{\dagger}\sigma_3\,, \qquad 1 \le j \le 2N\,, &(38)\\ v_{N+j} &= \sigma^{-1}v_j\,, \qquad 1 \le j \le N\,. &(39)\end{aligned}$$

Differentiating both sides of (36) with $x$, and considering (5), we have

$$v_{j,x} = \left(-\frac{ik_j^2}{1+k_j^2}\sigma_3 + \beta_j I\right) v_j\,. \tag{40}$$

Similarly, we obtain

$$v_{j,t} = \left(-\frac{i}{2k_j^2}\sigma_3 + \alpha_j I\right) v_j\,, \tag{41}$$

where $\alpha_j$ and $\beta_j$ are any constants. From (40) and (41), we have

$$v_j = e^{\left(-\frac{ik_j^2}{1+k_j^2}\sigma_3+\beta_j I\right)x+\left(-\frac{i}{2k_j^2}\sigma_3+\alpha_j I\right)t} v_{j,0}\,, \qquad 1 \le j \le N\,. \tag{42}$$

For simplicity, we often assume that $v_{j,0}(1 \le j \le N)$ is a nonzero constant vector. If we let $k_j = \xi_j + i\eta_j$ and $v_{j,0} = (e^{\alpha_{j0}+i\beta_{j0}}, 1)^T$, then $v_j$ can be rewritten as

$$v_j = e^{\epsilon_j}\left(e^{(z_j+i\varphi_j)/2}, e^{-(z_j+i\varphi_j)/2}\right)^T\,, \qquad 1 \le j \le N\,, \tag{43}$$

where

$$\begin{aligned} \epsilon_j &= \frac{\alpha_j t + \beta_j x + (\alpha_{j0} + i\beta_{j0})}{2}\,,\\ z_j &= \frac{4\xi_j\eta_j}{\left(1+\xi_j^2-\eta_j^2\right)^2 + 4\xi_j^2\eta_j^2}x - \frac{2\xi_j\eta_j}{\left(\xi_j^2+\eta_j^2\right)^2}t + \alpha_{j0}\,,\\ \varphi_j &= -\frac{2\left(\xi_j^2-\eta_j^2\right)\left(1+\xi_j^2-\eta_j^2\right)+8\xi_j^2\eta_j^2}{\left(1+\xi_j^2-\eta_j^2\right)^2 + 4\xi_j^2\eta_j^2}x - \frac{\xi_j^2-\eta_j^2}{\left(\xi_j^2+\eta_j^2\right)^2}t + \beta_{j0}\,. \end{aligned} \tag{44}$$

## 5. Exact solutions

In order to regularize the RH problem (20), now we introduce the following rational matrix:

$$\chi_j = I - \frac{k_j - \hat{k}_j}{k - \hat{k}_j} T_j\,, \qquad \chi_j^{-1} = I + \frac{k_j - \hat{k}_j}{k - \hat{k}_j} T_j\,, \qquad j = 1, \ldots, n\,, \tag{45}$$

where we assume that $u_j$ ($j = 1, \ldots, n$; $n = 2N$) is a nonzero column vector and

$$T_j = \frac{u_j u_j^\dagger}{u_j^\dagger u_j}\,, \qquad j = 1, \ldots, n\,. \tag{46}$$

$T_j$ is a projection operator and we have

$$T_j^2 = T_j\,, \qquad \mathrm{tr}\, T_j = 1\,, \qquad T_j^\dagger = T_j\,, \qquad \mathrm{rank}\, T_j = 1\,. \tag{47}$$

From (47), we easily know that $T_j$ is a first order matrix, and similar to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then there exists a reversible matrix $Q_j$, satisfying $Q_j T_j Q_j^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and

$$\det\left(\chi_j^{-1}\right) = \det\left(Q_j \chi_j^{-1} Q_j^{-1}\right) = \begin{vmatrix} 1 + \frac{k_j - \hat{k}_j}{k - k_j} & 0 \\ 0 & 1 \end{vmatrix} = \frac{k - \hat{k}_j}{k - k_j}\,. \tag{48}$$

Similarly, we obtain

$$\det(\chi_j) = \frac{k - k_j}{k - \hat{k}_j}\,. \tag{49}$$

Since $k = k_j$ is a single zero of $\det P_1$, we can have $\det(P_1(x,t,k)\chi_j^{-1})|_{k=k_j} \neq 0$ at $k = k_j$. So the RH problem (20) is regularized at $k = k_j$.

Similarly, $k = \hat{k}_j$ is a single zero of $\det P_2$, then $\det(\chi_j P_2(x,t,k))|_{k=\hat{k}_j} \neq 0$ at $k = \hat{k}_j$. So the RH problem (20) is regularized at $k = \hat{k}_j$.

In the following, we will regularize the RH problem (20) for all zeros. Firstly, we let

$$\begin{aligned} \Gamma(k) &= \chi_n(k)\chi_{n-1}(k)\ldots\chi_2(k)\chi_1(k)\,, &(50)\\ P^+(x,t,k) &= \varphi_+(x,t,k)\Gamma(x,t,k)\,, &(51)\\ (P^-)^{-1}(x,t,k) &= \varphi_-(x,t,k)\Gamma(x,t,k)\,. &(52) \end{aligned}$$

Then $\varphi_\pm$ satisfy the regular RH problem

$$\varphi_-^{-1}(x,t,k)\varphi_+(x,t,k) = \Gamma P^- P^+ \Gamma^{-1} = \Gamma G(k) \Gamma^{-1}\,. \tag{53}$$

To solve Eq. (1), we choose the jump matrix $G$ to be the $2 \times 2$ identity matrix which corresponds to the reflectionless case. Then, we have

$$\varphi_-^{-1}\varphi_+ = I\,. \tag{54}$$

Based on the definition of $\Gamma$, we have $\Gamma \to I$ with $k \to \infty$ noting the canonical normalization condition (21), we have $\varphi_+ = I$ and $P^+(x,t,k) = \Gamma(x,t,k)$. Next, we factorize $\Gamma(x,t,k)$

$$\begin{aligned}\Gamma &= \left(I - \frac{k_n - \hat{k}_n}{k - \hat{k}_n}T_n\right)\dots\left(I - \frac{k_2 - \hat{k}_2}{k - \hat{k}_2}T_2\right)\left(I - \frac{k_1 - \hat{k}_1}{k - \hat{k}_1}T_1\right)\\ &= I - \sum_{\ell=1}^{n}\sum_{j=1}^{n}\frac{\left(M^{-1}\right)_{\ell j}}{k - \hat{k}_j}y_\ell y_j^\dagger\,,\end{aligned} \tag{55}$$

where $M_{j\ell} = \frac{1}{k_\ell - \hat{k}_j}y_j^\dagger y_\ell$, $(M^{-1})_{\ell j}$ is the $(\ell,j)$-element of $M^{-1}$, and $M = (M_{j\ell})$. In the following, we will prove (55). We assume that $k_j \in \mathbb{C}\setminus\mathbb{R}$, $j = 1,2,\dots,n$ are different complex parameters and denote

$$A_j(k) = I - \frac{k_j - \bar{k}_j}{k - \bar{k}_j}T_j\,, \tag{56}$$

where $T_j$ is defined as (46). Then we have

$$\det A_j(k) = \frac{k - k_j}{k - \bar{k}_j}\,, \qquad A_j(k)^{-1} = A_j\left(\bar{k}\right)^\dagger\,, \qquad A_j\left(\bar{k}\right)^\dagger = I + \frac{k_j - \bar{k}_j}{k - k_j}T_j\,. \tag{57}$$

Letting

$$\Gamma(k) = A_n(k)A_{n-1}(k)\dots A_2(k)A_1(k)\,, \tag{58}$$

we have

$$\begin{aligned}\Gamma(k)^{-1} &= A_1(k)^{-1}A_2(k)^{-1}\cdots A_{n-1}(k)^{-1}A_n(k)^{-1}\\ &= A_1\left(\bar{k}\right)^\dagger A_2\left(\bar{k}\right)^\dagger\dots A_{n-1}\left(\bar{k}\right)^\dagger A_n\left(\bar{k}\right)^\dagger\\ &= \left[A_n\left(\bar{k}\right)A_{n-1}\left(\bar{k}\right)\dots A_2\left(\bar{k}\right)A_1\left(\bar{k}\right)\right]^\dagger\\ &= \Gamma\left(\bar{k}\right)^\dagger\,.\end{aligned} \tag{59}$$

We easily know that

— $\Gamma(k)$ is a meromorphic function for $k \in \mathbb{C}$,

— $\Gamma(k)$ have only $n$ singularities: $\bar{k}_1, \ldots, \bar{k}_n$,

— every singularity $\bar{k}_j$ is a single pole of $\Gamma(k)$.

Letting

$$\Gamma_j = \lim_{k \to \bar{k}_j} \left(k - \bar{k}_j\right) \Gamma(k) , \tag{60}$$

and

$$\Gamma_0(k) = \Gamma(k) - \sum_{j=1}^{n} \frac{\Gamma_j}{k - \bar{k}_j} , \tag{61}$$

we have

$$\lim_{k \to \bar{k}_j} \left(k - \bar{k}_j\right) \Gamma_0(k) = 0 . \tag{62}$$

This means that the Laurent expansion of $\Gamma_0(k)$ at $k = \bar{k}_j$ has no negative power items. In other words, $\Gamma_0(k)$ is analytic at $k = \bar{k}_j$. On the other hand, $\Gamma_0(k)$ is analytic at $k \in \mathbb{C} \setminus \{\bar{k}_1, \cdots, \bar{k}_n\}$, then $\Gamma_0(k)$ is an integral function at $k \in \mathbb{C}$. From the definition of $\Gamma_0(k)$, we can know

$$\lim_{k \to \infty} \Gamma_0(k) = I . \tag{63}$$

Based on the Liouville theorem, we obtain

$$\Gamma_0(k) = I , \qquad k \in \mathbb{C} . \tag{64}$$

By (61), we have

$$\Gamma(k) = I + \sum_{j=1}^{n} \frac{\Gamma_j}{k - \bar{k}_j} . \tag{65}$$

Owing to (58) and (60), we have

$$\begin{aligned} \Gamma_j &= A_n\left(\bar{k}_j\right) \cdots A_{j+1}\left(\bar{k}_j\right) \lim_{k \to \bar{k}_j} \left(k - \bar{k}_j\right) A_j(k) A_{j-1}\left(\bar{k}_j\right) \cdots A_1\left(\bar{k}_j\right) \\ &= -\left(k_j - \bar{k}_j\right) A_n\left(\bar{k}_j\right) \ldots A_{j+1}\left(\bar{k}_j\right) P_j A_{j-1}\left(\bar{k}_j\right) \ldots A_1\left(\bar{k}_j\right) \\ &= -\left(k_j - \bar{k}_j\right) A_n\left(\bar{k}_j\right) \ldots A_{j+1}\left(\bar{k}_j\right) \frac{u_j u_j^\dagger}{u_j^\dagger u_j} A_{j-1}\left(\bar{k}_j\right) \ldots A_1\left(\bar{k}_j\right) . \end{aligned} \tag{66}$$

Letting

$$x_j = A_n\left(\bar{k}_j\right)\dots A_{j+1}\left(\bar{k}_j\right)\frac{u_j}{u_j^\dagger u_j}\,, \qquad y_j^\dagger = u_j^\dagger A_j\left(\bar{k}\right)A_{j-1}\left(\bar{k}_j\right)\dots A_1\left(\bar{k}_j\right)\,, \tag{67}$$

we get

$$\Gamma(k) = I - \sum_{j=1}^{n}\frac{k_j-\bar{k}_j}{k-\bar{k}_j}x_j y_j^\dagger\,. \tag{68}$$

From (59), we find

$$\Gamma(k)^{-1} = I + \sum_{j=1}^{n}\frac{k_j-\bar{k}_j}{k-k_j}y_j x_j^\dagger\,. \tag{69}$$

By (68) and (69), we have

$$(k-k_\ell)\left(I - \sum_{j=1}^{n}\frac{k_j-\bar{k}_j}{k-\bar{k}_j}x_j y_j^\dagger\right)\left(I + \sum_{j=1}^{n}\frac{k_j-\bar{k}_j}{k-k_j}y_j x_j^\dagger\right) = (k-k_\ell)I\,. \tag{70}$$

When $k \to k_\ell$, we have

$$\left(I - \sum_{j=1}^{n}\frac{k_j-\bar{k}_j}{k_\ell-\bar{k}_j}x_j y_j^\dagger\right)y_\ell x_\ell^\dagger = 0 \tag{71}$$

and

$$\left(I - \sum_{j=1}^{n}\frac{k_j-\bar{k}_j}{k_\ell-\bar{k}_j}x_j y_j^\dagger\right)y_\ell x_\ell^\dagger x_\ell = 0\,. \tag{72}$$

Since $x_\ell$ is a nonzero vector, we have $x_\ell^\dagger x_\ell \neq 0$. Then we get

$$\left(I - \sum_{j=1}^{n}\frac{k_j-\bar{k}_j}{k_\ell-\bar{k}_j}x_j y_j^\dagger\right)y_\ell = 0\,. \tag{73}$$

Namely, we have

$$\Gamma(k_\ell)y_\ell = 0 \tag{74}$$

and

$$y_\ell = \sum_{j=1}^{n}\frac{k_j-\bar{k}_j}{k_\ell-\bar{k}_j}x_j y_j^\dagger y_\ell\,. \tag{75}$$

We define $D_j$ and $M_{j\ell}$ as following:

$$D_j = k_j - \bar{k}_j\,, \qquad M_{j\ell} = \frac{1}{k_\ell - \bar{k}_j} y_j^\dagger y_\ell\,. \tag{76}$$

Then (75) can be rewritten as

$$y_\ell = \sum_{j=1}^{n} x_j D_j M_{j\ell}\,. \tag{77}$$

We introduce four matrixes:

$$\begin{cases} D = \operatorname{diag}\{D_1, D_2, \ldots, D_n\}\,, & M = (M_{j\ell})_{n\times n}\,, \\ \check{Y} = [y_1, y_2, \ldots, y_n]\,, & \check{X} = [x_1, x_2, \ldots, x_n]\,. \end{cases} \tag{78}$$

Then (77) can be rewritten as

$$\check{Y} = \check{X} D M\,, \qquad \check{X} D = \check{Y} M^{-1}\,. \tag{79}$$

Every column of $\check{X} D = \check{Y} M^{-1}$ is

$$\left(k_j - \bar{k}_j\right) x_j = \sum_{\ell=1}^{n} y_\ell \left(M^{-1}\right)_{\ell j}\,. \tag{80}$$

We substitute (80) into (68),

$$\mathit{\Gamma}(k) = I - \sum_{\ell=1}^{n}\sum_{j=1}^{n} \frac{\left(M^{-1}\right)_{\ell j}}{k - \bar{k}_j} y_\ell y_j^\dagger\,, \qquad M_{j\ell} = \frac{1}{k_\ell - \bar{k}_j} y_j^\dagger y_\ell \tag{81}$$

considering the zeros of $\det P_1$ and $\det P_2$, we get $\hat{k} = \bar{k}$. This means that (55) is equal to (81).

When $G = I$, we can get $P^+(x,t,k) = \mathit{\Gamma}(x,t,k)$. Then we know $\mathit{\Gamma}(x,t,k)$ solves also (5). From (74), we have

$$\begin{aligned} \mathit{\Gamma}(x,t,k_\ell)\left(y_{\ell,x} + \frac{ik_j^2}{1+k_j^2}\sigma_3 y_\ell\right) &= 0\,, \\ \mathit{\Gamma}(x,t,k_\ell)\left(y_{\ell,t} + \frac{i}{2k_j^2}\sigma_3 y_\ell\right) &= 0\,. \end{aligned}$$

Since $k_\ell$ is a single zero of $\det \mathit{\Gamma}(k)$, the solution space of linear problem $\mathit{\Gamma}(k_\ell)X = 0$ is one-dimensional space. Thus, the vector $y_{\ell,x} + \frac{ik_j^2}{1+k_j^2}\sigma_3 y_\ell$

and $y_{\ell,t} + \frac{i}{2k_j^2}\sigma_3 y_\ell$ are linearly related with $v_\ell$. Therefore, without loss of generality, we might let $m = \ell$ and $y_\ell = v_m$ at (81). Similarly, from (59) and (74), we can also let $y_j^\dagger = \hat{v}_j$. We have

$$\begin{aligned} P_1(k) &= I - \sum_{m=1}^{2N}\sum_{j=1}^{2N} \frac{v_m \hat{v}_j \left(M^{-1}\right)_{mj}}{k - \hat{k}_j}, \\ P_2(k) &= I + \sum_{m=1}^{2N}\sum_{j=1}^{2N} \frac{v_m \hat{v}_j \left(M^{-1}\right)_{mj}}{k - k_m}, \end{aligned} \tag{82}$$

where $(M^{-1})_{mj}$ is the $(m,j)$-entry of $M^{-1}$ and $M = (M_{mj})_{2N\times 2N}$ is an invertible matrix

$$M_{mj} = \frac{\hat{v}_m v_j}{k_j - \hat{k}_m}, \qquad 1 \le m, j \le 2N\,.$$

Therefore, the matrices $P_1^{(1)}$ and $P_1^{(2)}$ can be obtained from (82) as

$$\begin{aligned} P_1^{(1)} &= -\sum_{m=1}^{2N}\sum_{j=1}^{2N} v_m \hat{v}_j \left(M^{-1}\right)_{mj}, \\ P_1^{(2)} &= -\sum_{m=1}^{2N}\sum_{j=1}^{2N} \hat{k}_j v_m \hat{v}_j \left(M^{-1}\right)_{mj}. \end{aligned} \tag{83}$$

Then $N$-soliton solutions are obtained for the nonlinear wave equation (1) as follows:

$$\begin{aligned} u &= -\partial_x \left( \sum_{m=1}^{2N}\sum_{j=1}^{2N} v_m \hat{v}_j \left(M^{-1}\right)_{mj} \right)_{12} - 2i \left( \sum_{m=1}^{2N}\sum_{j=1}^{2N} v_m \hat{v}_j \left(M^{-1}\right)_{mj} \right)_{12}, \\ w &= 1 - 2i \left( \sum_{m=1}^{2N}\sum_{j=1}^{2N} v_m \hat{v}_j \left(M^{-1}\right)_{mj} \right)_{21} \partial_x \left( \sum_{m=1}^{2N}\sum_{j=1}^{2N} v_m \hat{v}_j \left(M^{-1}\right)_{mj} \right)_{12} \\ &\quad -2i\partial_t \left( \sum_{m=1}^{2N}\sum_{j=1}^{2N} \hat{k}_j v_m \hat{v}_j \left(M^{-1}\right)_{mj} \right)_{11}. \end{aligned} \tag{84}$$

In particular, for $N = 1$ in formula (84), one-soliton solution of Eq. (1) takes the form of

$$u=\frac{8\xi_1\eta_1\left((\xi_1-i(\eta_1+1))e^{\frac{4\xi_1\eta_1 t}{\left(\xi_1^2+\eta_1^2\right)^2}}-(\xi_1+i(\eta_1-1))e^{\frac{8\xi_1\eta_1 x}{\xi_1^4+2\xi_1^2\left(\eta_1^2+1\right)+\left(\eta_1^2-1\right)^2}}\right)\exp\left(-\frac{i\left(2x\left(\xi_1^2+\eta_1^2\right)^2+t(\xi_1+i\eta_1)^2+t\right)}{\left(1+(\xi_1+i\eta_1)^2\right)(\xi_1-i\eta_1)^2}\right)}{\left(\eta_1^2+(\xi_1-i)^2\right)\left((\xi_1-i(\eta_1-1))e^{\frac{8\xi_1\eta_1 x}{\xi_1^4+2\xi_1^2\left(\eta_1^2+1\right)+\left(\eta_1^2-1\right)^2}}-(\xi_1+i(\eta_1+1))e^{\frac{4\xi_1\eta_1 t}{\left(\xi_1^2+\eta_1^2\right)^2}}\right)^2},$$

$$\begin{aligned} w=&\Bigg(\left(\xi_1^2+\eta_1^2\right)^2\left(\left(\xi_1^2+(\eta_1+1)^2\right)e^{\frac{8\xi_1\eta_1 t}{\left(\xi_1^2+\eta_1^2\right)^2}}+\left(\xi_1^2+(\eta_1-1)^2\right)e^{\frac{16\xi_1\eta_1 x}{\xi_1^4+2\xi_1^2\left(\eta_1^2+1\right)+\left(\eta_1^2-1\right)^2}}\right)\\ &-2\left(\xi_1^6+\xi_1^4\left(\eta_1^2+1\right)-\xi_1^2\eta_1^2\left(\eta_1^2+14\right)-\eta_1^6+\eta_1^4\right)\exp\left(4\xi_1\eta_1\left(\frac{t}{\left(\xi_1^2+\eta_1^2\right)^2}+\frac{2x}{\xi_1^4+2\xi_1^2\left(\eta_1^2+1\right)+\left(\eta_1^2-1\right)^2}\right)\right)\Bigg)\\ &\Bigg/\left(\xi_1^2+\eta_1^2\right)^2\left((-\xi_1+i(\eta_1+1))\,e^{\frac{4\xi_1\eta_1 t}{\left(\xi_1^2+\eta_1^2\right)^2}}+(\xi_1+i\,(\eta_1-1))\,e^{\frac{8\xi_1\eta_1 x}{\xi_1^4+2\xi_1^2\left(\eta_1^2+1\right)+\left(\eta_1^2-1\right)^2}}\right)\\ &\left((\xi_1-i\,(\eta_1-1))\,e^{\frac{8\xi_1\eta_1 x}{\xi_1^4+2\xi_1^2\left(\eta_1^2+1\right)+\left(\eta_1^2-1\right)^2}}-(\xi_1+i\,(\eta_1+1))\,e^{\frac{4\xi_1\eta_1 t}{\left(\xi_1^2+\eta_1^2\right)^2}}\right), \end{aligned} \tag{85}$$

where $k_1=\xi_1+i\eta_1\in D_+$, $k_1\neq 0$, $k_1\neq\pm i$, and $\alpha_0=\beta_0=0$.

To illustrate one-soliton solution, we choose $\xi_1 = 1$, $\eta_1 = 1$, $\alpha_0 = \beta_0 = 0$. Therefore, the profiles of the solutions are plotted in Fig. 1 to Fig. 6.

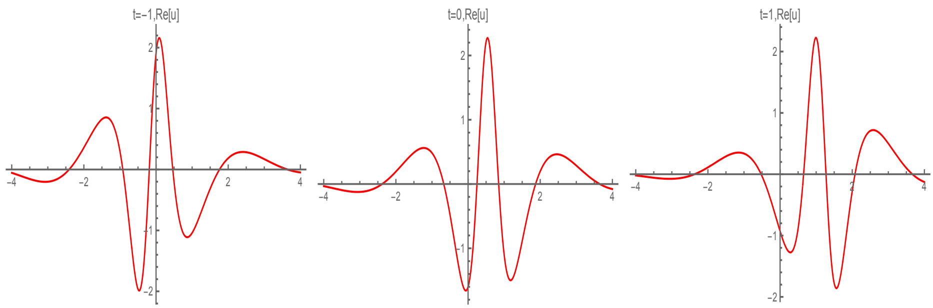


Fig. 1. Re[$u$] in (85) with the parameters chosen as $\xi_1 = 1$, $\eta_1 = 1$, $\alpha_0 = \beta_0 = 0$.

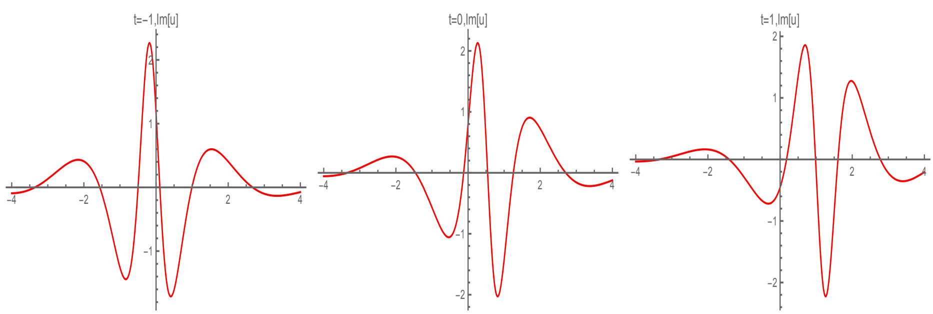


Fig. 2. Im[$u$] in (85) with the parameters chosen as $\xi_1 = 1$, $\eta_1 = 1$, $\alpha_0 = \beta_0 = 0$.

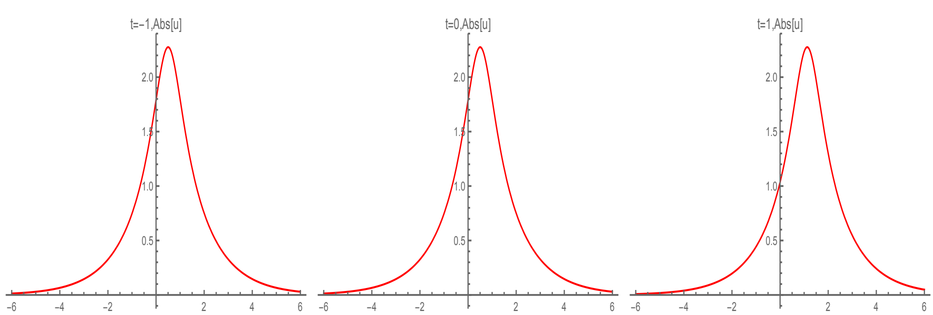


Fig. 3. Abs[$u$] in (85) with the parameters chosen as $\xi_1 = 1$, $\eta_1 = 1$, $\alpha_0 = \beta_0 = 0$.

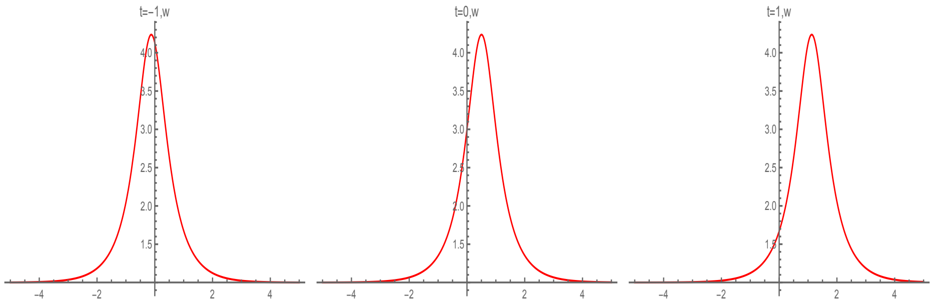


Fig. 4. $w(x,t)$ in (85) with the parameters chosen as $\xi_1 = 1$, $\eta_1 = 1$, $\alpha_0 = \beta_0 = 0$.

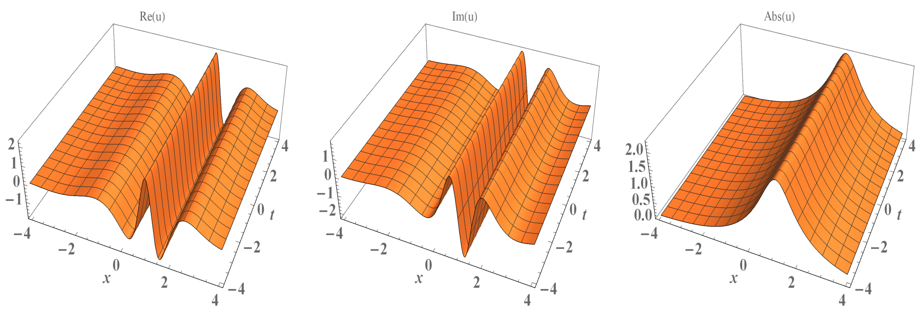


Fig. 5. Re[$u$], Im[$u$], Abs[$u$] in (85) with the parameters chosen as $\xi_1 = 1$, $\eta_1 = 1$, $\alpha_0 = \beta_0 = 0$.

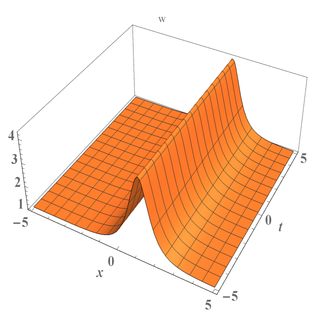

Fig. 6. $w(x,t)$ in (85) with the parameters chosen as $\xi_1 = 1$, $\eta_1 = 1$, $\alpha_0 = \beta_0 = 0$.

**Remark.** Equation (2) has an equivalent form of

$$\widehat{Y}_x = \widehat{U}\widehat{Y}\,, \qquad \widehat{Y}_t = \widehat{V}\widehat{Y}\,,$$

with

$$\widehat{U} = \begin{pmatrix} i\lambda^{-1} & (1+\lambda^{-1})u \\ \bar{u} & -i\lambda^{-1} \end{pmatrix},$$
$$\widehat{V} = \frac{1}{2}\begin{pmatrix} i(1+\lambda)w & (1+\lambda)(uw+iu_t) \\ \lambda(\bar{u}w - i\bar{u}_t) & -i(1+\lambda)w \end{pmatrix},$$

where $\lambda = -\frac{1}{k^2} - 1$, $k \neq \pm i, k \neq 0$. Through the following gauge transformation:

$$Y = T\widehat{Y}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-ik} \end{pmatrix} \quad (k \neq \pm i),$$

we have Eq. (2).

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