

THE ROLE OF NONHYDRODYNAMIC MODES IN BJORKEN FLOW*

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When hydrodynamics is pushed to the limits of its applicability, the influence of transient nonhydrodynamic modes becomes important. As hydrodynamics plays an important role in modelling heavy-ion collisions, understanding the nonhydrodynamic sector of QCD is a worthy, but difficult, goal. In this contribution, I describe the role that nonhydrodynamic modes play in toy models of the expanding quark-gluon plasma, with a focus on my work in kinetic theory.

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1. Introduction

Nonhydrodynamic modes, as opposed to hydrodynamic ones, are transient effects that quickly decay. Naturally, they have not been the main interest in studies of hydrodynamics. However, in the last decade or so, motivated in part by studies of thermalization in AdS/CFT and by the application of hydrodynamics in modelling heavy-ion collisions, they have become a target of study [1].

They play multiple roles. In hydrodynamics as an effective field theory, they act as a UV regulator, ensuring that the theory does not violate causality. In microscopic models, they carry fine grained information on the microscopic physics and their decay sets the timescale for when the system hydrodynamizes. This is an in principle different criteria than the classical understanding of the regime of hydrodynamics, which is as an expansion in gradients around thermal equilibrium.

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The gradient expansion will play a central role here as it has a subtle interplay with nonhydrodynamic modes, a relation which the theory of resurgence describes. These relations require determining the gradient expansion to very high orders, which so far has been done only in toy models.

In this contribution, I describe the structure of nonhydrodynamic modes in some toy models of the expanding quark–gluon plasma. BRSSS and $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) will set the stage for what my own research has focused on: kinetic theory in the relaxation time approximation (RTA).

2. Nonhydrodynamic modes in BRSSS and in $\mathcal{N} = 4$ SYM

2.1. Nonhydrodynamic modes as a regulator

The modern understanding of hydrodynamics is as an effective field theory close to equilibrium. Equations of motion come from conservation of the energy-momentum tensor. The latter is parametrized by all terms built out of hydrodynamic fields such as energy density \mathcal{E} , local rest frame u^μ and their gradients. All terms up to some number of gradients allowed by the symmetries of the situation should be included.

In relativistic theories, the naive application of this program suffers from acausality. Modes with sufficiently large wavenumber k propagate faster than light. While these modes are outside the regime of applicability of hydrodynamics, they need to be dealt with in order to handle the theory numerically. Müller, Israel and Stewart cured this by introducing a new mode acting as a regulator in this regime.

From this perspective, nonhydrodynamic modes are unphysical. Whenever the regulator plays a role, one has pushed the theory outside of its regime of applicability. However, when hydrodynamics is considered as a limit of a microscopic theory, such nonhydrodynamic modes can be interpreted as additional physical content.

2.2. Bjorken flow

One of the main simplifying assumptions here is that of Bjorken flow, which effectively reduces the dimensionality from $3 + 1$ to $0 + 1$. We assume isotropy and homogeneity in directions transverse to the beam line and boost symmetry in the beam direction. In proper time and rapidity coordinates, these symmetries restrict the form of the energy-momentum tensor to

$$T^{\mu\nu} = \text{diag} \{ \mathcal{E}(\tau), \mathcal{P}_L(\tau), \mathcal{P}_T(\tau), \mathcal{P}_T(\tau) \} . \quad (1)$$

In addition, with a conformal equation of state and the conservation of $T^{\mu\nu}$, the pressures can be expressed in terms of \mathcal{E} . Everything is encoded in $\mathcal{E}(\tau)$,

but it turns out to be useful to instead study the dimensionless pressure anisotropy

$$\mathcal{A}(\tau) = \frac{\mathcal{P}_T(\tau) - \mathcal{P}_L(\tau)}{\mathcal{E}(\tau)}, \quad (2)$$

as a function of the dimensionless time variable $w = T\tau$.

2.3. The nonhydrodynamic mode in BRSSS

The multiple roles of nonhydrodynamic modes in hydrodynamics can be illustrated most clearly in the second order conformal hydrodynamic theory BRSSS [2]. This section follows Refs. [3, 4] closely. The pressure anisotropy satisfies a first order differential equation

$$C_\tau w \left(1 + \frac{\mathcal{A}}{12}\right) \mathcal{A}' + \left(\frac{C_\tau}{3} + \frac{C_\lambda}{8C_\eta}\right) \mathcal{A}^2 + \frac{3}{2}w\mathcal{A} - 12C_\eta = 0, \quad (3)$$

where the C_i are dimensionless numbers related to the values of transport coefficients. One may solve this ODE by expanding in gradients (which in this setup is equivalent to an expansion in inverse w)

$$\mathcal{A}(w) = \sum_{n=1}^{\infty} a_n w^{-n}. \quad (4)$$

The resulting series is surprising in two ways. First, the coefficients a_n are completely fixed. There is no dependence on initial conditions, so this is not a general solution. Second, the coefficients grow factorially and the series diverges.

The first problem is easy to solve by considering perturbations around the gradient expansion. At leading order, one can add to the gradient expansion

$$\delta\mathcal{A}(w) \sim e^{-\frac{3}{2C_\tau}w} w^\beta, \quad (5)$$

where $\beta = \frac{C_\eta - 2C_\lambda}{C_\tau}$. This decaying contribution comes with a free parameter, the amplitude, which parametrizes different initial conditions. This is the nonhydrodynamic mode of BRSSS and we see here one of its roles: it carries information on initial conditions and its decay sets the timescale for when some universal behaviour (represented by the gradient expansion) takes over.

It is attractive to identify the all orders gradient expansion as “pure” hydrodynamics, encoding late time universal behaviour, while the transient nonhydrodynamic modes describe how the microscopic information decays away. For this, we need to understand how to make sense of the divergent

series. This can be done by the use of Borel transform. The transform makes the sum convergent by

$$\mathcal{A}(w) = \sum_{n=1}^{\infty} \frac{a_n}{w^n} \xrightarrow{\text{Borel transform}} \mathcal{A}_{\mathcal{B}}(\xi) = \sum_{n=1}^{\infty} \frac{a_n \xi^n}{n!}. \quad (6)$$

The series can now be summed and through the inverse Borel transform (which is the Laplace transform), one arrives at a resummed version of the gradient expansion. However, if the Borel transformed series has some non-analytic structure, the Laplace transform will be ambiguous. This is the case here. $\mathcal{A}_{\mathcal{B}}(\xi)$ has a sequence of poles which are plotted in Fig. 1. This dense series of poles is approximating a branch cut that one can determine to be of the form of $(\xi - \frac{3}{2C_\tau})^{-1-\beta}$. This induces an ambiguity in the Laplace transform corresponding to a term proportional to $e^{-\frac{3}{2C_\tau}w}w^\beta$. This precisely corresponds to the nonhydrodynamic perturbations in Eq. (5). Such relations between the divergent behaviour of a perturbative series and non-perturbative effects are formalized in the mathematical theory of *resurgence*.

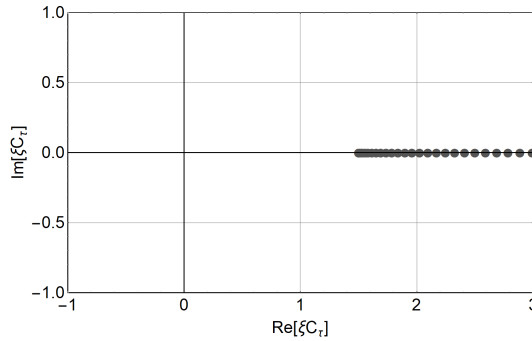


Fig. 1. Poles of the Borel transform of the gradient expansion of the pressure anisotropy in BRSSS. A series of poles are approximating a branch cut which gives rise to nonhydrodynamic ambiguities in the resummed gradient expansion.

2.4. Resurgence

The theory of resurgence [5] promotes perturbative series to what is called a trans-series, which includes nonperturbative terms like the exponentially decaying nonhydrodynamic modes. The trans-series takes the form of

$$\mathcal{A}(w) = \sum_{k=0}^{\infty} a_{k,0} w^{-k} + \sigma_1 e^{-S_1 w} w^{\beta_1} \sum_{k=0}^{\infty} a_{k,1} w^{-k} + \dots \quad (7)$$

One key fact is that the coefficients $f_{k,n}, S_n, \beta_n$ for different sectors are not independent. As we saw with the Borel transform, one can deduce information on the nonhydrodynamic modes from the gradient expansion. What remains independent, however, is the overall amplitude σ_n , which means the nonperturbative sectors carry some extra information. For more in-depth studies of resurgence in hydrodynamics, see Refs. [4, 6].

2.5. Nonhydrodynamic modes in $\mathcal{N} = 4$ SYM

Through the AdS/CFT correspondence, strongly coupled $\mathcal{N} = 4$ SYM can be used to study the non-equilibrium physics of gauge theories. Hydrodynamics emerges in the long-wavelength limit and nonhydrodynamic behaviour is included automatically without appeal to a regulator. As in BRSSS, one may calculate the gradient expansion in Bjorken flow and study the non-analytic features of the Borel transform [7], see Fig. 2. There are now two clear branch cuts and their locations show that they form a mode which is oscillating and exponentially decaying. Since the set of initial conditions of this theory is very rich, one would expect to find an infinite amount of different modes. There are some hints of more modes in Fig. 2 and indeed, a more involved resurgent analysis reveals this [8].

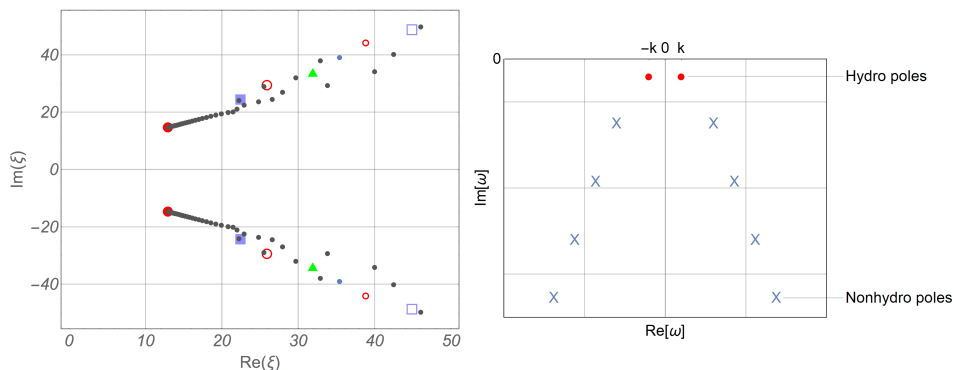


Fig. 2. Left: Poles of the Borel transform of the pressure anisotropy $\mathcal{A}_B(\xi)$ for $\mathcal{N} = 4$ SYM. Figure taken from [9]. Right: The non-analyticities of the sound channel correlator in thermal equilibrium in $\mathcal{N} = 4$ SYM. The features in thermal equilibrium can be mapped to the features in Bjorken flow, see Sec. 2.6. Note that the figures above should be rotated by 90° to be compared.

Apart from the number and behaviour of these modes, one of the most important differences between BRSSS and $\mathcal{N} = 4$ SYM is the physical interpretation of them. In the latter, they can be understood as the quasinormal modes of black holes. The fact the nonhydrodynamic sector of BRSSS does

not match the structure of $\mathcal{N} = 4$ SYM suggests that it will not be a good model when transients are important. With some modifications, the transient regime can be more faithfully reproduced [10, 11].

2.6. Where do the nonhydrodynamic modes come from?

The structure of transients in Bjorken flow can be understood and mapped to the analytic structure of two-point functions in thermal equilibrium [7, 12]. At any point in the non-equilibrium evolution, we may define a thermal state with the same effective temperature. Assuming that the system relaxes locally at a rate determined by this thermal state, the effect over a finite time interval is to modify decay behaviour as

$$\exp\left[-\frac{\tau}{\tau_{\text{rel}}}\right] \rightarrow \exp\left[-\int_0^\tau \frac{d\tau'}{\tau_{\text{rel}}(\tau')}\right]. \quad (8)$$

In the above theories, which are conformal, τ_{rel} is inversely proportional to the temperature. At leading order in the gradient expansion in Bjorken flow, $T \propto \tau^{-1/3}$ so we have $\int_0^\tau \frac{d\tau'}{\tau_{\text{rel}}(\tau')} \sim \frac{3}{2}\tau^{2/3}$. This result implies that single poles of correlators in thermal equilibrium map to nonhydrodynamic modes in Bjorken flow with a modified power law and a shift of decay rate by a factor of 3/2. For $\mathcal{N} = 4$ SYM, this maps the transients in Bjorken flow to the quasinormal modes of black holes in thermal equilibrium.

3. Kinetic theory in RTA

The kinetic theory applies to an entirely different regime than $\mathcal{N} = 4$ SYM, namely weakly coupled particles whose statistics is governed by the Boltzmann equation. This equation includes a complicated collision term encoding the scattering of particles. To repeat the above analysis in kinetic theory, we greatly simplify the collision term by employing the relaxation time approximation (RTA). In this approximation, we assume that the effect of scattering is to relax toward local equilibrium on a timescale τ_{rel} . In this setup, the Boltzmann equation reduces to

$$\partial_\tau f(\tau, p) = \frac{f_{\text{eq}}(\tau, p) - f(\tau, p)}{\tau_{\text{rel}}}, \quad (9)$$

where f_{eq} is the equilibrium distribution function at an effective temperature set by the energy density $\mathcal{E} \sim T^4$. There is a large freedom in choosing τ_{rel} and to compare with the conformal theories above, we will assume $\tau_{\text{rel}} = \gamma T^{-1}$, where γ is a constant.

Before we go to Bjorken flow, let us recall the structure of two-point functions in RTA in thermal equilibrium, studied in [13, 14]. In Fig. 3, the analytic structure of the retarded sound channel correlator is shown. Compared to the simple nonhydrodynamic poles of the theories above, RTA has a branch cut corresponding to a decay rate given by τ_{rel} . This cut comes from the contributions of the microscopic particles building up the macroscopic wave. An interesting question here is how the branch cut in equilibrium manifests itself in the structure of transients in Bjorken flow. Moreover, given that the freedom of initial conditions amounts to specifying the distribution function, we should expect an infinite number of transients.

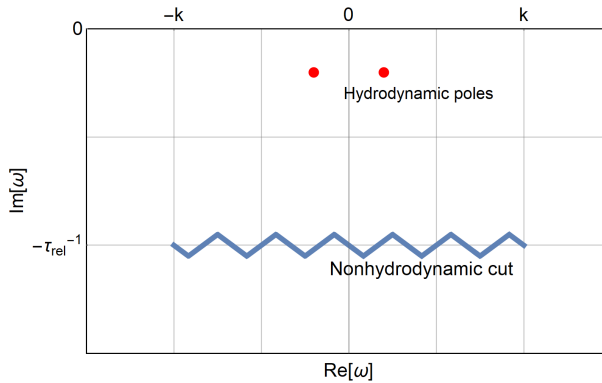


Fig. 3. The retarded correlator in the sound channel in equilibrium in RTA kinetic theory for momentum-independent relaxation time. In contrast to the single poles of BRSSS and $\mathcal{N} = 4$ SYM, the nonhydrodynamic structure in RTA involves a branch cut.

Using Eq. (9), one can derive an integral equation directly for the energy density [15, 16]

$$\mathcal{E}(\tau) = D(\tau, \tau_0) \mathcal{E}_0(\tau) + \frac{1}{2} \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\text{rel}}(\tau')} \mathcal{E}(\tau') D(\tau, \tau') H\left(\frac{\tau'}{\tau}\right), \quad (10)$$

where

$$D(\tau, \tau_0) = \exp \left[- \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\text{rel}}(\tau')} \right], \quad H(q) = q^2 + \frac{\arctan \sqrt{\frac{1}{q^2} - 1}}{\sqrt{\frac{1}{q^2} - 1}}, \quad (11)$$

and initial conditions are encoded in $\mathcal{E}_0(\tau)$. The function $H(q)$, which is to be evaluated between 0 and 1, does not look so interesting but it turns out that its analytic structure in the complex plane is of crucial importance

to interpret the gradient expansion. Equation (10) allows us to study the nonhydrodynamic modes in three different ways: by numerical solutions for various initial conditions, by the gradient expansion, and by directly plugging in an Ansatz for the transient modes.

Let us start with the gradient expansion. It is again divergent [17, 18], so let us look at the analytic structure of the Borel transform of the pressure anisotropy. The surprising result is shown in Fig. 4. On the real axis, we see a branch cut, whose decay rate matches the expectation from the branch cut in equilibrium. In addition, there are cuts off the axis, which in the previous cases signaled the presence of oscillatory nonhydrodynamic modes. However, this is not the case here! In this theory, this kind of resummation of the gradient expansion is misleading, as explained in Sec. 3.1.

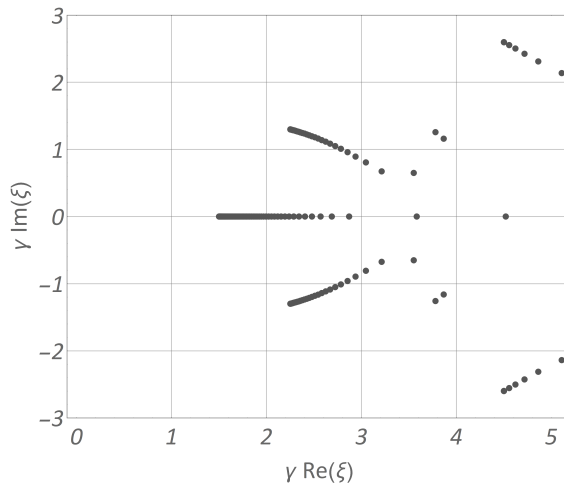


Fig. 4. Poles of the Borel transformed pressure anisotropy $\mathcal{A}_B(\xi)$ for RTA in boost-invariant flow. The branch cut on the real axis represents an infinite number of decaying nonhydrodynamic modes, distinguished by different power laws. The off-axis structures are not physical modes, but arise from imaginary time contours in Eq. (10).

Given the three branch cuts closest to the origin, the subleading ones can be understood as nonlinear combinations of the first three. With two of these unphysical, one can only identify a single physical nonhydrodynamic mode. Where are the rest that are needed to represent the initial conditions?

This becomes clear when plugging in the Ansatz $e^{-Sw}w^\beta$ and determining the possible values of S and β . One finds only one decay rate possible, $S = \frac{3}{2\gamma}$. What saves us, however, is the power law β , which allows for an infinite set of solutions. Interestingly, all except the leading power law have a non-zero imaginary part. This implies that one will get oscillatory

behaviour in $\log(w)$ as

$$\Re(\sigma w^\beta) \propto w^{\Re(\beta)} \cos(\theta + \Im(\beta) \log(w)). \quad (12)$$

In Fig. 4, these modes which are distinguished by different power laws are stacked on top of each other.

Lastly, Eq. (10) allows for very accurate numerical solutions from which one can verify these conclusions. Since the gradient expansion represents something independent of initial conditions, given two different solutions, the difference between them will be purely transient. In this way, one can confirm the presence of at least the two leading transients, and that the oscillatory modes do not contribute.

3.1. Unphysical modes

The gradient expansion in RTA shows unphysical modes in its Borel transform. The reason can be understood from the structure of Eq. (10). In this integral equation, we have not yet explicitly mentioned what the contour between τ_0 and τ is supposed to be. Of course, we understand that it should be along the real axis, but when evaluating the gradient expansion no use is made of the specific contour. This leads to the possibility that the gradient expansion can be sensitive to nonhydrodynamic modes related to different choices of the contour. Indeed, as we showed in Ref. [18], the poles of the function $H(\tau'/\tau)$ in the complex plane conspire with the damping function $D(\tau, \tau_0)$ to produce precisely the off-axis cuts in Fig. 4. To see such modes in the solutions to the equation, one would have to consider paths in imaginary time. However, they still contribute to physical quantities in that their presence is seen in the gradient expansion, which when truncated at low orders serves as a good approximation at late times. This is analogous to ghost instantons in quantum field theory [19].

3.2. Non-conformal relaxation time

It is not so difficult to generalize the previous results to a relaxation time of the form of $\tau_{\text{rel}} \propto T^{-\Delta}$ for $\Delta < 3$ [20]. S and β shift around slightly, the cuts moving around in the Borel plane but with no qualitative change in the structure, see Fig. 5. As $\Delta \rightarrow 3$, what used to be the late time perfect fluid solution, $T \sim \tau^{-1/3}$ becomes unstable.

To summarize, the nonhydrodynamic contributions to $\mathcal{E}(\tau)$ takes the form of

$$e^{-\frac{w}{1-\Delta/3}} w^{\beta + \frac{4\Delta}{45(1-\Delta/3)^2}}, \quad (13)$$

where β satisfies $M(\beta(1 - \Delta/3) - \Delta/3) = 0$ and

$$M(z) = \frac{{}_3F_2\left(1, \frac{z}{2} + 2, \frac{z}{2} + 2; \frac{z}{2} + \frac{5}{2}, \frac{z}{2} + 3; 1\right)}{2z^2 + 14z + 24} + \frac{1}{2(z + 4)}. \quad (14)$$

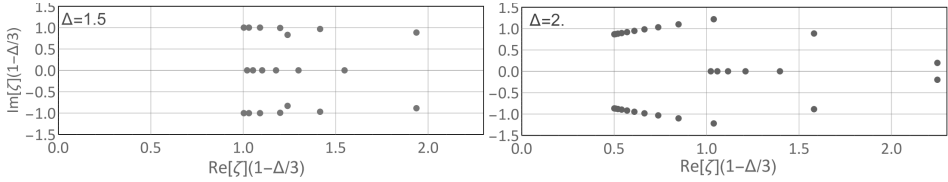


Fig. 5. Borel plane in RTA for nonconformal relaxation time $\tau_{\text{rel}} \propto T^{-\Delta}$. As $\Delta \rightarrow 0$, the real nonhydrodynamic mode becomes more and more damped, but the unphysical ones approach the origin. At $\Delta = 3$, the late time perfect fluid solution becomes unstable.

4. Summary

Non-analytic features of correlators in thermal equilibrium can be divided into two classes: long-lived hydrodynamic features and transient nonhydrodynamic contributions. The latter acts as a regulator of acausal modes, sets the rate of divergence of the gradient expansion and parametrizes initial information that is lost in the hydrodynamic regime. Modes in thermal equilibrium map to modes in Bjorken flow and they set the timescale when some universal behaviour takes over.

The techniques of Borel resummation that work well for BRSSS and $\mathcal{N} = 4$ SYM turn out to be misleading in kinetic theory in RTA. Analytic features of the equations lead to unphysical modes that do not contribute to physical observables. The branch cut structure of RTA (see Fig. 3) leads to a set of transients that are distinguished by power laws but not by decay rate. Both of these effects complicate the program of extracting nonperturbative information through resurgence.

In the expanding quark–gluon plasma, where the transition to the hydrodynamic stage is of great interest, such nonhydrodynamic features may play an important role. When designing a hydrodynamic theory that will be used close to its limits (such as large gradients or small systems [21]), it may not be enough to only tune transport coefficients, but one should also model the nonhydrodynamic sector. In this contribution, I have described this sector for a few simple, although very different, theories. Going beyond these toy models, nonhydrodynamic modes have been investigated in RTA with momentum-dependent relaxation time [14] and in EKT for ϕ^4 [22].

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