TOLMAN–BONDI–LEMAÎTRE SPACETIME WITH A GENERALISED CHAPLYGIN GAS

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(Received March 28, 2019; accepted October 8, 2019)

The Tolman–Bondi–Lemaître-type of inhomogeneous spacetime with generalised Chaplygin gas equation of state given by $p = -\frac{A}{\rho^\alpha}$, where α is a constant, is investigated. We get an inhomogeneous spacetime at the early stage but at the late stage of the universe, the inhomogeneity disappears with suitable radial co-ordinate transformation. For the large scale factor, our model behaves like ΛCDM-type which is in accord with the recent WMAP studies. We have calculated $\frac{\partial\rho}{\partial r}$ and it is found to be negative for $\alpha > 0$ which is in agreement with the observational analysis. A striking difference with Chaplygin gas ($\alpha = 1$) lies in the fact that with any suitable co-ordinate transformation, our metric cannot be reduced to the Einstein–de Sitter type of homogeneous spacetime in dust distribution as it is possible for the Chaplygin gas. We have also studied the effective deceleration parameter and find that the desired feature of *flip* occurs at the late universe. It is seen that the flip time depends explicitly on α. We also find that flip is not synchronous occurring earlier at the outer shells, thus offering a natural path against occurrence of the well-known shell crossing singularity. This is unlike the Tolman–Bondi case with perfect gas, where one has to impose stringent external conditions to avoid this type of singularity. We further observe that if we adopt separation of variables method to solve the field equations, the inhomogeneity in matter distribution disappears. The whole situation is later discussed with the help of Raychaudhury equation and the results are compared with previous cases. This work is the generalisation of our previous article, where we have taken $\alpha = 1$.

DOI:10.5506/APhysPolB.50.1555

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1. Introduction

From the growing number of observational data of high-redshift and luminosity-distance relation of type IA supernovae in the last decade [1, 2], we know that when they are interpreted in the framework of Einstein's field equations and the standard FRW type of universe (homogeneous and isotropic), we are left with the only alternative that the universe is currently passing through an accelerated phase of expansion where baryonic matter paradoxically contributes only four percent of the total energy budget. Moreover, if we have faith in Einstein's theory, the FRW model dictates that one should hypothesize at once a peculiar and rather unphysical type of matter field (DE) [3] with a very large negative pressure clearly violating the energy conditions, to explain the late acceleration.

In the existing literature, a fairly good number of DE models are proposed, but very little is precisely suggested about the nature and origin of it. Nowadays, the DE problem remains one of the major open problems of theoretical physics [4]. On the way of searching for possible solutions of this problem, various models have been explored during the last few decades, referring to new exotic forms of matter, *e.g.*, quintessence [5, 6], phantom [7, 8], holographic models [9], string theory landscape [10, 11], the Born–Infeld quantum condensate [12], the modified gravity approaches [13, 14], inhomogeneous spacetime [15, 16], various types of higher dimensional theories *etc.* (readers interested in more detail may refer to [17–20] for a comprehensive overview of existing theoretical models). The one which attracted huge attention is the Chaplygin gas (CG) inspired model [21–24], obeying an EoS, $p = -\frac{A}{\rho}$. Although the model is very successful in explaining the SNe Ia data, it shows that the CG fails to explain the tests connected with structure formation and observed strong oscillations of matter power spectrum [25]. To overcome the problems, it is generalized (GCG) [26] with the addition of an arbitrary constant as

$$p = -\frac{A}{\rho^{\alpha}}\,, \tag{1}$$

where both $A(A > 0)$ and α are constants. Here, α is constrained in the range of $0 < \alpha < 1$ in order to have an acoustic speed that is at the most luminal for perturbation [27] and also for the best fit with observations [28–30]. Another bottleneck stems from the fact that the basic inferences from the ΛCDM and GCG are essentially the same and thus one cannot chose between the two from the experimental angle. One more point of concern is the fact that the accelerating phase coincides with the period when the inhomogeneities in the matter distribution at length scales < 10 Mpc become significant so that the universe can no longer be approximated as homogeneous at these scales. Moreover, one may point out that homogeneity and isotropy of the geometry are not essential ingredients to establish a number of relevant results in relativistic cosmology. One need not be too sacrosanct about these concepts so as to sacrifice basic physics (energy conditions, for example) in relativistic cosmology. On the other hand, if we forgo the concepts of homogeneity and isotropy, the observational data do not force us to imply an accelerating expansion of the universe, or even if the cosmic expansion is accelerating, it does not necessarily point to an existence of a dark energy. Thus, a parallel line of activities has emerged to explain the observational findings without introducing the concept of dark energy. A community of activists have started a sort of 'mission' to explain (sometimes with conflicting claims) the observational findings within inhomogeneous models. Given the complexities involved in dealing with inhomogeneous models, the simplest generalisation of FRW spacetime is the well-known Tolman–Bondi–Lemaître model which is also spherically symmetric but the spacetime is inhomogeneous and the acceleration is supposedly caused by the back reaction effects due to the inhomogeneities in the background FRW universe. It was shown that from observational point of view, their [15, 20] results become very similar to the predictions of CDM model.

The motivation for the present work may be summed up as follows: As pointed out earlier that following the discovery of the late acceleration of our cosmos and the subsequent inability of the standard models to explain the phenomenon within the context of Einstein's theory with standard perfect fluid, there have been a proliferation of proposals to reintroduce the idea of a cosmological constant, a quintessential field, higher dimensional theories, higher derivative models *etc.* However, all these suffer from the disqualification that the exotic fluid violate energy conditions and are also not physically viable.

An alternative line of approach is to address the problem in the realm of inhomogeneous cosmology, such that the back reaction coming from the extra terms due to inhomogeneity may trigger and drive the acceleration without being forced to invoke the presence of any exotic fluid and a vigorous search for compatibility of late acceleration with inhomogeneous model ensued. However, the journey is not free from controversies and failures. Returning to the idea of back reaction, Kolb *et al.* [31] argued, using perturbative techniques, that when observed from the centre of perturbation, the expansion rate is large and sometimes may accelerate. The work later got credence from similar analysis of Wiltshire [32] and Carter *et al.* [33] where the universe is modeled as underdense bubble in an Einstein–de Sitter universe and predictions tally with those of ΛCDM. However, it is later pointed out [34, 35] that the claim is seriously flawed as domain of validity of perturbation is extrapolated to a regime where perturbative analysis breaks down as also constraints are violated.

Thus, it points to the fact that acceleration cannot be explained with the help of inhomogeneities alone. Therefore, we have thought it fit to explore the phenomenon of late acceleration in inhomogeneous model with the help of now popular Chaplygin gas to see if the two are compatible *i.e.* if one can explain acceleration in this framework too.

As is common in all Chaplygin-types of models, our field equations are amenable to closed form solutions only at the extremal cases. Unlike the FRW models, all the physical parameters are here both space- and time-dependent and all our solutions reduce to our earlier work [23] when $\alpha = 1$.

The organization of work is as follows: in Section 2, we write the field equations of our inhomogeneous spacetime with a generalized Chaplygin gas as a matter field and find the detail solutions in Section 3. The solution described by our equation (25) is unique and may be termed as the generalised Einstein–de Sitter metric (ED) and one cannot directly revert to the well-known ED metric with any coordinate transformation. At the late stage of evolution, we get the solution similar to ΛCDM model. We also calculate the acceleration flip in our spacetime, which depends both on space and time. In contrast to the homogeneous case, the flip is evidently not synchronous. Each shell characterised by an r-constant hypersurface has its own instant of flip.

For any inhomogeneous dynamics, we come across two important singularities — shell crossing and shell focusing. We have noted that in our case, shells with higher value of r start accelerating earlier and thus shell crossing singularity is naturally avoided. For completeness, we contrast our inferences with those obtained from the Raychawdhury equation [36], and the paper ends with a brief discussion in Section 4.

2. Field equations and its integrals

$$\mathrm{d}s^2 = \mathrm{d}t^2 - e^{\lambda(t,r)}\, \mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) , \tag{2}$$

where the scale factor, $R(t,r)$ depends on both time and space coordinates (t,r) respectively. As inhomogeneous equations in GTR are, in general, very difficult to solve analytically we assume for mathematical simplicity that $g_{00} = 1$.

In comoving coordinate system, the energy momentum tensor for the above defined coordinates is given by

$$T^{\mu}_{\nu} = (\rho + p)\delta^{\mu}_{0}\delta^{0}_{\nu} - p\delta^{\mu}_{\nu} , \tag{3}$$

where $\rho(t,r)$ is the matter density and $p(t,r)$ is the pressure. The fluid consists of successive shells marked by r, whose local density is time-dependent over the successive hypersurfaces. The function $R(t,r)$ describes the location of the shells characterized by r at the time t. Einstein's field equations, subject to the rescaled gauge,

$$R(0,r) = r\,, \tag{4}$$

gives the following independent equations for metric (2) and energy momentum tensor (3) as

$$-\frac{e^{-\lambda}}{R^2}\left(2RR'' + R'^2 - RR'\lambda'\right) + \frac{1}{R^2}\left(R\dot{R}\dot{\lambda} + \dot{R}^2 + 1\right) = \rho\,, \tag{5}$$

$$-e^{-\lambda}\frac{R'^2}{R^2} + \frac{1}{R^2}\left(2R\ddot{R} + \dot{R}^2 + 1\right) = -p\,, \tag{6}$$

$$\frac{e^{-\lambda}}{R^2}\left(2RR'' + R'^2 - RR'\lambda'\right) + \frac{1}{R^2}\left(R\dot{R}\dot{\lambda} + \dot{R}^2 + 1\right) = -p\,, \tag{7}$$

$$2\dot{R}' - \dot{\lambda}R' = 0\,. \tag{8}$$

Here, prime and a dot overhead denote space and time derivative respectively.

Solving equation (8), we get

$$e^{\frac{\lambda(t,r)}{2}} = \frac{R'}{f(r)}\,, \tag{9}$$

where $f(r)$ is an arbitrary function of r such that $f(r) > 0$.

Since the WMAP and other recent data [37, 38] point to a nearly flat universe in the current era, we take $f(r) = 1$ such that the field equations finally reduce to the following two independent equations as:

$$\frac{\dot{R}^2}{R^2} + 2\frac{\dot{R}'}{R'}\frac{\dot{R}}{R} = \rho\,, \tag{10}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = -p\,. \tag{11}$$

The conservation equation leads to

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{1}{e^{\frac{\lambda}{2}}R^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{\lambda}{2}}R^2\right)(\rho + p) = 0\,. \tag{12}$$

For our case, we take a matter field, given by equation (1) along with (12) and we get

$$\dot{\rho} + \frac{1}{e^{\frac{\lambda}{2}}R^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{\lambda}{2}}R^2\right)\left(\rho - \frac{A}{\rho^{\alpha}}\right) = 0 \tag{13}$$

which, on integration, gives

$$\rho = \left[A + \frac{C(r)}{\left(e^{\frac{\lambda}{2}} R^2\right)^{1+\alpha}} \right]^{\frac{1}{1+\alpha}} , \tag{14}$$

where $C(r)$ is a function of integration. Now, putting equation (9), we get

$$\rho = \left[A + \frac{C(r)}{(R' R^2)^{1+\alpha}} \right]^{\frac{1}{1+\alpha}} . \tag{15}$$

With the help of equation (10), we finally obtain

$$\frac{\dot{R}^2}{R^2} + 2\frac{\dot{R}'}{R'}\frac{\dot{R}}{R} = \left[A + \frac{C(r)}{(R' R^2)^{1+\alpha}} \right]^{\frac{1}{1+\alpha}} . \tag{16}$$

This is the main equation in our future analysis but unlike the homogeneous models, $C(r)$ also depends on space. As it is well-known, the resulting field equations with the Chaplygin-type of matter field do not, in general, offer any closed type of solutions and in what follows, we see that we have to study some extremal cases only. Following Moffat [39], the present authors, in an earlier communication [23], have taken the expression of Hubble parameter as

$$H = \frac{2}{3} H_\perp + \frac{1}{3} H_r , \tag{17}$$

where

$$H_\perp = \frac{\dot{R}}{R} \tag{18}$$

and

$$H_r = \frac{\dot{R}'}{R'} \tag{19}$$

which may be taken as a measure of the local expansion rate in the perpendicular and radial directions, respectively. Now, we can write the deceleration parameter

$$q_\perp = -\frac{1}{H_\perp^2}\frac{\ddot{R}}{\dot{R}} . \tag{20}$$

From equation (15), another important physical quantity, $\frac{\partial \rho}{\partial r}$ (a sort of measure of inhomogeneity), comes out to be

$$\rho' = \frac{\partial \rho}{\partial r} = -\frac{C(r)}{1+\alpha} \frac{(1+\alpha)\left(\frac{R''}{R'} + 2\frac{R'}{R}\right) - \frac{C'}{C(r)}}{\left(R^2 R'\right)^{1+\alpha} \rho^\alpha} . \tag{21}$$

For realistic mass distribution, $\rho' < 0$ implying

$$(1+\alpha)\left(\frac{R''}{R'}+2\frac{R'}{R}\right) > \frac{C'}{C(r)}\,. \tag{22}$$

If we consider $C(r)$ to be a true constant, then from equation (21), we see that $\rho' < 0$ as expected. Otherwise, we have to know the form of $C(r)$ to get an idea regarding the negativity of ρ'. We have chosen here two simple forms of $C(r)$ as *(i)* power law and *(ii)* exponential to check the negativity of ρ' in the next section.

3. Solutions

As pointed out earlier, parent equation (16) admits hypergeometric solutions only in general. So we have to take some special cases only.

3.1. Case A: $R(t,r)$ is very small

When the scale factor $R(t,r)$ is relatively small, *i.e.*, at the early stage of the universe, from equation (16) we get dust-dominated universe for $C(r) = \left(\frac{4}{3}\alpha r^{3\alpha-1}\right)^{1+\alpha}$ yielding

$$R(t,r) = r^{\alpha}\left[t+t_0(r)\right]^{\frac{2}{3}}\,, \tag{23}$$

where $t_0(r)$ is an arbitrary function of integration depending on r.

With this expression of $R(t,r)$, the pressure vanishes. Moreover, for isotropic expansion ($e^{\frac{\lambda}{2}} = R$), we get $\rho \sim \frac{1}{R^3}$ (in an r-constant hypersurface) as in the FRW universe. Interestingly, expression (23) is not exactly Tolman–Bondi-like because we are dealing with a generalised Chaplygin gas-type exotic fluid and our line element reduces to

$$\mathrm{d}s^2 = \mathrm{d}t^2 - r^{2(\alpha-1)}\left[t+t_0(r)\right]^{\frac{4}{3}}\left\{\alpha^2\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\right\}\,. \tag{24}$$

If we further assume that $t_0(r)$ vanishes or becomes a *true* constant (in that case a time translation is necessary), then we get

$$\mathrm{d}s^2 = \mathrm{d}t^2 - r^{2(\alpha-1)}t^{\frac{4}{3}}\left\{\alpha^2\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\right\}\,. \tag{25}$$

The spacetime described by equation (25) is unique and one may look upon it as a modified Einstein–de Sitter metric for the inhomogeneous spacetime. There is a striking difference between the spacetime described by equation (25) and that in our work [23] referred to earlier for $\alpha = 1$. In our previous work with pure Chaplygin gas ($\alpha = 1$), the additional assumption of $t_0(r) = 0$ reduces the metric to a homogeneous Einstein–de Sitter case with dust distribution in the flat space ($R^{\frac{2}{3}}$). However here, $t_0(r) = 0$ does not reduce the metric to any homogeneous form. For that, we need an additional assumption of $\alpha = 1$. Therefore, the generalised Chaplygin gas does not admit any homogeneous distribution in the Tolman–Bondi metric. From equation (15), we get the expression of density as

$$\rho(t,r) \approx \frac{\sqrt{C}(r)}{(R'R^2)^{1+\alpha}} = \frac{4\alpha}{3r\,[t+t_0(r)]\left[\alpha\frac{\{t+t_0(r)\}}{r}+\frac{2}{3}t_0'\right]}\,. \tag{26}$$

If we calculate the deceleration parameter $q_\perp$ using equations (20) and (23), we get $q_\perp = \frac{1}{2}$ implying a dust-dominated universe. From equation (26), we have checked the signature of ρ' given by

$$\rho' = -\frac{8\alpha\left[(3\alpha+1)\{t_0(r)+t\}\,t_0'(r) + rt_0'(r)^2 + r\{t_0(r)+t\}\,t_0''(r)\right]}{\{t_0(r)+t\}^{\,2}\{3\alpha t_0(r)+2rt_0'(r)+3\alpha t\}^{\,2}}\,. \tag{27}$$

Equation (27) shows that ρ' is always negative for positive value of α as desired. This equation further ensures that α should be greater than zero.

3.2. Case B: $R(t,r)$ is very large

Type 1: In the late stage of evolution, the second term of the RHS of equation (16) vanishes and we get

$$\frac{\dot{R}^2}{R^2} + 2\frac{\dot{R}'}{R'}\frac{\dot{R}}{R} = A^{\frac{1}{1+\alpha}}\,. \tag{28}$$

(a) A straightforward integration of equation (28) gives $R(t,r)$ as

$$R(t,r) = R_0\exp\left[\sqrt{\frac{A^{\frac{1}{1+\alpha}}}{3}}(t+r)\right]\,. \tag{29}$$

This is the well-known de Sitter-type of solution generalised to inhomogeneous spacetime with $A^{\frac{1}{1+\alpha}}$ simulating as Λ, the cosmological constant. It may be pointed out at this stage that the beauty of the idea of the Chaplygin gas lies in the fact that it unifies both the dark matter and dark energy concept in different limits producing an early dust-dominated and an accelerating phase at the late stage of the evolution. It is found that this late stage expansion mimics the ΛCDM model. At this stage, a comparison to an earlier work of Moffat [40] of the LTB model with cosmological constant may be relevant. Our key equation (16) yields solution (29) for large scale factor which is strikingly similar to the Moffat result [40]. However, the essential difference lies in the fact that while Moffat assumed *a priori* a cosmological constant in his analysis, in our case, it manifests itself at a late stage of evolution. Moreover, a simple radial coordinate transformation

$$\bar{r} = R_0 \exp\left[\sqrt{\frac{A^{\frac{1}{1+\alpha}}}{3}} r\right] \tag{30}$$

reduces metric (2) to

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \exp\left(2\sqrt{\frac{A^{\frac{1}{1+\alpha}}}{3}}\, t\right)\left\{\mathrm{d}\bar{r}^2 + \bar{r}^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right\} . \tag{31}$$

At the late stage of evolution, it is seen that with suitable transformation of radial co-ordinate (equation (30)), we get the de Sitter-type metric with homogeneous spacetime. Thus, it may be concluded that for large $R(t,r)$, the inhomogeneity may disappear as expected.

One can also see from equation (15) that for the late universe

$$\rho \simeq A^{\frac{1}{1+\alpha}} + \frac{C(r)}{(1+\alpha)A^{\frac{\alpha}{1+\alpha}}}\frac{1}{(R'R^2)^{1+\alpha}} , \tag{32}$$

$$p \simeq -A^{\frac{1}{1+\alpha}} + \frac{\alpha}{1+\alpha}\frac{C(r)}{A^{\frac{\alpha}{1+\alpha}}}\frac{1}{(R'R^2)^{1+\alpha}} . \tag{33}$$

This may be viewed as a combination of a cosmological constant $A^{\frac{1}{1+\alpha}}$ with a type of matter representing a ΛCDM model. Moreover, in the asymptotic limit ($R \sim \infty$), we get $p = -\rho = -A^{\frac{1}{1+\alpha}}$ for this Chaplygin type of gas, corresponding to an empty universe with a cosmological constant.

In this case, the deceleration parameter $q_\perp = -1$, which shows an acceleration at the late stage. Now, we can calculate ρ' using equation (32) and we get

$$\frac{\partial\rho}{\partial r} = -\frac{C(r)}{(1+\alpha)A^{\frac{\alpha}{1+\alpha}}}\frac{1}{(R'R^2)^{1+\alpha}}\left\{(1+\alpha)\left(\frac{R''}{R'} + 2\frac{R'}{R}\right) - \frac{C'}{C(r)}\right\} \tag{34}$$

which is consistent with inequality condition (22) for $\rho' < 0$. With the help of equation (29), condition (22) reduces to $\sqrt{3}(1+\alpha)A^{\frac{1}{2(1+\alpha)}} > \frac{C'}{C(r)}$. Since $C(r)$ is a positive integration constant, it may be true constant or may be a function of r. If the integration constant $C(r) \equiv C$ is a true constant, then $\rho' < 0$. On the other hand, if $C(r)$ depends on r such that $C(r) \propto e^{\gamma r}$, which gives $\sqrt{3}(1+\alpha)A^{\frac{1}{2(1+\alpha)}} > \gamma$ and under this condition $\rho' < 0$.

(b) Alternatively, one may also get another type of solution of (28) as

$$R(t,r) = R_0 \sinh^{\frac{2}{3}} w(t+r)\,, \tag{35}$$

where $w = \frac{\sqrt{3}}{2}A^{\frac{1}{2(1+\alpha)}}$. Unlike the previous work [23], this result does not contain any explicit reference of α, being absorbed in the expression of w.

Now, using equations (20) and (35), we get the deceleration parameter as

$$q_\perp = \frac{3}{2}\,\mathrm{sech}^2\, w(t+r) - 1\,. \tag{36}$$

Figure 1 shows that the *flip* occurs early at greater value of r, *i.e.*, velocity increases for greater r. The flip time τ_c can be calculated from equation (36) when $q_\perp = 0$ and we get

$$\tau_c = \frac{2}{\sqrt{3}}A^{-\frac{1}{2(1+\alpha)}}\,\mathrm{sech}^{-1}\left(\sqrt{\frac{2}{3}}\right) - r\,. \tag{37}$$

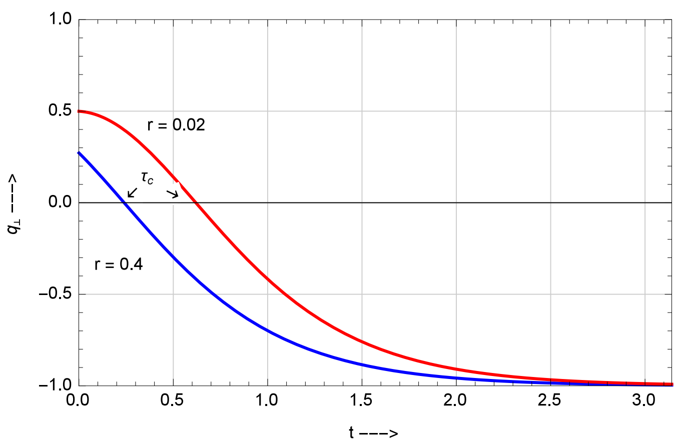


Fig. 1. The variation of $q_\perp$ *vs.* t is shown taking $A = 2$ and $\alpha = 1$.

As expected, the flip time (τ_c) explicitly depends on α. The variation of τ_c with α depends on magnitude of A. If $A > 1$, the τ_c increases as α increases, *i.e.*, late flip for large α, on the other hand, for $A < 1$, *i.e.*, the conclusion is just the reverse. For $A = 1$, τ_c is independent of α for r-constant hypersurface. The variation of τ_c with α for different values of A is shown in figure 2.

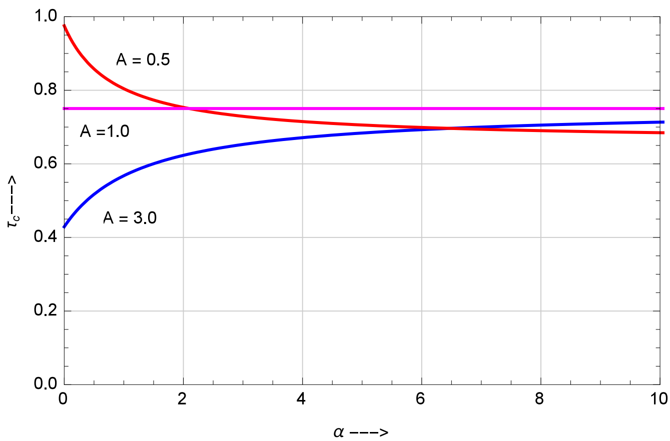


Fig. 2. τ_c with α for different value od A are shown taking $r = 0.01$.

Another important conclusion coming out of equation (37) has not escaped our notice. As is customary in any inhomogeneous evolutions, this equation shows that all physical quantities including instant of flip depend on both space and time co-ordinate. So each shell characterised by an r-constant hypersurface has its own flip time. Moreover, we further observe that shells with higher values of r start accelerating earlier than those with lower values of r. This is a good news because it avoids the well-known shell crossing singularity associated with any inhomogeneous evolution. This is unlike the Tolman–Bondi case with perfect gas, where one has to impose stringent external conditions to avoid this type of singularity.

Next, we have to check the signature of ρ'. Using condition (22), we may write $(1+\alpha)\{\tanh w(t+r) + \coth w(t+r)\} > \gamma$ for $\rho' < 0$, where we have taken $C(r) = e^{\gamma r}$.

Type 2: Now, we attempt to solve equation (16) using the method of separation of variables. Let $R(t,r) = a(t)g(r)$. From equation (16), we get

$$3\frac{\dot{a}^2}{a^2} = \left(A + \frac{B}{a^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}}, \tag{38}$$

where

$$B=\frac{C(r)}{\left(g'g^2\right)^{1+\alpha}}\,. \tag{39}$$

However, the LHS of equation (38) depends on time only which dictates that B must be a true constant.

Then, using equations (21) and (39), a long but straightforward calculation shows that $\rho'=0$ (C may be a function of r or a true constant), implying that the matter field is homogeneous in this case. May not be out of space to point out that one of the authors discussed, *albeit* in a different context, the same situation and got similar results [41].

3.3. Temporal solution

Equation (38) gives the hypergeometric solution of $a(t)$ with t. The solution and other features are the same as homogeneous case [20] at the late stage of evolution, *i.e.*, $a(t)$ is large in this case and equation (38) becomes (neglecting higher order terms)

$$3\frac{\dot a^2}{a^2}=A^{\frac{1}{1+\alpha}}+\frac{B}{(1+\alpha)A^{\frac{\alpha}{1+\alpha}}}a^{-3(1+\alpha)}\,. \tag{40}$$

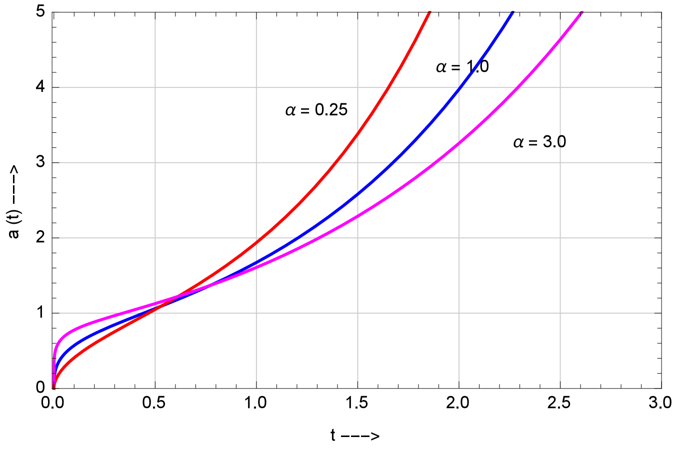


Fig. 3. The variation of $a(t)$ *vs.* t is shown taking $A=5$ and $B=5$.

Solving equation (40), we get the solution

$$a(t)=a_0\sinh^m\omega t\,, \tag{41}$$

where $a_0=\left\{\frac{B}{A(1+\alpha)}\right\}^{\frac{1}{3(1+\alpha)}}$; $m=\frac{2}{3(1+\alpha)}$ and $\omega=\frac{\sqrt{3}}{2}(1+\alpha)A^{\frac{1}{2(1+\alpha)}}$.

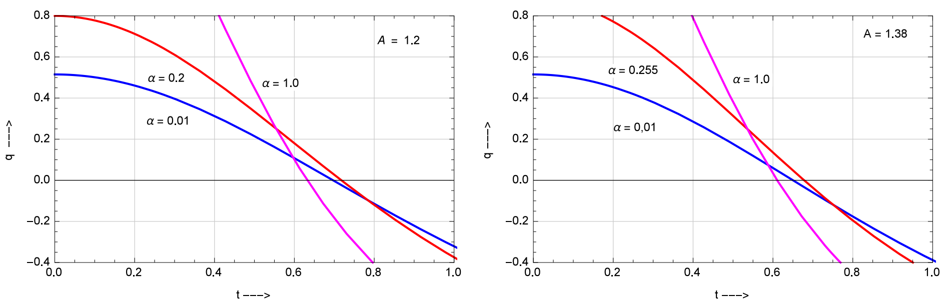
From equation (41), we get the deceleration parameter

$$q = \frac{1 - m\cosh^2 \omega t}{m\cosh^2 \omega t}\,. \tag{42}$$

Equation (42) shows that the exponent m determines the evolution of q. A little analysis of equation (42) shows that *(i)* if $m > 1$, we get only acceleration, no *flip* occurs in this condition, but for $m > 1$, it gives $-\frac{1}{3} > \alpha$, which is physically unrealistic, since previously we have shown $\alpha > 0$. *(ii)* Again, if $0 < m < \frac{2}{3}$, it gives early deceleration and late acceleration and in this condition $\alpha > 0$, so the desirable feature of *flip* occurs which agrees with the observational analysis for a positive value of α.

Figure 4 shows the variation of q with t for different values of α where flip occurs. It is seen that the flip time (t_c) is different for different values of α but this change is not monotonous. We would like to focus on the occurrence of late flip because all observational evidences suggest that accelerating phase is a recent phenomena. It is interesting to note that the late flip also depends on the value of A. In figure 4, we have taken two values of A, where we get the maximum t_c for corresponding value of α, *e.g.*, for $A = 1.2$, we get the $(t_c)_{\max}$ at $\alpha = 0.20$ and for $A = 1.38$, it comes out to be $\alpha = 0.255$. In this context, correspondence to an earlier work of Campo [28] is relevant where he also got similar results while dealing with Generalised Chaplygin gas. It is interesting to mention that we also got similar results in our earlier work [30] although in a different context. From figure 4, we find that flip occurs later at this range of α in conformity with observational analysis. In this case, the *flip* time (t_c) will be

$$t_c = \frac{1}{\omega}\cosh^{-1}\left(\sqrt{\frac{1}{m}}\right)\,. \tag{43}$$



(a) (b)

Fig. 4. The variation of q and t for different values of α with $B = 1$. (a) Figure shows that the maximum value of t_c at $\alpha = 0.2$; (b) t_c becomes maximum at $\alpha = 0.255$.

Using equation (43), we have drawn figure 5 where the variation of t_c with α for different value of A is shown. It is seen that the variation of t_c with α is not monotonous. When the value of α is small, t_c increases with α; after a certain value of α, t_c decreases as α increases. That means we get a maximum value of t_c for a different value of A. As a trial case, we see the following data table where we have seen the maximum value of t_c for a different value of A with corresponding α.

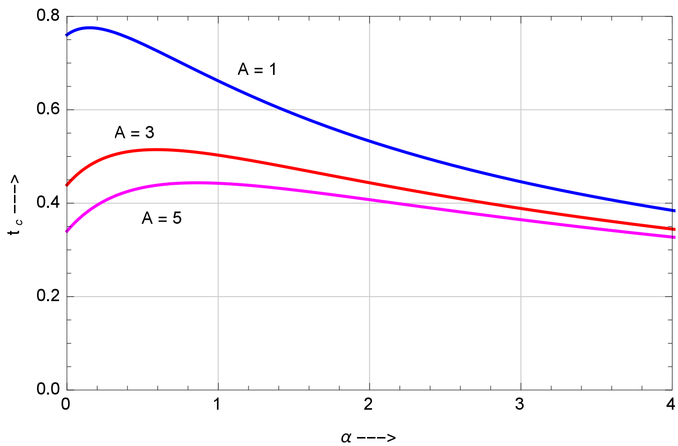


Fig. 5. The graphs clearly show that flip time depends on α.

From Table I, we find that the value of $(t_c)_{\max}$ is larger for the smaller value of A. From the observational point of view, it has previously been seen that this corresponds to a value of $\alpha \sim 0.25$. The table further shows that for $\alpha = 0.255$, the $(t_c)_{\max}$ will be 0.6780 when we consider the value of $A = 1.38$.

TABLE I

Table for α and t_c.

A	1.2	1.3	1.35	1.38	1.4
$(t_c)_{\max}$	0.7177	0.6945	0.6840	0.6780	0.6742
α	0.200	0.233	0.246	0.255	0.260

3.4. Radial solution

$$g(r) = \left\{ \frac{3}{B^{\frac{1}{1+\alpha}}} \int C(r)^{\frac{1}{1+\alpha}} \mathrm{d}r \right\}^{\frac{1}{3}} . \tag{44}$$

Now, we have two options: *(i)* $C(r)$ is a function of r only and *(ii)* $C(r)$ is a true constant.

(i) We may choose the simplest form of $C(r)$:

(a) $C(r) = r^{\beta}$, where β is a constant. Equation (44) reduces to

$$\int C(r)^{\frac{1}{1+\alpha}} \mathrm{d}r = \frac{1+\alpha}{1+\alpha+\beta} r^{\frac{1+\alpha+\beta}{1+\alpha}}, \tag{45}$$

and we get

$$g(r) = \left\{ \frac{3(1+\alpha)}{(1+\alpha+\beta)B^{\frac{1}{\alpha+1}}} \right\}^{\frac{1}{3}} r^{\frac{1+\alpha+\beta}{3(1+\alpha)}}. \tag{46}$$

(b) $C(r) = e^{\gamma r}$, where γ is a constant

$$\int C(r)^{\frac{1}{1+\alpha}} \mathrm{d}r = \frac{1+\alpha}{\gamma} e^{\frac{\gamma r}{1+\alpha}} \tag{47}$$

which gives

$$g(r) = \left\{ \frac{3(1+\alpha)}{\gamma B^{\frac{1}{\alpha+1}}} \right\}^{\frac{1}{3}} e^{\frac{\gamma r}{3(1+\alpha)}}. \tag{48}$$

(ii) When $C(r)$ is a true constant, *i.e.*, $C(r) \equiv C$, the expression of $g(r)$ is given by

$$g(r) = 3^{\frac{1}{3}} \left(\frac{C}{B} \right)^{\frac{1}{3(1+\alpha)}} r^{\frac{1}{3}}. \tag{49}$$

3.5. General solution

Now, the general solution will be

$$R(t,r) = \left[\frac{3}{\{A(1+\alpha)\}^{\frac{1}{1+\alpha}}} \int C(r)^{\frac{1}{1+\alpha}} \mathrm{d}r \right]^{\frac{1}{3}} \sinh^{m} \omega t. \tag{50}$$

Using equations (45), (47) and (50), we can write the general solution in the following form:

(i) $C(r)$ is a function of r:

(a) $C(r)=r^{\beta}$:

$$R(t,r)=\left\{\frac{3}{(1+\alpha+\beta)}\right\}^{\frac{1}{3}}\left\{\frac{(1+\alpha)^{\alpha}}{A}\right\}^{\frac{1}{3(1+\alpha)}} r^{\frac{1+\alpha+\beta}{3(1+\alpha)}}\sinh^{m}\omega t\,. \tag{51}$$

(b) $C(r)=e^{\gamma r}$:

$$R(r,t)=\left\{\frac{3}{\gamma}\right\}^{\frac{1}{3}}\left\{\frac{(1+\alpha)^{\alpha}}{A}\right\}^{\frac{1}{3(1+\alpha)}} e^{\frac{\gamma r}{3(1+\alpha)}}\sinh^{m}\omega t\,, \tag{52}$$

and

(ii) $C(r)\equiv C$:

$$R(r,t)=3^{\frac{1}{3}}\left\{\frac{C}{A(1+\alpha)}\right\}^{\frac{1}{3(1+\alpha)}} r^{\frac{1}{3}}\sinh^{m}\omega t \tag{53}$$

when we put $\beta=0$ into equation (51), $C(r)$ becomes constant (unity) and equations (51) and (53) are identical.

If we calculate both $q_{\perp}$ and t_c, we get the same expressions (42) and (43), respectively, because we are using the method of separation of variables to calculate the solution of $R(t,r)$.

It is to be mentioned that the considered here $C(r)$ is proportional to both power law and exponential function of r. Actually, these type of assumptions-based on some solutions of $R(t,r)$, *e.g.*, in equation (23), we see that $R(r)\propto r^{\alpha}$, on the other hand, we get exponential relation in equation (30); in a different work, Moffat [42] got the same type of exponential function of r.

4. Raychaudhuri equation

For the sake of completeness, we have contrasted the results obtained so far with those obtained from the well-known Raychaudhuri equation [36], given by

$$\theta_{,\mu}v^{\mu}=\dot{v}^{\mu}_{;\mu}-2\left(\sigma^{2}-\omega^{2}\right)-\frac{1}{3}\theta^{2}+R_{\nu\eta}v^{\nu}v^{\eta}\,, \tag{54}$$

where the terms have their usual significance. For our irrotational system, it reduces to

$$\theta^{2}q=6\sigma^{2}+12\pi G\left(\rho+3p\right)\,. \tag{55}$$

With the help of equations (1), (15) and (55), we finally get for deceleration parameter

$$\theta^2 q = 6\sigma^2 + 12\pi G\left[-2A + \frac{C(r)}{(R'R^2)^{1+\alpha}}\right]\left[A + \frac{C(r)}{(R'R^2)^{1+\alpha}}\right]^{-\frac{\alpha}{1+\alpha}} \tag{56}$$

and for shear scalar

$$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{1}{3}\left(H_r - H_\perp\right)^2 . \tag{57}$$

4.1. Case A: early stage

At the early phase of this evolution when the scale factor $R(r,t)$ is small enough, equation (56) reduces to

$$\theta^2 q = 6\sigma^2 + 12\pi G\,\frac{[C(r)]^{\frac{1}{1+\alpha}}}{R'R^2} . \tag{58}$$

It follows from equation (58) that q, the deceleration factor, is always positive. So accelerated expansion is absent in this dust-dominated phase though inhomogeneity is present here. The same conclusion was obtained previously using equation (23), where $q_\perp = \frac{1}{2}$. Interestingly, this result is very similar to the work of Alnes *et al.* [35].

4.2. Case B: late stage

Type I: If we consider the late stage of evolution *i.e.*, $R(t,r)$ is large enough in this phase, the second term of the RHS of equation (16) vanishes and we get from equation (56)

$$\theta^2 q = 6\sigma^2 - 24\pi G A^{\frac{1}{1+\alpha}} . \tag{59}$$

(a) When we use the scale factor given by equation (29), the shear scalar becomes $\sigma^2 = 0$. Equation (59) reduces to

$$\theta^2 q = -24\pi G A^{\frac{1}{1+\alpha}} . \tag{60}$$

It gives accelerating universe at the late stage. In the previous section, we get the same conclusion with the help of equation (29), where the value of $q_\perp = -1$.

(b) Again, if we consider the expression of the scale factor given by equation (35), the shear scalar becomes $\sigma^2 = \frac{8}{3}\omega^2 \text{cosech}^2\left[2\omega\left(r+t\right)\right]$ and $A = (\frac{4}{3}\omega)^{(1+\alpha)}$. Equation (59) reduces to

$$\theta^2 q = 16\omega^2 \text{cosech}^2\left[2\omega\left(r+t\right)\right] - 32\pi G\omega^2 \,. \tag{61}$$

Figure 6 shows σ^2 *vs.* t for r-constant hypersurface. In this graph, we have seen that as t increases, σ^2 decreases, *i.e.*, when $t \to \infty$, $\sigma^2 \to 0$. Thus, initially, it represents the decelerating universe and after *flip*, we get acceleration in line with current observational result. It is to be mentioned that the expressions of σ^2 and $\theta^2 q$ seem to be identical with our previous work [23] but are not exactly the same because here the expression for ω contains α.

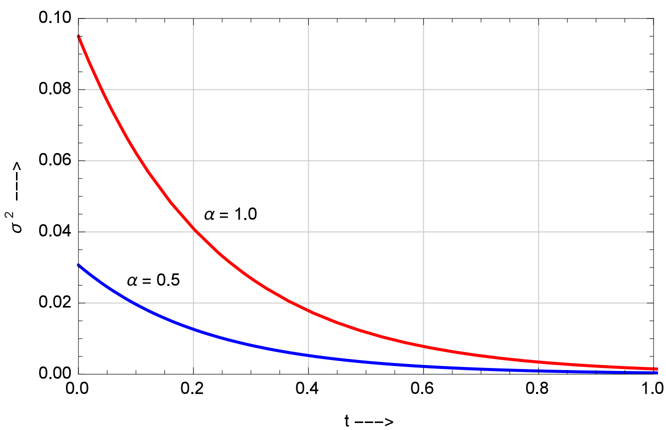


Fig. 6. The variation of σ^2 *vs.* t is shown taking $A = 2$ and $r = 1$.

Type II: Again, if we consider first order approximation of equation (50), neglecting higher order terms, we get

$$\theta^2 q = 6\sigma^2 + \frac{24\pi G}{A^{\frac{\alpha}{1+\alpha}}}\left[-A + \frac{1+3\alpha}{2(1+\alpha)}\frac{C(r)}{(R'R^2)^{1+\alpha}}\right] . \tag{62}$$

If we consider $R(t,r) = a(t)g(r)$, then from equation (51), it follows that $\sigma = 0$. Now, equation (56) reduces to

$$\theta^2 q = \frac{24\pi G}{A^{\frac{\alpha}{1+\alpha}}}\left[-A + \frac{1+3\alpha}{2(1+\alpha)}\frac{B}{a^{3(1+\alpha)}}\right] . \tag{63}$$

It follows from equation (63) that flip occurs when $a(t) = \left\{\frac{1+3\alpha}{2(1+\alpha)}\frac{B}{A}\right\}^{\frac{1}{3(1+\alpha)}}$. Now, $q < 0$, at $a(t) > \left\{\frac{1+3\alpha}{2(1+\alpha)}\frac{B}{A}\right\}^{\frac{1}{3(1+\alpha)}}$ *i.e.*, acceleration takes place in this case.

Therefore, we get early deceleration and late acceleration here. This also follows from equation (42) for $\alpha > 0$.

5. Concluding remarks

We have considered a Tolman–Bondi–Lemaître-type of inhomogeneous spacetime with a generalised Chaplygin gas equation of state. There is a proliferation of articles on accelerating universe with Chaplygin EoS in homogeneous spacetime but scant attention has been paid so far to address the problem in inhomogeneous spacetime. One intriguing problem is that accelerating phase supposedly starts at the period when inhomogeneities in the distribution in the universe at length scale < 10 Mpc can no longer be ignored. This primarily motivates us to investigate the matter in inhomogeneous spacetime. The salient features of other findings may be briefly summed up as:

(i) Using our field equations being highly nonlinear with contributions from both inhomogeneity and the generalised Chaplygin-type of matter field, we have been able to get the solutions in a closed form at extreme cases only, *i.e.*, at early and late stages of the universe. In the former case, we have seen that $\frac{\partial \rho}{\partial r}$ is always negative for $\alpha > 0$. From the theoretical point of view, we may conclude that the α should be positive which is in agreement with the observational analysis. Here, C is a function of r, *i.e.*, $C(r) = \left(\frac{4}{3}\alpha r^{3\alpha-1}\right)^{1+\alpha}$. Interestingly, we have seen that the deceleration parameter $q_{\perp} = \frac{1}{2}$ represents dust-dominated universe.

(ii) In a different context, the scale factor $R(t,r)$ has been calculated at asymptotic range *i.e.*, at late stage of the universe. At the extreme case with suitable transformation of radial co-ordinate, the solution resembles de Sitter-type metric with homogeneous spacetime (see equation (31)). Thus, it may be concluded that at late stage of the universe, inhomogeneity may disappear as expected.
Further, the integration function C may be either a true constant or a function of r. If we consider C as a true constant, then $\frac{\partial \rho}{\partial r} < 0$ as desired for a regular distribution in each case. Otherwise, if $C \equiv C(r)$, we have to take particular forms of $C(r)$ and ρ' may be negative under certain restriction.

(iii) Another area of interest is the spacetime described by equation (25). This is a unique result in the sense that for pure Chaplygin gas ($\alpha = 1$), one can reduce equation (25) to the well-known Einstein–de Sitter case with some additional assumption. However, for the generalised Chaplygin gas ($\alpha \neq 0$), similar assumption does not reduce it to any homogeneous spacetime.

(iv) From equation (35), it further follows that at the late era when *flip* occurs, the flip time (τ_c) depends explicitly on α. The variation of τ_c with α also depends on magnitude of A (figure 1). In this case, the flip occurs later for inner shells.
As it is well-known in an inhomogeneous model, all physical parameters depend on both space and time, including flip which evidently depends on time. It is not synchronous. The different shells characterised by r-constant hypersurfaces start accelerating at different instants of time. We have come across the phenomena of shell crossing singularity in inhomogeneous gravitational collapse. For an inhomogeneous expanding model with acceleration, this is particularly significant. Since our analysis shows that for a shell with a larger value of 'r' the velocity flip starts *earlier*, which is a good news for avoidance of *shell crossing singularity*. Thus, the Chaplygin gas inspired model offers a natural path against this singularity as opposed to the Tollman–Bondi case with *perfect gas*, where one has to impose a set of stringent external conditions.

(v) For the sake of completeness, we have adopted the separation of variable method to solve our key equation (16). Most of the authors explained Chaplygin gas considering extreme cases for temporal part. We have also studied the extremal form in Case A and Case B. Now, for large $R(t, r)$, we consider up to second term of the temporal part and then we are able to solve equation (29) in exact form. The solution of equation (29) was given in equation (30) which shows early deceleration as well as late acceleration. The desirable feature of flip occurs which agrees with the observational analysis for positive value of α. In this case, we find that the matter density becomes homogeneous *i.e.*, $\rho' = 0$ independent of the nature of C.

(a) One can also mention that the flip time (t_c) depends on the value of α but the dependance is not monotonic. Figure 4 shows the variation of q with t for different values of α where flip occurs. We have concentrated on the occurrence of a late flip because all observational probes point to a late accelerating phase. It is interesting to mention that the late flip also depends on the value of A. In figure 4, we have taken two values of A where we find the maximum t_c for corresponding value of α.

(b) To get the exact solution of the radial part represented by (44), we have to choose the expression of integration constant $C(r)$ as the simplest form *(i)* $C(r) = r^{\beta}$ and *(ii)* $C(r) = e^{\gamma r}$. However, if we consider $C(r)$ to be a true constant, interestingly, we get $R(t,r) \propto r^{\frac{1}{3}}$, *i.e.*, $R(t,r)$ is related to the power law expression of r.

(vii) We also have calculated $\theta^2 q$ with the help of Raychaudhury equation and showed that nature of q is the same for each case as in Section 3.

(viii) We further notice that in literature, there exist models generalising LTB with a cosmological constant. Our work essentially differs in that it is more general in nature because for a large scale factor, it reduces to that ΛCDM model, where $A^{\frac{1}{1+\alpha}}$ simulates Λ in equation (29).

As commented earlier in the introduction the Chaplygin gas (CG) scenario, besides its successful applicability in the accelerating universe paradigm, is also aesthetically satisfying in the sense that it beautifully synthesizes both matter and dark energy in a single whole, unlike the ΛCDM case which explains only a part of the evolution. Moreover, many workers including the present authors have also shown that the CG is thermodynamically stable [43] as well. One should also point out that the CG cosmology also suffers from serious shortcomings in its attempts to explain the large scale structure formation of the universe, inviting serious comments and criticisms. Without going into details (those interested in more details are referred to [25] and [26]), we would like to mention that the value of the square sound velocity $c_{\rm s}^2$ here comes out to be very small which is shown to produce unphysical oscillations giving finally rise to an exponential growth of current power spectrum of matter [25]. However, recent analysis has shown that one can circumvent this difficulty taking the generalized Chaplygin Gas [26]. Moreover, under the ΛCDM case, the $c_{\rm s}^2$ here though tiny remains positive throughout.

The present work suffers from another serious shortcoming in that we have so far not attempted to constrain the free model parameters with the help of observational data as is customary in relevant works in this field. The issue of compatibility of the obtained results with observational data will be addressed in our future work.

D.P. acknowledges the financial support of the Diamond Jubilee grant of Sree Chaitanya College, Habra.

REFERENCES

[1] A.G. Reiss *et al.*, *Astron. J.* **607**, 665 (2004).

[2] R. Amanullah *et al.*, *Astron. J.* **716**, 712 (2010).

[3] A. Kamada *et al.*, *Phys. Rev. Lett.* **119**, 111102 (2017).

[4] S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989).

[5] T. Chiba, N. Sugiyama, T. Nakamura, *Mon. Not. R. Astron. Soc.* **289**, L5 (1997) [arXiv:astro-ph/9704199].

[6] R. Caldwell, R. Dave, P.J. Steinhardt, *Phys. Rev. Lett.* **80**, 1582 (1998) [arXiv:astro-ph/9708069].

[7] R.R. Caldwell, R. Dave, P.J. Steihardt, *Phys. Rev. Lett.* **80**, 1582 (1998) [arXiv:astro-ph/9908168].

[8] R.R. Caldwell, *Phys. Lett. B* **545**, 23 (2002) [arXiv:astro-ph/9908168].

[9] M. Li, *Phys. Lett. B* **603**, 1 (2004) [arXiv:hep-th/0403127].

[10] L. McAllister, E. Silverstein, *Gen. Relativ. Gravitation* **40**, 565 (2008) [arXiv:0710.2951 [hep-th]].

[11] J. Polchinski, arXiv:hep-th/0603249.

[12] E. Elizalde, J.E. Lidsey, S. Nojiri, S.D. Odintsov, *Phys. Lett. B* **574**, 1 (2003) [arXiv:hep-th/0307177].

[13] A. Shafieloo, V. Sahni, A.A. Starobinsky, *Phys. Rev. D* **80**, 101301 (2009) [arXiv:0903.5141 [astro-ph.CO]].

[14] M.I. Wanas, Dark Energy: Is it of Torsion Origin? in: Proc. of 1st Middle-East and Africa Regional IAU Meeting (MEARIM-2008), A.A. Hady, M.I. Wanas (Eds.).

[15] A. Krasiński, C. Hellaby, K. Bolejko, M.-N. Célérier, *Gen. Relativ. Gravitation* **42**, 2453 (2010) [arXiv:0903.4070 [gr-qc]].

[16] S. Chatterjee, *J. Cosmol. Astropart. Phys.* **03**, 014 (2011) [arXiv:1012.1706 [gr-qc]].

[17] K. Bamba *et al.*, *Astrophys. Space Sci.* **342**, 155 (2012) [arXiv:1205.3421 [gr-qc]].

[18] M. Li, X.-D. Li, S. Wang, Y. Wang, *Front. Phys.* **8**, 828 (2013) [arXiv:1209.0922 [astro-ph.CO]].

[19] J. Yoo, Y. Watanable, *Int. J. Mod. Phys. D* **21**, 1230002 (2012) [arXiv:1212.4726 [astro-ph.CO]].

[20] D. Panigrahi, S. Chatterjee, *Gen. Relativ. Gravitation* **40**, 833 (2008) [arXiv:0709.0374 [grqc]].

[21] D. Panigrahi, S. Chatterjee, *Gravit. Cosmol.* **17**, 18 (2011) [arXiv:1006.0476 [gr-qc]]; D. Panigrahi, *AIP Conf. Proc.* **1316**, 459 (2010).

[22] A. Kamenschik, U. Moschella, V. Pasquier, *Phys. Lett. B* **511**, 265 (2001).

[23] D. Panigrahi, S. Chatterjee, *J. Cosmol. Astropart. Phys.* **10**, 002 (2011).

[24] D. Panigrahi, S. Chatterjee, *Int. J. Mod. Phys. D* **21**, 1250079 (2012).
[25] H. Sandvik, M. Tegmark, M. Zaladarriaga, I. Waga, *Phys. Rev. D* **69**, 123524 (2004).
[26] M.C. Bento, O. Bertolami, A.A. Sen, *Phys. Lett. B* **575**, 172 (2003).
[27] J.C. Fabris, S.V.B. Goncalves, E.V. De Souza, *Gen. Relativ. Gravitation* **34**, 53 (2002).
[28] S. del Campo, *J. Cosmol. Astropart. Phys.* **11**, 004 (2013) [arXiv:1310.4988 [astro-ph.CO]].
[29] B.C. Paul, P. Thakur, *J. Cosmol. Astropart. Phys.* **11**, 052 (2013) [arXiv:1306.4808 [astro-ph.CO]].
[30] D. Panigrahi, B.C. Paul, S. Chatterjee, *Gravit. Cosmol.* **21**, 83 (2015) [arXiv:1305.7204 [gr-qc]].
[31] E.W. Kolb, S. Matarrese, A. Notari, A. Riotto, arXiv:hep-th/0503117.
[32] D.L. Wiltshire, arXiv:gr-qc/0503099.
[33] B.M.N. Carter *et al.*, arXiv:astr-ph/0504192.
[34] M. Giovannini, *Phys. Lett. B* **634**, 1 (2006).
[35] H. Alnes, A. Morad, Ø. Gron, *J. Cosmol. Astropart. Phys.* **01**, 007 (2007).
[36] A.K. Raychaudhuri, *Phys. Rev.* **98**, 1123 (1955).
[37] S. Räsänen, *J. Cosmol. Astropart. Phys.* **11**, 003 (2006) [arXiv:astro-ph/0607626].
[38] G. Hinshaw *et al.*, *Astrophys. J. Suppl.* **208**, 19 (2013) [arXiv:1212.5226 [astro-ph.CO]].
[39] J.W. Moffat, *J. Cosmol. Astropart. Phys.* **05**, 001 (2006) [arXiv:astro-ph/0505326].
[40] J.W. Moffat, arXiv:astro-ph/0504004.
[41] S. Chatterjee, B. Bhui, *Mon. Not. R. Astron. Soc.* **247**, 57 (1990).
[42] J.W. Moffat, arXiv:astro-ph/0606124.
[43] D. Panigrahi, *Int. J. Mod. Phys. D* **24**, 1550030 (2015) [arXiv:1405.7667 [gr-qc]].