ANALYSIS OF THE BREAKDOWN OF EXPONENTIAL DECAYS OF RESONANCES

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A simple model of alpha decay with the Dirac delta potential is studied. The model leads to breakdown of the exponential decay and to the power law behavior at asymptotic times. Time dependence of the survival probability of the particle in the potential well is analyzed numerically with two methods: integration of the Green's function representation and numerical solution of the time-dependent Schrödinger equation. The numerical results confirm power law with exponent $n = 3$ after the turnover into the non-exponential decay regime. Moreover, oscillations of non-escape probability are observed in the intermediate stage of the process. The simple alpha-decay model is compared to the results of the Rothe–Hintschich–Monkman experiment which was the first experimental proof of violation of the exponential law.

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1. Introduction

The exponential law in alpha decay was first explained by Gamow in 1928 [1]. In his reasoning, eigenfunctions with complex eigenenegies are present, which initially caused some concerns about validity of this approach. Nevertheless, this method leads to correct predictions about the exponential time dependence and the Geiger–Nutall law. Although the exponential decay law provides a very good description for quasi-stationary states, it is only an approximate solution. It was observed by Khalfin in 1958 [2] that the exponential behavior cannot hold for all times $t \in (0, \infty)$. In particular, for asymptotic times $t \to \infty$, the survival probability $S(t)$ of a state decreases slower than any exponential function [2–4]

$$S(t) \geq A\,\mathrm{e}^{-bt^q} \tag{1}$$

if the energy distribution density is bounded from below or above ($A > 0$, $b > 0$ and $0 < q < 1$). The first detailed description of the delta-barrier potential discussed in this paper was done by Winter [5] in 1961. He shows that

non-escape probability in such a system decreases exponentially with time first, then oscillations of probability occur and, finally, non-escape probability behaves like a power of time.

It is important to distinguish between *survival* and *non-escape* probability. The non-escape probability $P(t)$ of a state initially confined in a potential well ($\psi(x,0) = 0$ for $x > R$) is the probability that a particle remains in the well at a time t

$$P(t) = \int_0^R \mathrm{d}x \; |\psi(x,t)|^2 \,. \tag{2}$$

The survival probability is the probability that *the state* remains the same

$$S(t) = |\, \langle \psi(t) | \psi(0) \rangle \,|^2 = \left| \int_0^R \mathrm{d}x \; \psi(x,t)^* \psi(x,0) \right|^2 . \tag{3}$$

In particular, one can think of a situation such that the state evolves through different, even orthogonal states, but all these states are confined in the well. Then, $S(t) < 1$ while $P(t) = 1$. Thus, survival and non-escape probabilities are related, yet distinct quantities.

Several models of systems that exhibit violation of the exponential law were studied (see *e.g.* [6–9]). Among the physical phenomena described by these models, one can find: spontaneous decay in two-level systems, alpha-decay, single photon ionization of atoms, *etc.* For a certain model of a phenomenon, it can be shown that both, the non-escape and the survival probability obey power law for sufficiently large times [2, 3, 5–7, 10–14]

$$S(t),\; P(t) \sim \begin{cases} \mathrm{e}^{-t/\tau}; & \text{for } t \lesssim t_{\text{breakdown}} \,, \\ t^{-n}; & \text{for } t \gtrsim t_{\text{breakdown}} \,. \end{cases} \tag{4}$$

The exponent n depends on the phenomenon and the model. Typically, it takes values between 1 and 4.

It is shown, *e.g.* in [13], that both $P(t)$ and $S(t)$ decrease exponentially first and behave like a power of time for large times. Therefore, for the purpose of understanding the origin of exponential decay and deviation from exponential behavior, we will focus on survival probability $S(t)$, because it is simpler. Let us consider the time evolution of a state [2–4, 15]

$$|\psi(t)\rangle = \int_{-\infty}^{\infty} \mathrm{d}E \, \mathrm{e}^{-iEt/\hbar} c(E) \, |E\rangle \;, \tag{5}$$

where $|E\rangle$ are eigenstates of the Hamiltonian. The amplitude that the initial state has not decayed at time t is

$$a(t) = \langle\psi(0)|\psi(t)\rangle = \int\limits_{-\infty}^{\infty} \mathrm{d}E\, \mathrm{e}^{-iEt/\hbar}\omega(E)\,. \tag{6}$$

Here, $\omega(E) = c^*(E)c(E)$ is the energy distribution. If the energy density is modeled by the Breit–Wigner distribution

$$\omega(E) = \frac{1}{2\pi}\frac{\Gamma}{(E-E_0)^2 + (\Gamma/2)^2}\,, \tag{7}$$

the amplitude equals

$$a(t) = \exp\left(-\frac{-iE_0t}{\hbar} - \frac{\Gamma|t|}{2\hbar}\right)\,. \tag{8}$$

Thus, in this case, the time dependence of the survival probability is given by an exponential function. However, the energy density must be bounded from below in order for the ground state to exist. Distribution (7) is therefore approximate, some corrections in $\omega(E)$ must be included, and new terms will also appear in the formulas for the amplitude and the probability. These terms decrease slower than the exponential function and become dominant for large times [2].

Although the power-like evolution is expected at large times, its experimental observation is not trivial. If the exponential stage of a process lasts 20 or more lifetimes, the non-escape probability may be too low to be measured. This would require accuracy of the order of $\mathrm{e}^{-20} \approx 10^{-9}$ or better, but no high-energy resonance is measured with such a precision. Therefore, discovery of decay processes which cease to follow the exponential behavior sufficiently early is a major theoretical and experimental challenge. The first experimental proof of the turnover into the non-exponential decay regime, based on the measurements of the luminescence decays of dissolved organic materials, was found only quite recently, namely in 2006 [12]. Another proof of non-exponential behavior at large times was found in the scattering process $\alpha^+\alpha \to {}^8\mathrm{Be}(2^+) \to \alpha^+\alpha$ [16]. It is somehow indirect in a sense that the energy density was measured, the difference from the Breit–Wigner distribution was observed, but the survival probability was obtained from these calculations.

The paper is organized as follows. Section 2 provides an analysis of the simple, one-dimensional model of alpha decay from [6]. Section 3 includes numerical results for this model. In particular, behavior in exponential and power law regimes, and in the intermediate stage between them is compared

with our predictions. For the first time, calculations for this theoretical model are compared to the results of the Rothe–Hintschich–Monkman experiment [12]. Quite satisfactory agreement is found. The final remarks and discussion are presented in Section 4.

2. Model

The alpha-decay model studied in this paper assumes a potential

$$V(x) = \begin{cases} \frac{\lambda\hbar^2}{2ma}\delta(x-a) & \text{for } x > 0\,, \\ +\infty & \text{for } x \leq 0\,. \end{cases} \tag{9}$$

Analysis of this potential can be found in [5, 6, 10, 11]. Following Gamow's reasoning [1], one could use an outgoing wave Ansatz in the region outside the potential well, $\varphi_2(x) = B\mathrm{e}^{ikx}$ for $x > a$. Such an assumption leads to a discrete spectrum of complex eigenenergies E_n and the eigenfunctions which decay exponentially in time. The most troublesome corollary of this approach is the fact that the eigenfuctions diverge when $x \to \infty$ (because Im $k_n < 0$), so are not normalizable.

The way out of this problem is to treat the process as a scattering one, *i.e.* an incoming wave must be included in the solution for the outer region, $\varphi_2(x) = \mathrm{e}^{-ikx} + B\mathrm{e}^{ikx}$ for $x > a$. It can be easily seen that, taking into account the boundary conditions, the most general form of the solution inside the potential well is $\varphi_1(x) = A \sin kx$ $(0 < x < a)$. From the matching conditions at $x = a$, one finds that the spectrum of eigenenergies is continuous and real. The coefficients $A = A(k)$ and $B = B(k)$ are

$$A(k) = -\frac{2ika}{ka + \lambda\mathrm{e}^{ika}\sin ka}\,, \tag{10}$$

$$B(k) = -\frac{ka + \lambda\mathrm{e}^{-ika}\sin ka}{ka + \lambda\mathrm{e}^{ika}\sin ka}\,. \tag{11}$$

Under such a choice of coefficients, the basis is normalized as [11]

$$\int_{x=0}^{\infty} \varphi_k(x)\varphi^*_{k'}(x)\,\mathrm{d}x = 2\pi\delta(k-k')\,.$$

With the aid of the Green's function formalism, one can find the time-dependent wavefunction inside the well [6]

$$\psi(x,t) = \frac{1}{2\pi}\int_0^{\infty} \mathrm{e}^{-\frac{i\hbar}{2m}k^2t}\phi(k)|A(k)|^2 \sin kx\,\mathrm{d}k\,, \tag{12}$$

where

$$\phi(k) \equiv \int_0^a \psi(x',0) \sin kx' \, \mathrm{d}x' \tag{13}$$

depends on the initial wavefunction. When $\lambda \gg 1$, the poles of analytic continuation of $|A(k)|^2$

$$W(k) = -A(-k)A(k) = \frac{4k^2a^2}{(ka)^2 + \lambda ka \sin(2ka) + \lambda^2 \sin^2(ka)} \tag{14}$$

are very close to the real axis. Thus, the leading contribution to integral (12) comes from points on the real axis that lie in the vicinity of poles of $W(k)$. To proceed, one would like to expose the role of poles and get rid of the oscillatory factor $\mathrm{e}^{-(i\hbar/2m)k^2t}$ in Eq. (12). This can be achieved by shifting the integration contour. After rotation in the clockwise sense by 45°, one obtains

$$\psi(x,t) = \mathrm{e}^{-i\pi/4} \int_0^\infty \mathrm{e}^{-\frac{\hbar k^2}{2m}t} f\left(\mathrm{e}^{-i\pi/4}k, x\right) \, \mathrm{d}k + \sum_{n=1}^\infty C(k_n, x) \, \mathrm{e}^{-\frac{i\hbar}{2m}k_n^2 t} \,, \tag{15}$$

where

$$f(k,x) \equiv \frac{1}{2\pi}\phi(k)W(k) \sin kx \,. \tag{16}$$

The first term in Eq. (15) is the *background integral* and $-C(k_n, x)/2\pi i$ are residues of $f(k,x)$ at the poles $k = k_n$. In the above formula, the first term has power behavior at large times, while the second one corresponds to the exponential decays. When the background integral in Eq. (15) can be neglected, the non-escape probability is roughly equal to

$$P_{\text{poles}}(t) \approx \sum_{n=1}^\infty c_n \mathrm{e}^{-\Gamma_n t/\hbar} \,, \tag{17}$$

where

$$c_n = \int_0^a |C(k_n, x)|^2 \mathrm{d}x \tag{18}$$

and

$$\Gamma_n = -\mathrm{Im}\left(\hbar^2 k_n^2/m\right) \,. \tag{19}$$

However, for sufficiently large times, the contribution of the background integral to the non-escape probability follows the power law

$$P_{\text{background}}(t) \sim \frac{m^3a^6}{\lambda^4 t^3} \,, \tag{20}$$

and eventually dominates over the pole contribution. The time when it happens, *i.e.* the time of the breakdown of the exponential decay, can be estimated to be $t_{\text{breakdown}} \sim 10(\hbar/\Gamma_1)\ln\lambda$ [6].

The problem of a particle in potential (9) is related to the case of an infinite potential well. When $\lambda \gg 1$, the two problems are almost the same and, as can be seen from Eq. (14), the poles correspond to the momenta which satisfy $\sin(k_n a) \approx 0$. This is the well-known condition in the case of the infinite well. Thus, there is a correspondence between the n^{th} excited state

$$\psi^{(n)}(x) = N \sin\left(\frac{n\pi x}{a}\right) \tag{21}$$

and the n^{th} pole (or the n^{th} contributions in the sums in Eqs. (15) and (17)). This relation should also hold approximately for finite λs. In particular, if the initial wavefunction is chosen to be $\psi(x,0) = \psi^{(n)}(x)$, c_n is expected to be the largest coefficient in sum (17). The values of the weights c_n for $\psi^{(n)}(x)$ taken as the initial condition, with $n = 1,\ 2,\ 3,\ 4$, are presented in Table I. To obtain these numbers, one needs to calculate residues of $f(k,x)$, Eq. (16) and use Eq. (18).

TABLE I

Dependence of weights c_n on the number n of the excited state $\psi^{(n)}$ (Eq. (21)) used as the initial condition ($\lambda = 8$).

n	c_1	c_2	c_3	c_4	c_5
1	1.012	0.022	0.005	0.002	0.001
2	0.016	1.059	0.060	0.013	0.006
3	0.005	0.036	1.148	0.106	0.023
4	0.003	0.013	0.057	1.270	0.157

All above statements will hold in 3 dimensions, for $\ell = 0$. This is because the radial equation for $rR(r)$ ($R(r)$ — radial part of the wavefunction), when $\ell = 0$, is the same as the time-independent Schrödinger equation in 1 dimension, $rR(r) \to 0$ at $r = 0$ [17], the matching conditions are clearly the same and, when calculating the probability, the extra r^{-2} factor cancels out with r^2 from the Jacobian.

3. Consequences of the model

3.1. Numerical results

The wavefunction is calculated with two methods. The first one is a direct numerical solution of the time-dependent Schrödinger equation. In this approach, a Gaussian potential barrier of width Δ centered around $x = a$ is used instead of a Dirac delta potential to have a smooth function.

Then, the results are extrapolated to $\Delta \to 0$. Another way to obtain $\psi(x,t)$ is to use the Green's function representation of the wavefunction, Eq. (15). The two methods are compared for times of the order of 1–2 lifetimes. They provide compatible results, *e.g.* the difference between their predictions for $|\psi(0.6a,\ 0.4\tau_0)|^2$ is about 0.1% (when $\lambda = 8$). Here, $\tau_0 \equiv m(\lambda a)^2/(2\pi^3\hbar)$ is a characteristic time unit of the process. For larger times, in particular when the breakdown of the exponential decay occurs, only the Green's function approach is used.

It is interesting that direct numerical solution can be done for times exceeding 30 half-lives [18]. Yet, in this paper, we restrict to only 1–2 lifetimes, since the main purpose of the direct simulation is to compare its results with the Green's function method.

To find the time dependence of the non-escape probability, the wavefunction is calculated from Eq. (15) at 100 equally spaced points from $x = 0$ to $x = a$, and then the non-escape probability, $P(t) = \int_0^a |\psi(x,t)|^2\, \mathrm{d}x$ is obtained by the trapezoid method (to simplify the calculations). In this part, we assume the initial wavefunction is $\psi(x,0) = \sqrt{2/a}\sin(\pi x/a)$.

The time dependence of the non-escape probability is presented in Fig. 1. The results of the fit of the exponential function to the early time behavior and the power function for large times are shown in Fig. 2 for $\lambda = 1$. The time axis in Fig. 1 is chosen such that the initial slope of all curves is -1, *i.e.* the time is plotted in units of τ, where τ is the lifetime of a decaying state corresponding to a given λ. Values of $\tau = \tau(\lambda)$ are found from the fit.

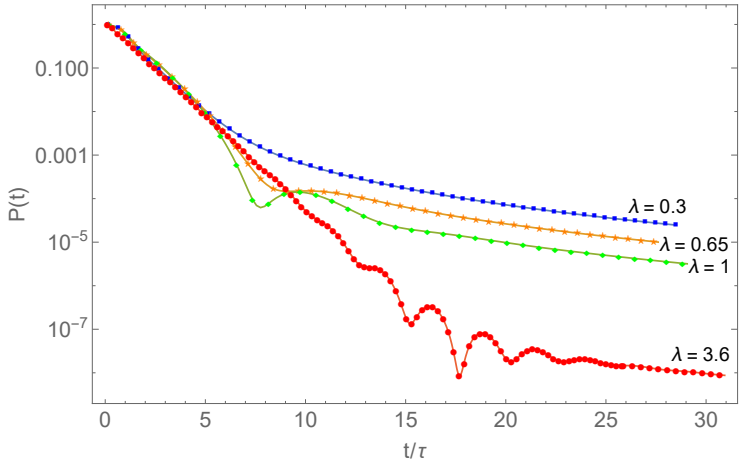
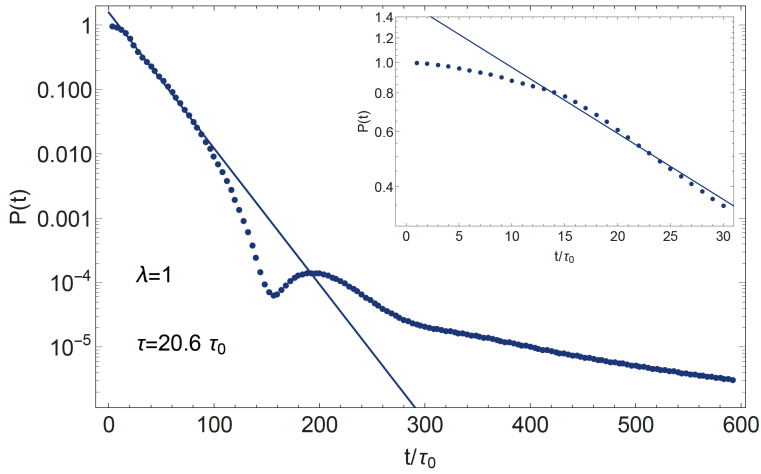
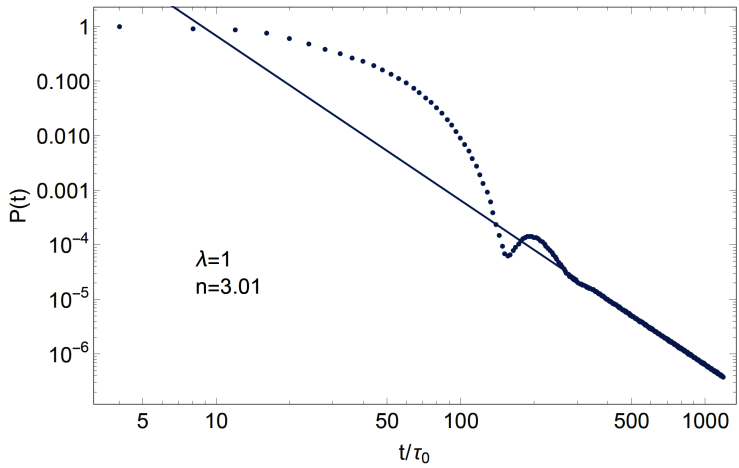


Fig. 1. Logarithmic plot of the time dependence of the non-escape probability for $\lambda = 0.3,\ 0.65,\ 1,\ 3.6$. The horizontal axis is chosen such that the initial slope of all curves is -1, *i.e.* the time is plotted in units of τ, where τ is the lifetime of a decaying state corresponding to a given λ.



(a) $\lambda = 1$, logarithmic plot



(b) $\lambda = 1$, log-log plot

Fig. 2. Time dependence of the non-escape probability. Very close to $t = 0$, $P(t)$ deviates from exponential function (inset in Fig. 2 (a)), then its time dependence is exponential (Fig. 2 (a)) and, after a single oscillation, it follows power law at large times (Fig. 2 (b)). Exponential and power functions are fitted in appropriate regimes (solid lines).

Both the exponential and power law regimes can be easily observed, so the theoretical predictions from [6] are confirmed in these numerical calculations. The exponent n of the power law $P(t) = Bt^{-n}$ agrees with the theoretical value $n = 3$ within uncertainty of the fit, as shown in Table II.

TABLE II

Exponent fitted in the power law regime of time dependence of the non-escape probability for various λ.

λ	n
0.3	3.010 ± 0.017
0.65	2.996 ± 0.020
1	2.992 ± 0.012
3.6	3.000 ± 0.040

Moreower, for very small times, the observed behavior is not exponential, but initially $\dot{P}(0) = 0$ (see the inset in Fig. 2 (a)). The phenomenon that the decay of a quasi-stationary state is not exponential in this regime is known from the theory [3, 6, 19] and was seen in experiment for transition between two $^9Be^+$ ground-state hyperfine levels [20] and for quantum tunneling of ultra-cold sodium atoms trapped in an optical lattice [21]. This effect is related to the *quantum Zeno paradox* [19, 22].

Another effect, which was noticed, is the presence of a single ($\lambda = 1$) or many ($\lambda = 3.6$) oscillations of the non-escape probability. They occur in the intermediate stage of the process, between the exponential and power law regimes. Similar oscillations were also observed in [3, 5, 10, 11, 13, 23]. In this region, the survival probability is not a monotonic function of time. Such behavior is caused by interference between the background integral part of Eq. (15) and the pole term when they are of comparable order. If one splits the wavefunction as in Eq. (15)

$$\psi(x,t) = \psi_{\rm bg}(x,t) + \psi_{\rm poles}(x,t)\,, \tag{22}$$

not only the total non-escape probability can be calculated, but also quantities

$$P_{\rm bg}(t) = \int_0^a |\psi_{\rm bg}(x,t)|^2\,{\rm d}x\,, \tag{23}$$

$$P_{\rm poles}(t) = \int_0^a |\psi_{\rm poles}(x,t)|^2\,{\rm d}x\,, \tag{24}$$

$$P_{\rm interf}(t) = \int_0^a \left(\psi^*_{\rm bg}(x,t)\psi_{\rm poles}(x,t) + {\rm h.c.}\right){\rm d}x\,. \tag{25}$$

They represent the separate contributions of the background, poles and interference term to the total non-escape probability. The interference term has an unexpected feature — it changes its sign abruptly. This explains

better the origin of the oscillations. For early times, the pole contribution dominates, and for asymptotic times, the contribution from the background integral is leading. Only for intermediate times $P_{\rm interf}(t)$ is comparable to the rest of the non-escape probability $P_{\rm poles}(t) + P_{\rm bg}(t)$. When it changes its sign in this intermediate stage, the derivative of $P(t)$ rapidly decreases (when $P_{\rm interf}(t)$ changes the sign from plus to minus) or increases (the opposite case). These abrupt changes of $P_{\rm interf}(t)$ cause the oscillations. Briefly, the reason for the oscillations is constructive and destructive interference between the pole and background contributions. Another, probably more physical, way of understanding these oscillations is interference of the outgoing and incoming waves included in our solution.

An important question, especially from the point of view of experimental observation of the breakdown of exponential decays of resonances, is what are the parameters for which turnover to the power law regime occurs at the earliest time. A good and general parameter for this purpose is a *Q-value* of a resonance

$$Q = \frac{\epsilon_n}{\Gamma_n} = -\frac{\mathrm{Re}\ E_n}{2\ \mathrm{Im}\ E_n}\,, \tag{26}$$

where $\epsilon_n = \mathrm{Re}\ E_n$ is the energy of a resonance and $\Gamma_n = -2\,\mathrm{Im}\ E_n$ — its width, *i.e.* $E_n = \epsilon_n - i\Gamma_n/2$. Time of the breakdown of the exponential behavior (in units of the lifetime) is proportional to the logarithm of Q [12], thus the lower the Q-value, the earlier it happens. On the other hand, if the Q-value is too small, the resonance is not well-formed. Thus, to have a well-formed resonance and to observe breakdown of the exponential law before there is a very low probability that the state has not decayed yet, a compromise must be made for the optimal choice of Q. Another way to check whether a resonance is still well-formed is to calculate the lifetime in two ways: from the fit (as in Fig. 2 (a)) or by finding the first pole of $A(k)$, Eq. (10) and calculating the corresponding lifetime

$$\tau_1 = \frac{\hbar}{\Gamma_1} = -\frac{m}{\mathrm{Im}\left(\hbar k_1^2\right)}\,. \tag{27}$$

If the resonance is well-formed and there is a single dominant pole, the two methods should provide similar results.

Analysis of quality of resonances for small λ is shown in Table III. From these data, one can infer that the optimal λ is roughly about 3.6. Certainly, this is a somehow subjective choice — the main criteria used for this decision are following. First, for $\lambda \gtrapprox 3.6$, the exponential function becomes a decent approximation for $P(t)$ for early times, which can be seen *e.g.* from the fact that lifetimes found from the fit and from the poles are in a good agreement. Second, one needs Q as low as possible to observe breakdown of exponential behavior for early times.

TABLE III

Q-values, lifetimes from the fit, lifetimes found from the first poles, and the relative difference between these two results for various λ.

λ	Q	τ/τ_0 (fit)	τ_1/τ_0 (poles)	[%]
0.3	0.208	204	119	53
0.65	0.454	47.0	33.9	32
1	0.667	20.8	17.6	17
3.6	2.48	3.55	3.48	2.0
5	3.79	2.64	2.57	2.7

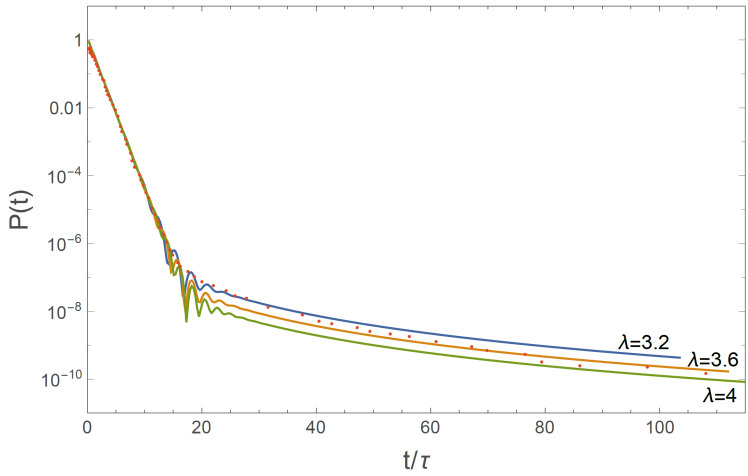
For this parameter, deviations from the exponential behavior start when the signal is $P(t)/P(0) \sim 10^{-5}$ of the initial value and the system enters the power law regime for $P(t)/P(0) \sim 10^{-8}$ (see Fig. 1).

3.2. Comparison with experiment

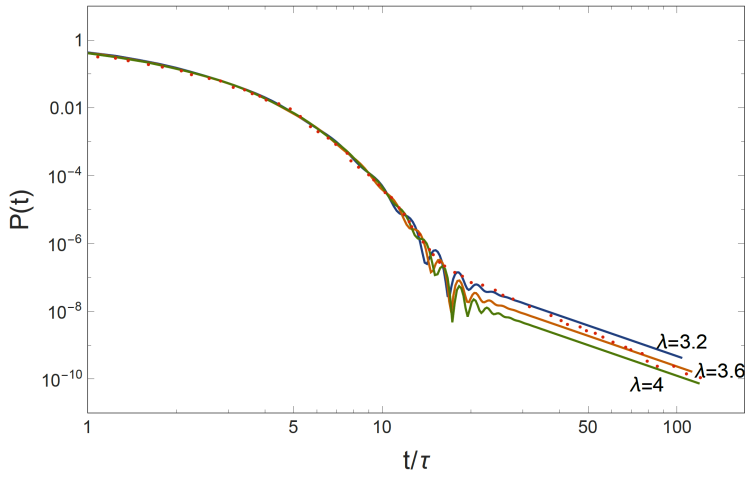
The main problem with observation of non-exponential regime in experiment is the fact that, in general, it occurs after many lifetimes, so the signal is too weak to be detected. Since the time of breakdown of exponential decay is lower for lower Q-values, a process with a broad energy spectrum is needed rather than a narrow, well-formed resonance. This fact made searches for non-exponential decays in nuclear and particle physics inefficient. Another physical system had to be used and the first successful experimental observation of violation of the exponential decay law was made by Rothe, Hintschich and Monkman [12] in 2005. It was found in atomic data, namely in measurements of the luminescent decays of dissolved organic materials. Both the impact of the solvent environment and the fact that large, organic molecules were used, leads to broadening of the energy spectrum. This effect is sufficient to observe the power law behavior all the way up to about 20 lifetimes (*e.g.* 17 for Rhodamine 6G and 11 for polyfluorene).

Their results can be compared to the predictions of the simple model studied here. The experimental data for Rhodamine 6G from [12] are compared with the curves from numerical calculations for the alpha-decay model in Fig. 3. In this plot, time is plotted in units of the lifetime τ of a decaying state. To translate data of [12] to Fig. 3, we fit exponential function in the exponential regime of the data and then use values of non-escape probability taken directly from Fig. 2 of [12] to plot $P(t/\tau)$. The value of λ, that fits the experimental results best, is

$$\lambda \approx 3.6\,. \tag{28}$$



(a) logarithmic scale



(b) log-log scale

Fig. 3. Comparison of the theoretical curves with the experimental data. The curves correspond to the time dependence of the non-escape probability calculated numerically for the alpha-decay model with a potential strength $\lambda = 3.2,\ 3.6,\ 4$. The points are experimental results for Rhodamine 6G [12].

For $\lambda = 3.6$, the lifetime found by fitting the exponential function to the theoretical curve is (see Table III) $\tau_{\rm th} \approx 3.55\ \tau_0$, where $\tau_0 = m(\lambda a)^2/(2\pi^3\hbar)$. The experimental value found in [12] reads $\tau_{\rm exp} = 3.9$ ns. Equating $\tau_{\rm th} = \tau_{\rm exp}$, one obtains after simple transformations and substitution of the numerical values

$$ma^2 = \frac{2\pi^3\hbar}{\lambda^2}\frac{\tau_{\text{exp}}}{\tau_{\text{th}}/\tau_0} \approx 1.2 \times 10^5\ m_p a_0^2\,, \tag{29}$$

where m_p is the mass of proton and a_0 is the Bohr radius. Large value of this constant, in these units, originates from the fact that the organic molecules used in the experiment are large and dissolved in the solvent, so both the mass and length scales are much larger than m_p and a_0. There is actually a very simple way to obtain this number: the molecular mass of Rhodamine 6G ($C_{28}H_{31}N_2O_3Cl$) is $A \approx 479$, the sum of atomic numbers of its constituents is $Z = 254$ and $AZ = 479 \times 254 \approx 1.2 \times 10^5$. Such a simple relation does not hold for other substances from the experiment, though. Still, even in those other cases AZ provides a reasonable estimate of the order of magnitude.

As can be seen in Fig. 3 (b), although the time dependence in the non-exponential region is well-described by a power function $P(t) = Bt^{-n}$, its exponent n is a bit larger for the experimental data than for the theoretical curves. The exponents of the power law found for various materials in the Rothe–Hintschich–Monkman experiment vary from 2 to 4 [12], which is not far from $n = 3$ predicted in the alpha-decay model (especially for Coumarin 450, $n \approx 2.9$), but it also indicates deviations from this very simple model. The physics of dissolved organic materials is certainly more complicated than the simple potential description, but apparently (9) provides a satisfactory first approximation.

There are also other models for decaying systems, *e.g.* the Onley–Kumar model [23], for which the state decays as t^{-4}. Therefore, while the Dirac delta potential provides a decent description for luminescence decay of Coumarin 450 or Rhodamine 6G, and important features of the theoretical model and of the experiment are in a satisfactory agreement, there exist more accurate models for other substances studied in [12].

4. Conclusions

In the present paper, the simple model of breakdown of exponential decay law was studied and compared for the first time with the experiment. Surprisingly, the model reproduces rather well the main features of the data. In this approximation, breakdown of the initial exponential time dependence occurs, and at large times, the process follows power behavior. These predictions, which originate from [6], were tested numerically to check analytic estimates and examine the low-λ behavior. The exponential and power law regimes were directly observed. The power law exponent was found to be $n = 3$ within uncertainty of the fit, as predicted. The breakdown of the exponential law was seen to occur at earlier times for low λ (low Q-value). On

the other hand, high Q-values mean better formation of resonances, so from experimental point of view, a compromise must be made. Such a balanced value is about $\lambda \approx 3.6$ ($Q \approx 2.5$). Moreover, oscillation of the non-escape probability in the intermediate stage of the process was observed. It originates from the interference between incoming and outgoing waves. However, the effect is likely to be model-dependent, in particular it is not observed in [12].

The results of the model studied here were compared to the experiment by Rothe, Hintschich and Monkman, which is the first experimental observation of turnover from the exponential time dependence to the power one. The simple model with a Dirac delta barrier successfully describes the experiment in both the exponential and power law regimes. To our knowledge, such a quantitative comparison between theory and experiment was never attempted before. An intriguing scaling in terms of the molecular mass and the atomic number was also found. Although physics of dissolved organic materials is more complex than this simple model and some deviations are observed, the one-dimensional model (9) provides a good, effective description.

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