

AHARONOV–BOHM EFFECT ON SPIN-0 SCALAR MASSIVE CHARGED PARTICLE WITH A UNIFORM MAGNETIC FIELD IN SOM–RAYCHAUDHURI SPACE-TIME WITH A COSMIC STRING

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We study the relativistic quantum dynamics of spin-0 massive charged particle in a Gödel-type space-time with electromagnetic interactions. We solve the Klein–Gordon equation subject to a uniform magnetic field in the Som–Raychaudhuri space-time with a cosmic string. In addition, we include a magnetic quantum flux into the relativistic quantum system, and obtain the energy eigenvalues and analyze an analogue of the Aharonov–Bohm (AB) effect.

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1. Introduction

The relativistic quantum dynamics of spin-0 and spin- $\frac{1}{2}$ particles have been investigated by several researchers. Spin-0 particles such as bosons, mesons are described by the Klein–Gordon equation and spin- $\frac{1}{2}$ particles such as fermions by the Dirac equation. The exact solutions of the wave equations are very important since they contain all the necessary information regarding the quantum system under consideration. However, analytical solutions are possible only in few cases, such as the hydrogen atom and harmonic oscillator [1, 2]. In recent years, many studies have been carried out to explore the relativistic energy eigenvalues and the corresponding wave-functions of these wave equations with or without external fields. The relativistic wave equations have been of current research interest for theoretical physicists [3, 4] including nuclear and high-energy physics [5, 6]. The relativistic quantum dynamics of spin-0 particles in the presence of external fields have been of great interest. The physical properties of the systems

are accessed by the solution of the Klein–Gordon equation with electromagnetic interactions [6, 7]. The electromagnetic interactions are introduced into the Klein–Gordon equation through the so-called minimal substitution, $p_\mu \rightarrow p_\mu - e A_\mu$, where e is the charge and A_μ is the four-vector potential of the electromagnetic field.

The relativistic quantum dynamics of spin-0 massive charged particles of mass M is described by the KG equation [8]

$$\left[\frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} g^{\mu\nu} D_\nu) - \xi R - M^2 \right] \Psi = 0, \quad (1)$$

where $D_\mu = \partial_\mu - i e A_\mu$ is the minimal substitution with e being the electric charge, A_μ is the potential of electromagnetic field, R is the scalar curvature, and ξ is the non-minimal coupling constant.

In recent years, several researchers have investigated the relativistic quantum dynamics of scalar particles in the background of the Gödel-type geometries. For example, the relativistic quantum dynamics of scalar particles [9], the Klein–Gordon oscillator with an external fields [10], scalar particles with a cosmic string [11], linear confinement of a scalar particle [12] (see also [13]), ground state of a bosonic massive charged particle in the presence of external fields in [14] (see also [15]). Furthermore, the relativistic quantum dynamics of a scalar particle in the Som–Raychaudhuri metric was investigated in [16, 17] and the similarity of the energy eigenvalues with the Landau levels in flat space was observed [1, 18]. The behavior of scalar particles with the Yukawa-like confining potential in the Som–Raychaudhuri space-time in the presence of topological defects was investigated in [19]. Other works are the scalar field subject to a Cornell potential [20], survey on the Klein–Gordon equation [21], bound states solution of spin-0 massive in a Gödel-type space-time with Coulomb potential [22]. In addition, spin-half particles have been studied in the Gödel-type space-time [9], in the Som–Raychaudhuri space-time with torsion and cosmic string [23], with topological defect [24], the Fermi field and Dirac oscillator in the Som–Raychaudhuri space-time [25], the Dirac Fermi field with scalar and vector potentials in the Som–Raychaudhuri space-time [26].

Our main motivation is to study the relativistic quantum dynamics of spin-0 scalar charged particles in the presence of an external fields including magnetic quantum flux in the Som–Raychaudhuri space-time with the cosmic string which was not studied in [11, 16]. We solve the Klein–Gordon equation in the considered framework and evaluate the energy eigenvalues and eigenfunctions, and analyze the relativistic analogue of the Aharonov–Bohm effect for bound states. We compare our results with [8, 11, 16] and see that the energy eigenvalues obtain here get modified due to the presence of various physical parameters.

2. Spin-0 scalar massive charged particles: The KG equation

Let us consider the following Som–Raychaudhuri (SR) space-time with a cosmic-string given by [11, 19, 22, 25, 26]

$$ds^2 = - (dt + \alpha \Omega r^2 d\phi)^2 + \alpha^2 r^2 d\phi^2 + dr^2 + dz^2, \quad (2)$$

where α and Ω characterize the cosmic string and the vorticity parameter of the space-time, respectively. The scalar curvature R of the space-time is given by

$$R = 2 \Omega^2. \quad (3)$$

We choose the four-vector potential of electromagnetic fields $A_\mu = (0, \vec{A})$ with

$$\vec{A} = (0, A_\phi, 0). \quad (4)$$

For geometry (2), KG equation (1) becomes

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \left\{ \frac{1}{\alpha r} \left(\frac{\partial}{\partial \phi} - i e A_\phi \right) - \Omega r \frac{\partial}{\partial t} \right\}^2 + \frac{\partial^2}{\partial z^2} - (M^2 + 2 \xi \Omega^2) \right] \Psi(t, r, \phi, z) = 0. \quad (5)$$

Since the line element is independent of time and symmetrical by translations along the z -axis, as well by rotations, it is reasonable to write the solution to Eq. (5) as

$$\Psi(t, r, \phi, z) = e^{i(-Et + l\phi + kz)} \psi(r), \quad (6)$$

where E is the energy of charged particle, $l = 0, \pm 1, \pm 2, \dots$ are the eigenvalues of the z -component of the angular momentum operator, and k are the eigenvalues of z -component of the linear momentum operator.

Substituting solution (6) into Eq. (5), we obtain the following equation for the radial wave function $\psi(r)$:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + E^2 - M^2 - k^2 - 2 \xi \Omega^2 - \frac{(l - e A_\phi)^2}{\alpha^2 r^2} - \Omega^2 E^2 r^2 - \frac{2 \Omega E}{\alpha} (l - e A_\phi) \right] \psi(r) = 0. \quad (7)$$

2.1. Interactions with a uniform magnetic field

Let us consider the electromagnetic four-vector potential associated with a uniform external magnetic field given by [8]

$$A_\phi = -\frac{1}{2} \alpha B_0 r^2 \quad (8)$$

such that the magnetic field is along the z -axis $\vec{B} = \vec{\nabla} \times \vec{A} = -B_0 \hat{k}$.

Substituting potential (8) into Eq. (7), we obtain the following radial-wave equation:

$$\psi''(r) + \frac{1}{r} \psi'(r) + \left[\lambda - \omega^2 r^2 - \frac{l^2}{\alpha^2 r^2} \right] \psi(r) = 0, \quad (9)$$

where we define

$$\begin{aligned} \lambda &= E^2 - M^2 - k^2 - \frac{2(\Omega E + M \omega_c) l}{\alpha} - 2\xi \Omega^2, \\ \omega &= \sqrt{\Omega^2 E^2 + 2 M \omega_c \Omega E + M^2 \omega_c^2} = (\Omega E + M \omega_c), \\ \text{and } \omega_c &= \frac{e B_0}{2 M} \end{aligned} \quad (10)$$

is called the cyclotron frequency of the charged particle moving in the magnetic field.

Transforming $x = \omega r^2$ into the above Eq. (9), we obtain the following differential equation:

$$\psi''(x) + \frac{1}{x} \psi'(x) + \frac{1}{x^2} (-\xi_1 x^2 + \xi_2 x - \xi_3) \psi(x) = 0, \quad (11)$$

where

$$\xi_1 = \frac{1}{4}, \quad \xi_2 = \frac{\lambda}{4\omega}, \quad \xi_3 = \frac{l^2}{4\alpha^2}. \quad (12)$$

Comparing Eq. (11) with (A.1) in Appendix A, we get

$$\begin{aligned} \alpha_1 &= 1, & \alpha_2 &= 0, & \alpha_3 &= 0, & \alpha_4 &= 0, & \alpha_5 &= 0, \\ \alpha_6 &= \xi_1, & \alpha_7 &= -\xi_2, & \alpha_8 &= \xi_3, & \alpha_9 &= \xi_1, & \alpha_{10} &= 1 + 2\sqrt{\xi_3}, \\ \alpha_{11} &= 2\sqrt{\xi_1}, & \alpha_{12} &= \sqrt{\xi_3}, & \alpha_{13} &= -\sqrt{\xi_1}. \end{aligned} \quad (13)$$

Therefore, the energy eigenvalues expression after inserting Eqs. (12)–(13) into the Eq. (A.8) in Appendix A is

$$\begin{aligned} E_{n,l}^2 - 2\Omega \left(2n + 1 + \frac{|l|}{\alpha} + \frac{l}{\alpha} \right) E_{n,l} - M^2 - k^2 - 2\xi \Omega^2 \\ - 2M\omega_c \left(2n + 1 + \frac{|l|}{\alpha} + \frac{l}{\alpha} \right) = 0 \end{aligned} \quad (14)$$

with the energy eigenvalues associated with n^{th} radial modes is

$$E_{n,l} = \Omega \left(2n + 1 + \frac{l}{\alpha} + \frac{|l|}{\alpha} \right) \pm \left\{ \Omega^2 \left(2n + 1 + \frac{l}{\alpha} + \frac{|l|}{\alpha} \right)^2 + M^2 + k^2 + 2M\omega_c \left(2n + 1 + \frac{|l|}{\alpha} + \frac{l}{\alpha} \right) + 2\xi\Omega^2 \right\}^{\frac{1}{2}}, \quad (15)$$

where $n = 0, 1, 2, \dots$ and k is a constant.

The corresponding eigenfunctions is

$$\psi_{n,l}(x) = |N|_{n,l} x^{\frac{|l|}{2\alpha}} e^{-\frac{x}{2}} L_n^{\left(\frac{|l|}{\alpha}\right)}(x), \quad (16)$$

where $|N|_{n,l} = \left(\frac{n!}{\left(n + \frac{|l|}{\alpha}\right)!} \right)^{\frac{1}{2}}$ is the normalization constant and $L_n^{\left(\frac{|l|}{\alpha}\right)}(x)$ is the generalized Laguerre polynomials which are orthogonal over $[0, \infty)$ with respect to the measure with weighting function $x^{\frac{|l|}{\alpha}} e^{-x}$ as

$$\int_0^\infty x^{\frac{|l|}{\alpha}} e^{-x} L_n^{\left(\frac{|l|}{\alpha}\right)} L_m^{\left(\frac{|l|}{\alpha}\right)} dx = \frac{\left(n + \frac{|l|}{\alpha}\right)!}{n!} \delta_{nm}. \quad (17)$$

In [16], the Klein–Gordon equation in the Som–Raychaudhuri space-time without topological defects was studied. The energy eigenvalues are given by

$$E_{n,l} = \Omega (2n + 1 + l + |l|) \pm \sqrt{\Omega^2 (2n + 1 + l + |l|)^2 + M^2 + k^2}. \quad (18)$$

Thus, by comparing the result obtained in [16], we can see that the energy eigenvalues Eq. (15) get modified (increase) due to the presence of a uniform magnetic field B_0 , the topological defect parameter α , and the non-minimal coupling constant ξ with the background curvature in the relativistic system.

In [11], the Klein–Gordon equation in the Som–Raychaudhuri space-time with a cosmic string was studied. The energy eigenvalues are given by

$$E_{n,l} = \Omega \left(2n + 1 + \frac{l}{\alpha} + \frac{|l|}{\alpha} \right) \pm \sqrt{\Omega^2 \left(2n + 1 + \frac{l}{\alpha} + \frac{|l|}{\alpha} \right)^2 + M^2 + k^2}. \quad (19)$$

By comparing the result without external field as obtained in [11], we can see that the energy eigenvalues Eq. (15) get modified (increase) due to the presence of a uniform magnetic field B_0 and the non-minimal coupling constant ξ in the relativistic system.

In [8], the relativistic quantum dynamics of a charged scalar particles in the presence of an external fields in the cosmic string space-time was studied. The energy eigenvalues is given by

$$E_{n,l} = \pm \sqrt{M^2 + k^2 + 2 M \omega_c \left(n + \frac{1}{2} + \frac{|l|}{2\alpha} + \frac{l}{2\alpha} \right)}. \quad (20)$$

Again, by comparing the energy eigenvalues Eq. (15) with those in [8] or Eq. (20), here, we can see that the present energy eigenvalues get modified due to the presence of the vorticity parameter Ω of the space-time and the non-minimal coupling constant ξ with the background curvature.

2.2. Interactions with an external field including the magnetic quantum flux

Let us consider the system described in Eq. (7) in the presence of an external fields in the z -direction. We have assumed that the topological defects (*e.g.*, cosmic string) have an internal magnetic flux field (with magnetic flux Φ_B) [27–29]. The electromagnetic four-vector potential is given by the following angular component [10, 37]:

$$A_\phi = -\frac{1}{2} \alpha B_0 r^2 + \frac{\Phi_B}{2\pi}. \quad (21)$$

Here, $\Phi_B = \text{const.}$ is the internal quantum magnetic flux [27–29] through the core of the topological defects [28]. The three-vector potential in symmetric gauge is defined by $\vec{A} = \vec{A}_1 + \vec{A}_2$ such that $\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}_1 + \vec{\nabla} \times \vec{A}_2 = \vec{B} = -B_0 \hat{k}$. It is worth mentioning that this Aharonov–Bohm effect [30, 31] has been investigated in graphene [32], Newtonian theory [33], bound states of massive fermions [34], scattering of dislocated wave fronts [35], with torsion effects on a relativistic position-dependent mass system [36–38], and bound states of spin-0 massive charged particles [22, 39]. In addition, this effect has been investigated in the context of the Kaluza–Klein theory [40–45], and with a non-minimal Lorentz-violating coupling [46].

Substituting potential (21) into Eq. (7), we obtain the following equation:

$$\psi''(r) + \frac{1}{r} \psi'(r) + \left[\lambda_0 - \omega^2 r^2 - \frac{j^2}{r^2} \right] \psi(r) = 0, \quad (22)$$

where

$$\begin{aligned} \lambda_0 &= E^2 - M^2 - k^2 - 2(\Omega E + M \omega_c) j - 2\xi \Omega^2, \\ j &= \frac{(l - \Phi)}{\alpha}. \end{aligned} \quad (23)$$

Following the similar technique as done earlier, we obtain the relativistic eigenvalues associated with n^{th} radial modes

$$E_{n,l} = \Omega \left(2n + 1 + \frac{l - \Phi + |l - \Phi|}{\alpha} \right) \pm \left\{ \Omega^2 \left(2n + 1 + \frac{l - \Phi + |l - \Phi|}{\alpha} \right)^2 + k^2 + M^2 + 2m\omega_c \left(2n + 1 + \frac{l - \Phi + |l - \Phi|}{\alpha} \right) + 2\xi \Omega^2 \right\}^{\frac{1}{2}}. \quad (24)$$

Equation (24) is the energy spectrum of massive charged particles in the presence of an external uniform magnetic field including a magnetic quantum flux in the Som–Raychaudhuri space-time with a cosmic string. The energy eigenvalues depend on the cosmic string parameter α , the external magnetic field B_0 including the magnetic quantum flux Φ_B , and the non-minimal coupling constant ξ . We can see that the energy eigenvalues Eq. (24) get modified in comparison to the result of Eq. (15) due to the presence of the magnetic quantum flux Φ_B which causes shifts of the energy levels and gives rise to a relativistic analogue of the Aharonov–Bohm effect.

The wave functions are given by

$$\psi_{n,l}(x) = |N|_{n,l} x^{\frac{|l-\Phi|}{2\alpha}} e^{-\frac{x}{2}} L_n^{\left(\frac{|l-\Phi|}{\alpha}\right)}(x), \quad (25)$$

where $|N|_{n,l} = \left(\frac{n!}{(n + \frac{|l-\Phi|}{\alpha})!} \right)^{\frac{1}{2}}$ is the normalization constant and $L_n^{\left(\frac{|l-\Phi|}{\alpha}\right)}(x)$ is the generalized Laguerre polynomial.

Special case

We discuss a special case that corresponds to zero vorticity parameter, $\Omega \rightarrow 0$. In that case, the study space-time (2) reduces to a static cosmic string space-time.

Therefore, the radial-wave equation Eq. (22) becomes

$$\psi''(r) + \frac{1}{r} \psi'(r) + \left[E^2 - M^2 - k^2 - 2M\omega_c j - M^2 \omega_c^2 r^2 - \frac{j^2}{r^2} \right] \psi(r) = 0. \quad (26)$$

Transforming $x = M\omega_c r^2$ into Eq. (26), we obtain

$$\psi''(x) + \frac{1}{x} \psi'(x) + \frac{1}{x^2} (-\xi_1 x^2 + \xi_2 x - \xi_3) \psi(x) = 0, \quad (27)$$

where

$$\xi_1 = \frac{1}{4}, \quad \xi_2 = \frac{E^2 - M^2 - k^2 - 2M\omega_c j}{4M\omega_c}, \quad \xi_3 = \frac{j^2}{4}. \quad (28)$$

We obtained the following energy eigenvalues expression associated with n^{th} radial modes:

$$E_{n,l} = \pm \left(M^2 + k^2 + 4 M \omega_c \left(n + \frac{1}{2} + \frac{|l - \Phi|}{2\alpha} + \frac{(l - \Phi)}{2\alpha} \right) \right)^{\frac{1}{2}}, \quad (29)$$

where $n = 0, 1, 2, \dots$ and the corresponding eigenfunction is given by Eq. (25).

Equation (29) is the relativistic energy eigenvalue of a massive charged particle in the presence of an external fields including a magnetic quantum flux in static cosmic string space-time. Let us observe that the energy eigenvalue Eq. (29) in comparison to those result [8] gets modified due to the presence of the magnetic quantum flux Φ_B which causes shifts of the energy levels and gives rise to a relativistic analogue of the Aharonov–Bohm effect.

We can see in the above expressions of the energy eigenvalues Eqs. (24) and (29) that the angular momentum l is shifted

$$l_{\text{eff}} = \frac{1}{\alpha} (l - \Phi), \quad (30)$$

an effective angular momentum due to both the boundary condition, which states that the total angle around the string is $2\pi\alpha$, and the minimal coupling with the electromagnetic fields. We can see that the relativistic energy eigenvalues Eqs. (24) and (29) depend on the geometric quantum phase [27, 28]. Thus, we have that $E_{n,l}(\Phi_B + \Phi_0) = E_{n,l \pm \tau}(\Phi_B)$, where $\Phi_0 = \mp \frac{2\pi}{e} \tau$ with $\tau = 0, 1, 2, \dots$. This dependence of the relativistic energy eigenvalues on the geometric quantum phase gives rise to a relativistic analogue of the Aharonov–Bohm effect.

Formula (25) suggests that when the particle circles the string, the wavefunction changes according to

$$\Psi \rightarrow \Psi' = e^{2i\pi l_{\text{eff}}} \Psi = \exp \left\{ \frac{2\pi i}{\alpha} \left(l - \frac{e\Phi_B}{2\pi} \right) \right\} \Psi. \quad (31)$$

An immediate consequence of Eq. (31) is that the angular momentum operator may be redefined as

$$\hat{l}_{\text{eff}} = -\frac{i}{\alpha} \left(\partial_\phi - i \frac{e\Phi_B}{2\pi} \right), \quad (32)$$

where the additional term, $-\frac{e\Phi_B}{2\pi\alpha}$, takes into account the Aharonov–Bohm magnetic flux Φ_B (internal magnetic field).

3. Conclusions

In this paper, we have investigated spin-0 massive charged particles in the presence of an external fields including a magnetic quantum flux in the Som–Raychaudhuri space-time with a cosmic string. We have introduced the electromagnetic interactions into the Klein–Gordon equation through the minimal substitution. In Section 2.1, the Klein–Gordon field in the background of the Som–Raychaudhuri space-time with a cosmic string in the presence of external uniform magnetic field is considered and the final form of the radial wave equation is derived. We then solved it using the Nikiforov–Uvarov method and obtained the relativistic energy eigenvalues Eq. (15) and corresponding eigenfunctions Eq. (16). We have seen that the relativistic energy eigenvalues depend on the cosmic string (α), the parameter (Ω) that characterises vorticity of the space-time, the external magnetic field (B_0), and the non-minimal coupling constant (ξ). We have seen that the energy eigenvalues Eq. (15) get modified (increase) in comparison to those results obtained in [11, 16] due to the presence of an external uniform magnetic field as well as the cosmic string with the non-minimal coupling constant. We have also seen that the energy eigenvalues Eq. (15) in comparison to the result in [8] get modified (increase) due to the presence of vorticity parameter (Ω) of the space-time. In Section 2.2, we have considered an external uniform magnetic field including a magnetic quantum flux and driven the final form of the radial wave equation. We have solved this equation using the same method and obtained the relativistic energy eigenvalues Eq. (24) and corresponding eigenfunctions Eq. (25). The expression for the relativistic energy eigenvalues Eq. (24) reveals the possibility of establishing a quantum condition between the energy eigenvalues of a massive charged particle and the parameter that characterize the vorticity of the space-time (Ω). There we have discussed a special case that corresponds to zero vorticity parameter and seen that the energy eigenvalues Eq. (29) get modified (decrease) in comparison to the results in [8] due to the presence of a magnetic quantum flux. We have seen that the relativistic eigenvalues depend on the geometric quantum phase [27, 28] and we have that $E_{n,l}(\Phi_B + \Phi_0) = E_{n,l \mp \tau}(\Phi_B)$, where $\Phi_0 = \pm \frac{2\pi}{e} \tau$ with $\tau = 0, 1, 2, \dots$. This dependence of the energy eigenvalues on the geometric quantum phase gives rise to an analogue of the Aharonov–Bohm effect.

In this paper, we have shown some results which in addition to the previous results obtained in [8, 11, 16] present many interesting effects. This is the fundamental subject in physics and connection between these theories (gravitation and quantum mechanics) is not well-understood.

Appendix A

Brief review of the Nikiforov–Uvarov (NU) method

The Nikiforov–Uvarov method is helpful in order to find eigenvalues and eigenfunctions of the Schrödinger-like equation, as well as other second-order differential equations of physical interest. According to this method, the eigenfunctions of a second-order differential equation [47]

$$\frac{d^2\psi(s)}{ds^2} + \frac{(\alpha_1 - \alpha_2 s)}{s(1 - \alpha_3 s)} \frac{d\psi(s)}{ds} + \frac{(-\xi_1 s^2 + \xi_2 s - \xi_3)}{s^2(1 - \alpha_3 s)^2} \psi(s) = 0 \quad (\text{A.1})$$

are given by

$$\psi(s) = s^{\alpha_{12}} (1 - \alpha_3 s)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}} P_n^{\left(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10}-1\right)} (1 - 2\alpha_3 s), \quad (\text{A.2})$$

and the energy eigenvalues are

$$\alpha_2 n - (2n + 1)\alpha_5 + (2n + 1)(\sqrt{\alpha_9} + \alpha_3 \sqrt{\alpha_8}) + n(n - 1)\alpha_3 + \alpha_7 + 2\alpha_3 \alpha_8 + 2\sqrt{\alpha_8 \alpha_9} = 0. \quad (\text{A.3})$$

The parameters $\alpha_4, \dots, \alpha_{13}$ are obtained from the six parameters $\alpha_1, \dots, \alpha_3$ and ξ_1, \dots, ξ_3 as follows:

$$\begin{aligned} \alpha_4 &= \frac{1}{2}(1 - \alpha_1), & \alpha_5 &= \frac{1}{2}(\alpha_2 - 2\alpha_3), \\ \alpha_6 &= \alpha_5^2 + \xi_1, & \alpha_7 &= 2\alpha_4 \alpha_5 - \xi_2, \\ \alpha_8 &= \alpha_4^2 + \xi_3, & \alpha_9 &= \alpha_6 + \alpha_3 \alpha_7 + \alpha_3^2 \alpha_8, \\ \alpha_{10} &= \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, & \alpha_{11} &= \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3 \sqrt{\alpha_8}), \\ \alpha_{12} &= \alpha_4 + \sqrt{\alpha_8}, & \alpha_{13} &= \alpha_5 - (\sqrt{\alpha_9} + \alpha_3 \sqrt{\alpha_8}). \end{aligned} \quad (\text{A.4})$$

In a special case where $\alpha_3 = 0$, as in our case, we find the following:

$$\lim_{\alpha_3 \rightarrow 0} P_n^{\left(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10}-1\right)} (1 - 2\alpha_3 s) = L_n^{\alpha_{10}-1}(\alpha_{11} s), \quad (\text{A.5})$$

and

$$\lim_{\alpha_3 \rightarrow 0} (1 - \alpha_3 s)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}} = e^{\alpha_{13} s}. \quad (\text{A.6})$$

Therefore, the wave-function from (A.2) becomes

$$\psi(s) = s^{\alpha_{12}} e^{\alpha_{13} s} L_n^{\alpha_{10}-1}(\alpha_{11} s), \quad (\text{A.7})$$

where $L_n^{(\alpha)}(s)$ denotes the generalized Laguerre polynomial.

The energy eigenvalues equation reduces to

$$n\alpha_2 - (2n+1)\alpha_5 + (2n+1)\sqrt{\alpha_9} + \alpha_7 + 2\sqrt{\alpha_8\alpha_9} = 0. \quad (\text{A.8})$$

Note that the simple Laguerre polynomial is the special case of $\alpha = 0$ of the generalized Laguerre polynomial

$$L_n^{(0)}(s) = L_n(s). \quad (\text{A.9})$$

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