

A NEW NON-HERMITIAN QUADRATIC OPERATOR HAVING EXACT SOLUTION

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We report on the exact solution of a new modelled one-dimensional non-Hermitian quadratic operator generated using similarity transformation. In fact, the model Hamiltonian satisfies the T -symmetry condition *i.e.* $[H, T] = 0$. Apart from analytical study, we perform necessary computational work to verify the analytical results.

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1. Introduction

While dealing with the Hermitian operators [1]

$$H = H^+, \quad (1)$$

we all ensure that energy spectrum of the operators must be real. In other words, we call all the real eigenvalues of the operators as “unbroken spectra”. However, Bender and Boettcher [2] discovered that unbroken spectra can also be associated with Hamiltonian having PT -symmetry in nature *i.e.*

$$[H, PT] = 0. \quad (2)$$

In the above, P stands for parity operator having space reflection behaviour as

$$PxP^{-1} = -x, \quad (3)$$

$$P|x|P^{-1} = |x|, \quad (4)$$

$$PpP^{-1} = -p, \quad (5)$$

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and T stands for time reversal operator having the behaviour

$$TiT^{-1} = -i, \quad (6)$$

$$TpT^{-1} = -p, \quad (7)$$

$$TxT^{-1} = x, \quad (8)$$

$$T|x|T^{-1} = |x|. \quad (9)$$

Considering this, the commutation relation of

$$[x, p] = i \quad (10a)$$

remains invariant under PT operation. Similarly, one will see that the commutation relation defined by [3]

$$\frac{1}{(1 + \beta\lambda)}[x + i\beta p, p + i\lambda x] = i \quad (10b)$$

is also PT -invariant. In a previous study [3], this new commutation relation (Eq. (10b)) has been discussed using similarity transformation [4, 5], perturbation theory [3] and Lie algebraic analysis [6]. The above analysis is used for quadratic operators and the invariant of commutation relation is basically an indication on the energy invariance. We come across an interesting analysis on spectral nature of operator associated with $|x|$ in which Bender *et al.* [7] reported that certain PT -symmetry operator *i.e.*

$$H = p^2 + ix|x| \quad (11)$$

can yield broken spectra. In fact, this motivates one to study $|x|$ in connection with complex Hamiltonian. As reported above, we find that only operator $|x|$ is unaffected by either parity or time reversal transformation. Thus, we propose a new condition to get unbroken spectra *i.e.*

$$[H, T] = 0 \quad (12)$$

along with the following conditions:

$$HT|\Psi\rangle = TH|\Psi\rangle = E(T|\Psi\rangle), \quad (13)$$

$$T|\Psi\rangle = |\Psi\rangle. \quad (14)$$

It is worth mentioning here that the wave function of the operator H must be T -invariant. In order to justify our stand, we consider an exactly solvable quadratic operator. Prior to this, we also suggest a procedure how to generate the same using similarity transformation in the following.

2. Matrix model analysis

Let us consider a Hermitian matrix in the form of (2×2) as

$$H^{\text{Hermitian}} = H = \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix} \quad (15)$$

having eigenvalues

$$\lambda_1^H = 5.618 \quad \text{and} \quad \lambda_2^H = 3.382. \quad (16)$$

Using similarity transformation approach, the matrix S can be defined as

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}. \quad (17)$$

Now, we generate a new matrix as follows:

$$h = SHS^{-1}, \quad (18)$$

where h is given by

$$h = \begin{bmatrix} 3 & 1 \\ -1 & 6 \end{bmatrix}. \quad (19)$$

Further, the eigenvalues of this operator (Eq. (19)) are

$$\lambda_1^h = 5.618 \quad \text{and} \quad \lambda_2^h = 3.382. \quad (20)$$

However, h is non-Hermitian (*i.e.* $h \neq h^+$) but preserves real spectral nature. This non-Hermiticity can easily be visualised if one wrote it as

$$h = \begin{bmatrix} 3 & i \\ i & 6 \end{bmatrix}. \quad (21)$$

3. New non-Hermitian quadratic operator with $p \rightarrow p + i\beta|x|$

Now, let us consider an exactly solvable quadratic model Hermitian operator as

$$H_1|\Phi_n\rangle = [p^2 + \lambda x^2] |\Phi_n\rangle = E_n|\Phi_n\rangle = (2n+1)\sqrt{\lambda}|\Phi_n\rangle. \quad (22)$$

The wave function of the above operator is exactly known [1]. Now, we use similarity transformation as [8]

$$\langle\Phi_n|S_1^{-1} (S_1 H_1 S_1^{-1}) S_1|\Phi_n\rangle. \quad (23)$$

Hence, the new Hamiltonian ($h_1 = S_1 H_1 S_1^{-1}$) satisfies the eigenvalue relation

$$h_1 |\Psi_n\rangle = \varepsilon_n |\Psi_n\rangle. \quad (24)$$

The transformation operator is selected as

$$S_1 = e^{\frac{\beta|x|^2}{2}}, \quad (25)$$

then it is easy to see that

$$S_1 p S_1^{-1} \rightarrow p + i\beta|x|. \quad (26)$$

Hence, the commutation relation becomes invariant *i.e.*

$$[x, p + i\beta|x|] = i. \quad (27)$$

Therefore, the new Hamiltonian (h_1) becomes

$$S_1 H_1 S_1^{-1} = h_1 = p^2 + (\lambda - \beta^2) x^2 + i\beta(|x|p + p|x|). \quad (28)$$

Now, let us consider the case $\lambda \gg \beta^2$. Let $\lambda = 1 + \beta^2$, then the new non-Hermitian Hamiltonian reads as

$$h_1 = p^2 + x^2 + i\sqrt{\lambda - 1}(|x|p + p|x|). \quad (29)$$

As a model example, we consider $\lambda = 3$. Now, the h_1 (Eq. (28)) can be written as

$$h_1 = p^2 + x^2 + i\sqrt{2}(|x|p + p|x|) \quad (30)$$

having energy eigenvalue as

$$\varepsilon_n = (2n + 1)\sqrt{3}, \quad (31)$$

and wave function as

$$|\Psi_n\rangle = S_1 |\Phi_n\rangle. \quad (32)$$

4. Computation codes using MATLAB

In the following, we give the MATLAB code to generate the eigenvalues of the operator using the matrix diagonalisation method [9, 10]

$$N = 100; \quad n = 1 : N - 1; \quad m = \text{sqrt}(n); \quad (33)$$

$$x = [\text{diag}(m, -1) + \text{diag}(m, 1)] / \sqrt{2}; \quad (34)$$

$$p = i \times [\text{diag}(m, -1) - \text{diag}(m, 1)] / \sqrt{2}; \quad (35)$$

$$H = p^2 + x^2 + i \times \text{sqrt}(2) \times [p \times \text{sqrtm}(x^2) + \text{sqrtm}(x^2) \times p]; \quad (36)$$

$$\text{EigSort} = \text{sort}(\text{eig}(H)); \quad (37)$$

$$\text{EigSort}(1 : 10). \quad (38)$$

The first five eigenvalues of the operator (Eq. (30)) are tabulated in Table I.

TABLE I

Eigenvalues of non-Hermitian Hamiltonian (Eq. (30)).

Quantum number	Computed eigenvalue	Exact result
0	1.732 05	$\sqrt{3}$
1	5.196 15	$3\sqrt{3}$
2	8.660 25	$5\sqrt{3}$
3	12.124 35	$7\sqrt{3}$
4	15.588 45	$9\sqrt{3}$

5. Discussion and conclusions

In this paper, we proposed a new quadratic exactly solvable non-Hermitian model operator, whose energy eigenvalue and wave function can be calculated analytically. In fact, this new Hamiltonian is different from the standard PT -symmetric Hamiltonian as discussed above. We present a matrix model to derive the non-Hermitian T -symmetry model operator. Operator analysis involving $|x|$ and p has been carried out to generate the proposed new Hamiltonian using similarity transformation. In order to cross-check the theoretical analysis, we have given simple computer codes in matrix diagonalisation method using MATLAB. We hope both the beginners and experts will find it useful for further research in Mathematical Physics, Quantum Mechanics, 1-D field theory *etc.* One can extend the model on numerical calculation of any non-linear oscillator. For example, we find spectra of anharmonic (duffing) oscillator

$$H_3 = p^2 + x^2 + x^4, \quad (39)$$

and corresponding non-Hermitian Hamiltonian

$$h_3 = p^2 + i(|x|p + p|x|) + x^4 \quad (40)$$

remains the same. In the above model, we have briefly discussed unbroken spectra. It is worth mentioning here that like PT -symmetry breaking, the T -symmetry operator has also broken spectra. However, one does not know exactly the cause of broken spectra of some operators in Quantum Physics.

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REFERENCES

- [1] L.I. Schiff, «Quantum Mechanics, Third Edition», *Mc Graw-Hill*, Singapore 1985.
- [2] C.M. Bender, S. Boettcher, «Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry», *Phys. Rev. Lett.* **80**, 5243 (1998).
- [3] B. Rath, P. Mallick, «Zero energy correction method for non-Hermitian Harmonic oscillator with simultaneous transformation of co-ordinate and momentum: Wave function analysis under Iso-spectral condition», [arXiv:1501.06161 \[quant-ph\]](#); «Energy and Wave function Analysis on Harmonic Oscillator Under Simultaneous Non-Hermitian Transformations of Co-ordinate and Momentum: Iso-spectral case», *Open Phys.* **14**, 492 (2016).
- [4] F.M. Fernandez, «Non-Hermitian Hamiltonians and Similarity Transformations», *Int. J. Theor. Phys.* **55**, 843 (2016).
- [5] Z. Ahmed, «Pseudo-Hermiticity of Hamiltonians under gauge-like transformation: real spectrum of non-Hermitian Hamiltonians», *Phys. Lett. A* **294**, 287 (2002).
- [6] H.B. Zhang, G.Y. Jiang, G.C. Wang, «Unified algebraic method to non-Hermitian systems with Lie algebraic linear structure», *J. Math. Phys.* **56**, 072103 (2015).
- [7] C.M. Bender, D.W. Hook, L.R. Mead, «Conjecture on the analyticity of PT -symmetric potentials and the reality of their spectra», *J. Phys. A* **41**, 392005 (2008).
- [8] B. Rath, «Pseudo-hermiticity *versus* PT symmetry: spectral breakdown in nearly degenerate quantum energy eigenvalues», *Open Acc. J. Math. Theor. Phys.* **2**, 10 (2019).
- [9] H.J. Korsch, K. Rapedius, «Computations in quantum mechanics made easy», *Eur. J. Phys.* **37**, 055410 (2016).
- [10] B. Rath, «Iso-spectral Instability of Harmonic Oscillator: Breakdown of Unbroken Pseudo-Hermiticity and PT Symmetry Condition», *African Rev. Phys.* **10**, 427 (2015).