

QUARK AND GLUON CONDENSATES IN QCD REEVALUATED*

RENATA JORA

National Institute of Physics and Nuclear Engineering
P.O. Box MG-6, Bucharest-Măgurele, Romania
rjora@theory.nipne.ro

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We compute the quark and gluon condensates in QCD with N colors and N_f flavors based on the renormalization group equations and on the knowledge of a single-scale Λ which is directly related to Λ_{QCD} . For $N = 3$ and $N_f = 3$, our findings are in the good range for $0.2 \text{ GeV} \leq \Lambda \leq 0.36 \text{ GeV}$ and in excellent agreement with the results in the literature from sum rules for a value $\Lambda_{\text{QCD}} = 0.28 \text{ GeV}$.

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Quark and gluon condensates are vacuum expectations values of local operators. They are often considered in the context of quark–hadron dualities and calculated as such. However, the general view is that these condensates represent characteristic scales of the QCD vacuum that may be regarded as independent of the hadron degrees of freedom [1].

The quark condensate in QCD always indicates dynamical supersymmetry of the chiral symmetry $\text{SU}(3)_L \times \text{SU}(3)_R$ which is associated with the presence of the Nambu–Goldstone bosons. It is expected that the chiral group is broken down to $\text{SU}(3)_V$ such that all quark condensates are, at least in first order, the same.

For the case when one restricts the chiral group to $\text{SU}(2)_L \times \text{SU}(2)_R$, the relation between the quark condensates and the pions becomes evident in the context of the Gell-Mann–Oakes–Renner relation [2, 3]

$$f_\pi^2 m_\pi^2 = -2(m_u + m_d)\langle \bar{q}q \rangle. \quad (1)$$

Here, m_π is the mass of the pion, f_π is the pion decay constant, m_u and m_d are the current quark masses, and $\langle \bar{q}q \rangle$ is the vacuum quark condensate. Further relations of interest may be obtained from the current algebra [4] and the axial vector Ward–Takahashi identity.

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Although there is less information regarding the gluon condensate, both quark and gluon condensates have been computed in the literature in the lattice QCD approach [5], sum rules [6–11], and chiral models [12].

The purpose of the present work is to compute once again the quark and gluon condensates from the renormalization group equations of the quantum field theory of QCD without reference to the hadron structure and properties. Although usually in the literature the formation of the quark and gluon condensates are intertwined with confinement, here we will consider them as unrelated topics. We are less interested in finding the exact point in the space (N, N_f) where the condensates form and the actual phase transitions take place. Instead, we will aim to determine the values of these condensates for a specific value of the coupling constant in terms of the intrinsic scale of the Λ_{QCD} theory, which is considered known.

We consider QCD with N colors and N_f flavors. We work with the all order beta function inspired by the supersymmetric NSVZ beta function proposed in [13]

$$\beta(g) = \frac{\partial g}{\partial \ln(\mu)} = -\frac{g^3}{16\pi^2} \frac{\frac{11}{3}N - \frac{N_f}{3}(2 + \Delta_R \gamma_R)}{1 - \frac{g^2}{8\pi^2} \frac{17}{11}N}, \quad (2)$$

where

$$\Delta_R = 1 + \frac{17}{11} \frac{2N^2}{N^2 - 1}, \quad (3)$$

and

$$\gamma_R = -\frac{d \ln(m)}{d \ln(\mu)} = 3 \frac{N^2 - 1}{N} \frac{g^2}{16\pi^2}. \quad (4)$$

We will also need the anomalous dimension of the fermions wave function which is given by

$$\gamma_1 = \frac{g^2}{12\pi^2}. \quad (5)$$

We first consider the quark two-point correlator in the Fourier space which we will denote by $Q(p, M) = \frac{i}{p^\mu \gamma_\mu} f(\frac{p^\rho \gamma_\rho}{M})$, where f is an arbitrary function that we need to determine. We apply the renormalization group equations

$$\left[M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} + 2\gamma_1 \right] Q(p^\rho \gamma_\rho, M) = 0. \quad (6)$$

Here, M is the renormalization scale. We pick a specific point where $\beta(g) = \infty$ and, consequently, $(1 - \frac{g_1^2}{8\pi^2} \frac{17}{11}N) = 0$. Then either

$$\frac{\partial}{\partial g} Q(p, M) = 0, \quad (7)$$

or $Q(p, M)$ has a complicated dependence on g . We assume that the choice in Eq. (7) is valid. We solve Eq. (6) to find

$$Q_1 = \text{Tr} [Q(p, M)] = \text{Tr} \left[\frac{i}{p^\mu \gamma_\mu} a \left[\frac{p^\rho \gamma_\rho}{M} \right]^{2\gamma_1} \right], \quad (8)$$

where a is a constant determined as $a = 1$ from the renormalization condition $Q(p, M)|_{p^\rho \gamma_\rho = M} = \frac{i}{p^\mu \gamma_\mu}$.

The quark condensate can be then calculated as

$$\begin{aligned} \langle \bar{q}q \rangle &= v = \int \frac{d^4 p}{(2\pi)^4} Q_1 = i \int \frac{d^4 p_E}{(2\pi)^4} \text{Tr} [Q(p_E, M)] \\ &\approx -\frac{3}{4\pi^2} \int d(p_E^2) p_E^2 \frac{1}{M^{2\gamma_1}} (p_E^2)^{\gamma_1 - \frac{1}{2}} \\ &= -\frac{3}{4\pi^2} \frac{2}{2\gamma_1 + 3} M^3. \end{aligned} \quad (9)$$

Here, we integrated up to the scale M which corresponds to the coupling constant g_1 and the integral was performed in the Euclidean space.

The next step is to consider the two-point correlator in the Fourier space for the gluon field $G^{\mu\nu}(p^2, M^2) = -ig^{\mu\nu} \frac{1}{p^2} h(\frac{p^2}{M^2})$ (we work in the Feynman gauge), where again h is an arbitrary function, a solution of the associated renormalization group equation

$$\left[\frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} + 2\gamma_A \right] G^{\mu\nu}(p^2, M^2) = 0. \quad (10)$$

Here, γ_A is the anomalous dimension of the gluon wave function

$$\gamma_A = -\frac{3g^2}{16\pi^2}. \quad (11)$$

We consider the same fixed coupling constant g_1 and by the same arguments as for the quark condensate, we find

$$G_1 = \text{Tr} [G^{\mu\nu}(p^2, M^2)] = b4(N^2 - 1) \frac{-i}{p^2} \left(\frac{p^2}{M^2} \right)^{\gamma_A}. \quad (12)$$

The constant b is calculated as before from the renormalization condition at scale M as $b = 1$. The gluon condensate can then be computed approximately from

$$\begin{aligned} T &= \langle G_{\mu\nu}^a G^{a\mu\nu} \alpha \rangle = \int \frac{d^4 p}{(2\pi)^4} 2p^2 G_1 \frac{g_1^2}{4\pi} = \int \frac{d^4 p_E}{(2\pi)^4} 8(N^2 - 1) \left(\frac{p_E^2}{M^2} \right)^{\gamma_A} \\ &= \frac{1}{2\pi^2} \frac{1}{\gamma_A + 2} M^4 (N^2 - 1) \frac{g_1^2}{4\pi}. \end{aligned} \quad (13)$$

Here, $\alpha = \frac{g_1^2}{4\pi}$. Note that $G_{\mu\nu}^a G^{a\mu\nu} = -2A_\nu^a \partial^2 A_\nu^a + \dots$ and G_1 is proportional to the two gluon correlator and, therefore, in the Fourier space an extra factor of $2p^2$ corresponding to $-2\partial^2$ will appear in the integral in the first line of Eq. (13).

The scale M where the coupling constant is $g_1 = (\frac{88\pi^2}{51})^{\frac{1}{2}}$ is calculated by integrating Eq. (2) from Λ to M . The integral is done in first order as follows. One writes

$$\begin{aligned} \frac{dg}{d \ln(\mu)} &\approx -\frac{g^3}{16\pi^2} \frac{\frac{11}{3}N - \frac{2}{3}N_f}{1 - \frac{g^2}{8\pi^2} \frac{17}{11}N}, \\ \frac{dg}{-\frac{g^3}{16\pi^2}} \left[1 - \frac{g^2}{8\pi^2} \frac{17}{11}N \right] &\approx \left[\frac{11}{3}N - \frac{2}{3}N_f \right] d \ln(\mu), \\ d \left(\frac{8\pi^2}{g^2} \right) \left[1 - \frac{g^2}{8\pi^2} \frac{17}{11}N \right] &\approx \left[\frac{11}{3}N - \frac{2}{3}N_f \right] d \ln(\mu), \end{aligned} \quad (14)$$

and integrates from Λ which is the scale for which the contribution of the coupling constant to the left hand side of Eq. (14) (there is a small difference between Λ and what is called Λ_{QCD} which in the first approximation may be ignored) is zero to M where the coupling constant is g_1 . This leads to

$$\frac{1}{g_1^2} - \ln[g_1^2] \frac{1}{8\pi^2} \frac{17}{11}N \approx \frac{\frac{11}{3}N - \frac{2}{3}N_f}{8\pi^2} \ln\left(\frac{M}{\Lambda}\right), \quad (15)$$

from which one can determine

$$M \approx \Lambda \exp \left[\left[\frac{1}{g_1^2} - \ln[g_1^2] \frac{1}{8\pi^2} \frac{17}{11}N \right] \frac{8\pi^2}{\frac{11}{3}N - \frac{2}{3}N_f} \right]. \quad (16)$$

By specifying N and N_f , one can plot M as a function of Λ but instead, we shall use the expression in Eq. (16) directly.

In this paper, we determined the quark and gluon condensate based on first principle and on the knowledge of a single parameter Λ which we associate with what is usually called Λ_{QCD} . In order to do that, we needed a particular value of the coupling constant at some scale compatible with the beta function. Instead of the standard QCD beta function computed at five loops in [14], we used the all beta function inspired by the NSVZ beta function proposed by Pica and Sannino in [13]. Our choice had a double purpose: first, it simplified the calculations in a great measure, second, it made it easier to extrapolate our findings to the supersymmetric QCD similar calculations of the condensates. The latter analysis will be reserved for future work.

We considered standard QCD with $N = 3$ and $N_f = 3$.

In Fig. 1, we plotted the quark condensate in terms of the intrinsic scale of the theory Λ . For the range of $0.2 \leq \Lambda \leq 0.36$ GeV, we obtained $-0.005 \leq v \leq -0.03$ GeV³. In Fig. 2, we plotted the gluon condensate in terms of the intrinsic scale of the theory Λ . For the range of $0.2 \leq \Lambda \leq 0.36$ GeV, we got $0.017 \leq T \leq 0.18$ GeV⁴.

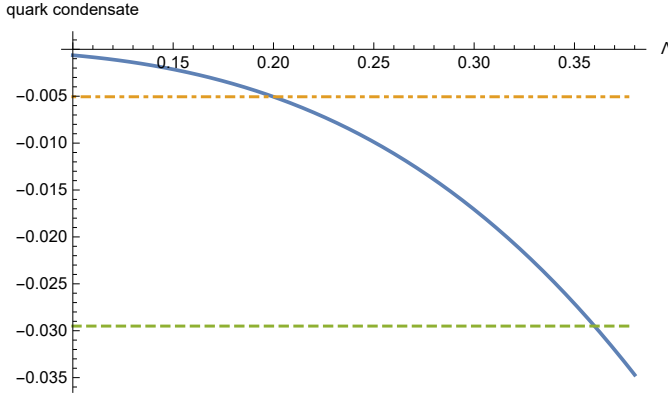


Fig. 1. Plot of the quark condensate $\langle \bar{q}q \rangle = v$ (in (GeV)³) in terms of Λ (in GeV), the intrinsic scale of the theory. The intersection of the thick line with the dot-dashed line indicates the value of the condensate for $\Lambda = 0.2$ GeV, whereas the intersection with the dashed line indicates the value of the condensate for $\Lambda = 0.36$ GeV.

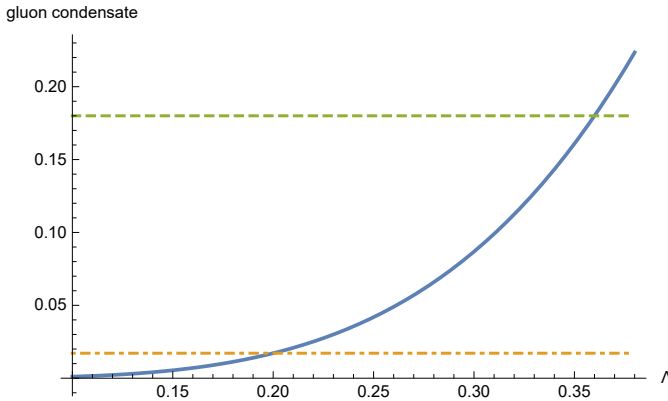


Fig. 2. Plot of the gluon condensate $\langle G_{\mu\nu}^a G^{a\mu\nu} \rangle = T$ (in (GeV)⁴) in terms of Λ (in GeV), the intrinsic scale of the theory. The intersection of the thick line with the dot-dashed line indicates the value of the condensate for $\Lambda = 0.2$ GeV, whereas the intersection with the dashed line indicates the value of the condensate for $\Lambda = 0.36$ GeV.

The gluon condensate was extracted for non-perturbative QCD from charmonium sum rules in [7]

$$\langle \alpha_s G^2 \rangle \approx 0.04 \text{ GeV}^4. \quad (17)$$

In the subsequent years, many other calculations were performed, some of them close to the value in Eq. (17), others like in lattice QCD, for example [5], with a larger range of values.

The light quark condensate has also been estimated in sum rules as [11]

$$\langle \bar{q}q \rangle = -(267 \pm 16 \text{ MeV})^3, \quad (18)$$

or lattice QCD [5]

$$\langle \bar{q}q \rangle = -(283 \pm 2 \text{ MeV})^3. \quad (19)$$

It might be more useful to find specific values of the condensates based on our knowledge of low-energy QCD. For that, we use the estimate for the gluon condensate in the sum rules approach as taken from [9] $\langle \frac{g^2}{4\pi} G^2 \rangle \approx 0.07 \text{ GeV}^4$ to determine $\Lambda_{\text{QCD}} = 0.28 \text{ GeV}$ and with this value, we computed the quark condensate $\langle \bar{q}q \rangle \approx -0.015 \text{ GeV}^3 = -(0.244 \text{ GeV})^3$. This compares very well with other results in the literature computed using other different methods [5–12].

The method can be used for determining the quark and gluon condensate for an arbitrary number of flavors and colors pending on the knowledge of the characteristic scale of the theory Λ .

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