SPIN POTENTIAL FOR RELATIVISTIC PARTICLES WITH SPIN 1/2

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We analyze algebraic structure of a relativistic semi-classical Wigner function of massive particles with spin 1/2 and show that it consistently includes information about the spin density matrix both in two-dimensional spin and four-dimensional spinor spaces. This result is subsequently used to explore various forms of equilibrium functions that differ by specific incorporation of spin potential. We argue that a scalar spin potential should be momentum dependent, while a tensor one may be a function of space-time coordinates only. This allows for the use of the tensor form in local thermodynamic relations. We furthermore show how scalar and tensor forms can be linked to each other.

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1. Introduction

In this work, we first analyze algebraic structure of a relativistic semi-classical Wigner function for massive particles with spin 1/2. We restrict our considerations to the leading order of expansion in $\hbar$ and show that it consistently includes information about the spin density matrix both in two-dimensional spin and four-dimensional spinor spaces. This consistency strongly relies on the fact that the two-by-two spin density matrix operates with quantities defined in the particle rest frame.

In the next step, we study equilibrium Wigner functions that differ by the form of spin potential. We demonstrate that a scalar spin potential frequently used in the literature [1–3] should be momentum dependent. In contrast, a tensor spin potential, introduced in new studies of hydrodynamics with spin [4], may be a function of space-time coordinates only. This allows for the use of the tensor form in local thermodynamic relations. We furthermore show how scalar and tensor forms can be linked to each other, provided the polarization effects are small.

Our results show that the introduction of a scalar spin potential is quite arbitrary. In contrast, the tensor form has much better physical motivation, as it plays a role of the Lagrange multiplier(s) coupled to angular momentum, which is a conserved quantity [5].

We expect that our results will be helpful for better understanding of equilibrium properties of particles with spin. This is important for development of hydrodynamic and kinetic theories for such systems, and is very much desirable in the context of the spin polarization measurements in heavy-ion collisions [6]. The latter revealed a non-zero effect for Λ hyperons, with a momentum dependence of polarization still waiting for a convincing theoretical explanation [7].

Our paper also clarifies the Lorentz structure of different quantities describing spin densities, therefore, it may be useful for future studies dealing with the relativistic spin dynamics. Although some of the formulas presented below were obtained earlier, to our knowledge, no attempt has been made before to directly link them all and explain their physical interpretation.

In Secs. 2–4, we analyze the structure of the Wigner functions not referring to any concept of equilibrium. Only in Secs. 5–9, which are central for our work, we discuss various equilibrium forms. We conclude in Sec. 10.

We use the metric tensor with the signature $(+,-,-,-)$ and the Levi-Civita symbol with $\epsilon^{0123} = +1$. The trace over spinor (spin) indices is denoted by tr_4 (tr_2). The conventions regarding the spinors and several useful relations are collected in Appendix A.

2. Semi-classical Wigner functions

Our starting point are Wigner functions for particles and antiparticles, $\mathcal{W}^{\pm}(x,k)$, obtained in the leading order of the semi-classical expansion [8]

$$\mathcal{W}^{+}(x,k) \;=\; \frac{1}{2}\sum_{r,s=1}^{2}\int \mathrm{d}P\,\delta^{(4)}\left(k-p\right)u^{r}(p)\bar{u}^{s}(p)f^{+}_{rs}(x,p)\,, \tag{1}$$

$$\mathcal{W}^{-}(x,k) \;=\; -\frac{1}{2}\sum_{r,s=1}^{2}\int \mathrm{d}P\,\delta^{(4)}\left(k+p\right)v^{s}(p)\bar{v}^{r}(p)f^{-}_{rs}(x,p)\,. \tag{2}$$

Here, m is the (anti)particle mass, k is the four-momentum, and $\mathrm{d}P$ is the Lorentz invariant integration measure $\mathrm{d}P = \mathrm{d}^3p/((2\pi)^3E_p)$, where $E_p = \sqrt{m^2+\boldsymbol{p}^2}$ is the on-mass-shell energy and $p^\mu = (E_p, \boldsymbol{p})$. The objects $u_r(p)$ and $v_r(p)$ are Dirac bispinors with the spin indices r and s running from 1 to 2 and the normalizations: $\bar{u}_r(p)u_s(p) = 2m\,\delta_{rs}$ and $\bar{v}_r(p)v_s(p) = -2m\,\delta_{rs}$. We note that a minus sign and a different ordering of spin indices are used in Eq. (2) compared to Eq. (1). The total Wigner function becomes a sum of the particle and antiparticle contributions $\mathcal{W}(x,k) = \mathcal{W}^+(x,k) + \mathcal{W}^-(x,k)$. One can easily check that $(\not{k} - m)\mathcal{W}(x,k) = 0$, as required for the leading-order term of the Wigner function in the $\hbar$ expansion [9–12].

The functions $\mathcal{W}^+(x,k)$ and $\mathcal{W}^-(x,k)$ can be expressed with the help of 16 independent generators of the Clifford algebra [9, 13]

$$\begin{aligned}\mathcal{W}^\pm(x,k) \;=\; \frac{1}{4}\big[&\mathcal{F}^\pm(x,k) + i\gamma_5\mathcal{P}^\pm(x,k) + \gamma^\mu\mathcal{V}^\pm_\mu(x,k)\\ &+\gamma_5\gamma^\mu\mathcal{A}^\pm_\mu(x,k) + \Sigma^{\mu\nu}\mathcal{S}^\pm_{\mu\nu}(x,k)\big]\,. \end{aligned} \tag{3}$$

Here, $\Sigma^{\mu\nu}$ is the Dirac spin operator, $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$. In the leading order of semi-classical expansion, one can check that only scalar and axial-vector coefficient functions are independent. The other coefficients are expressed in terms of $\mathcal{F}^\pm = \mathrm{tr}_4\left[\mathcal{W}^\pm(x,k)\right]$ and $\mathcal{A}^\pm_\mu = \mathrm{tr}_4\left[\gamma_\mu\gamma^5\mathcal{W}^\pm(x,k)\right]$ by the following expressions [9]:

$$\mathcal{P}^\pm(x,k) \;=\; -i\,\mathrm{tr}_4\left[\gamma^5\mathcal{W}^\pm(x,k)\right] = 0\,, \tag{4}$$

$$\mathcal{V}^\pm_\mu(x,k) \;=\; \mathrm{tr}_4\left[\gamma_\mu\mathcal{W}^\pm(x,k)\right] = \frac{k_\mu}{m}\mathcal{F}^\pm(x,k)\,, \tag{5}$$

$$\mathcal{S}^\pm_{\mu\nu}(x,k) \;=\; 2\,\mathrm{tr}_4\left[\Sigma_{\mu\nu}\mathcal{W}^\pm(x,k)\right] = -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}k^\alpha\mathcal{A}^{\pm\beta}(x,k)\,. \tag{6}$$

This set of equations should be supplemented by a subsidiary condition $k_\beta\mathcal{A}^{\pm\,\beta}(x,k) = 0$.

Using Eqs. (4)–(6) in the definition of the Wigner function, one finds

$$\mathcal{W}^\pm(x,k) \;=\; \frac{1}{4m}(m+\not{k})\left[\mathcal{F}^\pm + \gamma_5\gamma_\beta\mathcal{A}^{\pm\,\beta}\right]. \tag{7}$$

In this way, we reproduce Eq. (5.44) from Ref. [9] (note a different normalization and an opposite sign in front of $\mathcal{A}^{\pm\,\beta}$, which is a consequence of different conventions used in [9], see also [10]).

3. Spin density matrix

The functions $f^{+}_{rs}(x,p)$ and $f^{-}_{rs}(x,p)$ play a role of the spin density matrices[1]. They are two-by-two Hermitian matrices which can be generally decomposed as [14]

$$f^{\pm}_{rs}(x,p) = f^{\pm}_{0}(x,p)\left[\delta_{rs} + \boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})\cdot\boldsymbol{\sigma}_{rs}\right] . \tag{8}$$

Here, $\boldsymbol{\sigma}$ denotes a three-vector consisting of three Pauli matrices. The three-vector $\boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})$ can be interpreted as a spatial part of the polarization four-vector $\zeta^{\pm\mu}_{*}(x,p)$, with a vanishing zeroth component[2]

$$\zeta^{\pm\mu}_{*}(x,p) = \left(0, \boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})\right) . \tag{9}$$

The average polarization vector is defined by the formula

$$\left\langle \boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})\right\rangle = \frac{1}{2}\frac{\mathrm{tr}_2\left(f^{\pm}\boldsymbol{\sigma}\right)}{\mathrm{tr}_2\left(f^{\pm}\right)} = \frac{1}{2}\boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p}) . \tag{10}$$

Several important points should be emphasized here:

— The polarization three-vector $\boldsymbol{\zeta}^{\pm}_{*}$ describes spin polarization in the particle (antiparticle) rest frame (PRF), *i.e.*, in the frame where $p^{\mu} = (m,0,0,0)$. We denote this frame by asterisk [15].

— The measurements of the spin polarization vary between $-1/2$ and $+1/2$, hence, $|\boldsymbol{\zeta}^{\pm}_{*}| \leq 1$. The particle spin states with $|\boldsymbol{\zeta}^{\pm}_{*}| = 1$ correspond to pure states, while the cases with $|\boldsymbol{\zeta}^{\pm}_{*}| < 1$ desribe mixed states.

— The functions $f^{\pm}_{0}$ contain information averaged over spin degrees of freedom. Hence, it is tempting to write them as sums of the distributions of particles with spin up and down. We thoroughly discuss this point below.

— We stress that $\boldsymbol{\zeta}^{\pm}_{*}$ is a function of space-time coordinates and three-momentum of particles, $\boldsymbol{\zeta}^{\pm}_{*} = \boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})$. The quantity $\boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})$, after averaging over the space-time region where particles are produced, becomes a directly measured observable. This happens, for example, in the case of Λ spin polarization measured in heavy-ion collisions.

[1] We note here that, strictly speaking, the functions $f^{\pm}_{rs}(x,p)$ are phase-space density matrices rather than spin density matrices, as the latter should depend on momentum only.

[2] We follow here the arguments discussed in [14], where in the case of small polarization, the identification $\boldsymbol{P} = -2\boldsymbol{\zeta}^{\pm}_{*}$ should be made.

To transform the PRF components of any four-vector to the laboratory (LAB) frame, we use the so-called canonical boost $\Lambda^{\mu}_{\nu}(\boldsymbol{v}_p)$ (see, for example, Eq. (45) in Ref. [14]). In the case of the four-vector $\zeta^{\mu}_{*}(x,\boldsymbol{p})$, this leads to the formula

$$\zeta^{\mu}_{\pm} = \Lambda^{\mu}_{\nu}(\boldsymbol{v}_p)\zeta^{\nu}_{\pm *} = \left(\frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}^{\pm}_{*}}{m}, \boldsymbol{\zeta}^{\pm}_{*} + \frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}^{\pm}_{*}}{m(E_p+m)}\boldsymbol{p}\right) . \tag{11}$$

In the relativistic quantum mechanics and quantum field theory, one deals with the spin densities defined in the spinor space. It is interesting to show that expression (7) is proportional to such densities. With the explicit forms of matrix elements given in Appendix A, we find

$$\mathcal{F}^{\pm}(x,k) = 2m\int \mathrm{d}P\delta^{(4)}\left(k\mp p\right) f^{\pm}_{0}(x,p)\,, \tag{12}$$

$$\mathcal{A}^{\pm\beta}(x,k) = 2m\int \mathrm{d}P\delta^{(4)}\left(k\mp p\right) f^{\pm}_{0}(x,p)\zeta^{\pm\beta}(x,\boldsymbol{p})\,, \tag{13}$$

and

$$\mathcal{W}^{\pm}(x,k) = \pm\int \mathrm{d}P\delta^{(4)}(k\mp p) f^{\pm}_{0}(x,p)\rho^{\pm}(x,\boldsymbol{p})\,. \tag{14}$$

Here, we have introduced the four-dimensional matrices

$$\rho^{\pm}(x,\boldsymbol{p}) = \frac{1}{2}\left(\not{p}\pm m\right)\left(1+\gamma_5\not{\zeta}^{\pm}\right) , \tag{15}$$

which exactly agree with the definitions of the polarization spin matrices given in [16][3].

4. Scalar and axial components

Doing the integral over three-momentum in Eq. (12), one finds

$$\mathcal{F}(x,k) = \frac{4m}{(2\pi)^3}\delta\left(k^2-m^2\right)F(x,k) \tag{16}$$

with

$$F(x,k) = \left[\theta\left(k^0\right) f^{+}_{0}(x,k) + \theta\left(-k^0\right) f^{-}_{0}(x,-k)\right] . \tag{17}$$

[3] Note that the convention for γ_5 used in [16] differs by sign from ours, see Appendix A. Note also that our results are obtained by a straightforward calculation of the matrix elements rather than by a diagonalization of the matrix $f^{\pm}_{rs}$, what has been done in Ref. [2].

Similar decomposition can be obtained for the axial component, however, in this case, it is useful to introduce yet another form of the polarization vectors $\zeta^{\pm\beta}$. Since they are space-like, we can write them in the form of[4]

$$\zeta^{\pm\beta}(x,\boldsymbol{p}) = \pm\zeta^{\pm}(x,\boldsymbol{p})n^{\pm\beta}(x,\boldsymbol{p})\,, \tag{18}$$

where $n^{\pm\beta}(x,\boldsymbol{p})n^{\pm}_{\beta}(x,\boldsymbol{p}) = -1$, and

$$\zeta^{\pm}(x,\boldsymbol{p}) = \sqrt{-\zeta^{\pm\beta}(x,\boldsymbol{p})\zeta^{\pm}_{\beta}(x,\boldsymbol{p})} = \left|\boldsymbol{\zeta}^{\pm}_{*}\right|\,. \tag{19}$$

Here, we used the fact that the scalar product can be calculated in any frame and chose PRF. The explicit form of $n^{\mu}_{\pm}$ is

$$n^{\mu}_{\pm}(x,\boldsymbol{p}) = \pm\left(\frac{\boldsymbol{p}\cdot\boldsymbol{n}^{\pm}_{*}}{m}, \boldsymbol{n}^{\pm}_{*} + \frac{\boldsymbol{p}\cdot\boldsymbol{n}^{\pm}_{*}}{m(E_p+m)}\boldsymbol{p}\right)\,, \tag{20}$$

where

$$\boldsymbol{n}^{\pm}_{*}(x,\boldsymbol{p}) = \frac{\boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})}{|\boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})|} = \frac{\boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})}{\zeta^{\pm}(x,\boldsymbol{p})}\,. \tag{21}$$

We observe that the three-vectors $\boldsymbol{n}^{\pm}_{*}(x,\boldsymbol{p})$ describe the direction of mean polarization of particles with momentum $\boldsymbol{p}$ (measured in PRF), while the positive quantity $\zeta^{\pm}(x,\boldsymbol{p})$ defines the magnitude of spin polarization.

We stress again that the case of $\zeta^{\pm}(x,\boldsymbol{p}) = 1$ corresponds to a pure state, while the case of $\zeta^{\pm}(x,\boldsymbol{p}) < 1$ describes a mixed state. Thus, in most of the cases, the three-vector $\boldsymbol{n}^{\pm}_{*}(x,\boldsymbol{p})$ cannot be interpreted as an arbitrary quantization axis. It describes the *mean direction* obtained by measurements of spin projections of many particles along three independent directions.

Performing the integral over three-momentum in Eq. (13) and using the notation introduced above, one gets

$$\mathcal{A}^{\beta}(x,k) = \frac{4m}{(2\pi)^3}\delta\left(k^2-m^2\right)n^{\beta}(x,k)A(x,k)\,, \tag{22}$$

where

$$n^{\beta} = \theta\left(k^0\right)n^{+\beta}(x,\boldsymbol{k}) - \theta\left(-k^0\right)n^{-\beta}(x,-\boldsymbol{k}) \tag{23}$$

and

$$\begin{aligned} A(x,k) \;=\; & \left[\theta\left(k^0\right)f^{+}_{0}(x,k)\zeta^{+}(x,\boldsymbol{k})\right. \\ & \left.+\,\theta\left(-k^0\right)f^{-}_{0}(x,-k)\zeta^{-}(x,-\boldsymbol{k})\right]\,. \end{aligned} \tag{24}$$

[4] We note that the $\pm$ signs in definition (18) are conventional and the minus sign in (18) compensates the minus sign in the middle of the right-hand side of (23).

At this point, it is useful to compare our framework with previous, similar studies. In particular, one can check that Eq. (23) is consistent with expressions (26) and (27) obtained in Ref. [2], provided the vectors $\boldsymbol{n}_*^{\pm}(x,\boldsymbol{k})$ are identified with the vectors $\boldsymbol{n}^{\pm}$ defined therein. Our results also agree with Eqs. (26) and (28) in Ref. [1], if the vector $\boldsymbol{n}$ defined there is simultaneously equal to $\boldsymbol{n}_*^{+}(x,\boldsymbol{k})$ and $\boldsymbol{n}_*^{-}(x,-\boldsymbol{k})$. Thus, we agree with Ref. [1] only if $\boldsymbol{n}_*^{+}(x,\boldsymbol{k}) = \boldsymbol{n}_*^{-}(x,-\boldsymbol{k})$. The last condition represents a constraint on the most likely directions of polarization vectors for particles and antiparticles. We come back to their interpretation below Eq. (34).

Besides the two vectors $n^{\pm\beta}$, the system under consideration is described by the four scalar functions: $f_0^{\pm}$ and $\zeta^{\pm}$. They can be conveniently reorganized to describe particles with spins up and down along the direction set by unit vectors $n^{\pm\beta}$. This can be done with the help of the definition

$$f_{0s}^{\pm}(x,\pm k) = \tfrac{1}{2} f_0^{\pm}(x,\pm k)\left(1 + s\zeta^{\pm}(x,\pm\boldsymbol{k})\right) , \tag{25}$$

where $s = \pm 1$ denotes the spin direction. Note that we have $0 \leq \zeta^{\pm}(x,\pm\boldsymbol{k}) \leq 1$, hence $f_{0s}^{\pm}(x,\pm k)$ is positive if $f_0^{\pm}(x,\pm k) > 0$. Equation (25) allows us to rewrite Eqs. (17) and (24) as

$$\begin{aligned} F(x,k) \;=\; & \left[\theta\left(k^0\right)\left(f_{0+}^{+}(x,k) + f_{0-}^{+}(x,k)\right)\right. \\ & \left.+\theta\left(-k^0\right)\left(f_{0+}^{-}(x,-k) + f_{0-}^{-}(x,-k)\right)\right] \end{aligned} \tag{26}$$

and

$$\begin{aligned} A(x,k) \;=\; & \left[\theta\left(k^0\right)\left(f_{0+}^{+}(x,k) - f_{0-}^{+}(x,k)\right)\right. \\ & \left.+\theta\left(-k^0\right)\left(f_{0+}^{-}(x,-k) - f_{0-}^{-}(x,-k)\right)\right] . \end{aligned} \tag{27}$$

5. Equilibrium Wigner functions

So far, we have not addressed the fact that our Wigner function describes a system of particles with spin in equilibrium. As a matter of fact, different forms of such functions are proposed in the literature and the main aim of this work is to examine them and check their internal consistency connected with relativistic covariance and physical interpretation of the spin polarization measurements.

The optimal situation would be to derive an equilibrium form from the considerations that analyze either entropy production or the form of collision terms for particles with spin. As such calculations are not available at the moment, various discussions of the equilibrium for particles with spin have to make use of different arguments, usually combined together, to conclude

about the acceptable forms of the equilibrium functions. These functions necessarily invoke certain forms of the spin potential, hence, the issue of choosing the correct equilibrium form is connected with the introduction of the appropriate spin potential.

Some support in this respect comes from the analysis of kinetic theory with classical description of spin. With the arguments about the locality of the classical collision term, one can construct for this case an equilibrium distribution function that naturally involves a tensor spin potential [17]. We come back to this point below and turn to a discussion of specific equilibrium Wigner functions now.

6. Scalar spin potential

Since $f_0^\pm$ describes an average over the spin components, see Eq. (26), it seems natural to assume that $f_{0s}^\pm$ has the form of the standard equilibrium function depending on the flow vector u^μ, temperature T, chemical potential μ_e connected with the conservation of charge, and an additional scalar spin potential $\mu^\pm$ that controls the relative number of particles (plus sign) or antiparticles (minus sign) with spin up and down, namely,

$$\begin{aligned} f_{0s}^\pm(x,p) &= \frac{1}{2}\exp\left(-\frac{p\cdot u \mp \mu_e - s\mu^\pm}{T}\right) \\ &= \frac{1}{2}\exp\left(-\frac{p\cdot u \mp \mu_e}{T}\right)\exp\left(\frac{s\mu^\pm}{T}\right) \\ &\approx \frac{1}{2}\exp\left(-\frac{p\cdot u \pm \mu_e}{T}\right)\left(1+s\,\frac{\mu^\pm}{T}\right). \end{aligned} \tag{28}$$

Here, we have used the Boltzmann statistics[5]. The second line of Eq. (28) may be directly compared to Eq. (25). Since $\zeta^\pm$ depends in general on momentum, we conclude that $\mu^\pm/T$ should depend on momentum as well, a property which is not exhibited by (28). This connection is clearly seen for small values of $\mu^\pm/T$, when from the third line of Eq. (28) one gets

$$\mu^\pm = T(x)\zeta^\pm(x,\boldsymbol{p})\,. \tag{29}$$

Momentum-dependent spin chemical potential cannot be used in a traditional way in thermodynamic identities. Consequently, Eq. (28) has a very restricted range of applicability.

[5] A discussion of the Fermi–Dirac statistics is similar but much more involved.

The origin of the discussed difficulty is a simple fact that the spin polarization of relativistic massive particles is always defined in their rest frames, hence, different boosts should be applied to particles with different three-momenta in order to determine their spin polarization. This dependence is reflected in the momentum dependence of $\mu^{\pm}$, which eventually makes it a badly defined quantity from the thermodynamic point of view. Clearly, the problems outlined above disappear in the non-relativistic limit.

7. Tensor spin potential

In Ref. [18], the following local equilibrium Wigner functions were introduced:

$$\begin{aligned} f^{+}_{rs} &= \frac{1}{2m}\bar{u}_r(p)\exp\left[-p\cdot\beta+\xi_e+\frac{1}{2}\varpi_{\mu\nu}\Sigma^{\mu\nu}\right]u_s(p)\,, \\ f^{-}_{rs} &= -\frac{1}{2m}\bar{v}_s(p)\exp\left[-p\cdot\beta-\xi_e-\frac{1}{2}\varpi_{\mu\nu}\Sigma^{\mu\nu}\right]v_r(p)\,, \end{aligned} \tag{30}$$

where $\xi_e = \mu_e/T$ and $\varpi_{\mu\nu}$ is thermal vorticity defined by the expression $\varpi_{\mu\nu} = -(1/2)\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right)$ with $\beta^\mu = u^\mu/T$.

The equilibrium forms (30) were subsequently used in Ref. [4] to construct relativistic hydrodynamics of particles with spin 1/2. The main idea of Ref. [4] was to replace thermal vorticity in Eq. (30) by the spin polarization tensor $\omega_{\mu\nu}$, whose dynamics should be determined by the conservation of angular momentum (instead of being tightly connected with thermal vorticity). The spin polarization tensor can be identified with the ratio $\Omega_{\mu\nu}/T$, where $\Omega_{\mu\nu}$ plays a role of a tensor spin potential (both $\omega_{\mu\nu}$ and $\Omega_{\mu\nu}$ are rank two antisymmetric tensors that depend only on space and time coordinates, for brevity of notation we dominantly use $\omega_{\mu\nu}$ instead of $\Omega_{\mu\nu}$).

If the components of $\omega_{\mu\nu}$ are small, the form of equilibrium Wigner function advocated in Ref. [4] agrees with Eq. (7) where one should use [19]

$$\mathcal{F}^{\pm}_{\rm eq}(x,k) \;=\; 2m\int \mathrm{d}P\;\mathrm{e}^{-\beta\cdot p\pm\xi_e}\;\delta^{(4)}(k\mp p) \tag{31}$$

and

$$\mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) \;=\; -\int \mathrm{d}P\,\mathrm{e}^{-\beta\cdot p\pm\xi_e}\;\delta^{(4)}(k\mp p)\,\tilde{\omega}_{\mu\nu}\,p^\nu\,. \tag{32}$$

Here, $\tilde{\omega}_{\mu\nu}$ is the dual spin polarization tensor defined as $\tilde{\omega}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}$. One can notice that the approach proposed in Ref. [18] and extended in Ref. [4] introduces the same spin polarization tensor for particles and antiparticles which makes sense if they are all in common equilibrium.

The antisymmetric spin polarization tensor $\omega_{\mu\nu}$ can be always defined in terms of electric- and magnetic-like three-vectors in LAB frame, $\boldsymbol{e} = (e^1, e^2, e^3)$ and $\boldsymbol{b} = (b^1, b^2, b^3)$. In this case, following the electrodynamic sign conventions of [20], we write [14]

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix} . \tag{33}$$

The dual spin polarization tensor is obtained from the components of $\omega_{\mu\nu}$ by replacements $\boldsymbol{e} \to \boldsymbol{b}$ and $\boldsymbol{b} \to -\boldsymbol{e}$. In Ref. [14], it was demonstrated that

$$\boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p}) = -\frac{1}{2m}\left[E_p\,\boldsymbol{b} - \boldsymbol{p}\times\boldsymbol{e} - \frac{\boldsymbol{p}\cdot\boldsymbol{b}}{E_p+m}\boldsymbol{p}\right] . \tag{34}$$

Equation (34) shows that the spin polarization vectors of particles and antiparticles are indeed the same (in equilibrium described with the help of the tensor spin potential). This makes sense if they are in common equilibrium state. We have seen above that the condition $\boldsymbol{n}^{+}_{*}(x,\boldsymbol{k}) = \boldsymbol{n}^{-}_{*}(x,-\boldsymbol{k})$ is used in Ref. [1]. For the tensor spin potential this implies that in this case, $\boldsymbol{e} = 0$. The physical interpretation of this equation remains to be clarified. At the moment, we may notice that $\boldsymbol{e} = 0$ in the global equilibrium states with a rigid rotation [21].

Using Eq. (34) in Eq. (11) or by making a direct comparison of Eqs. (13) and (32) we find the identification

$$\zeta^{\pm}_{\mu}(x,\boldsymbol{p}) = -\frac{1}{2m}\tilde{\omega}_{\mu\nu}(x)p^{\nu} . \tag{35}$$

In Ref. [14], it was also shown that the right-hand side of Eq. (34) coincides with the value of the $\boldsymbol{b}$ field determined in PRF, namely,

$$\boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p}) = -\frac{1}{2}\boldsymbol{b}_{*}(x,\boldsymbol{p}) . \tag{36}$$

This is an interesting result indicating that for the spin polarization, only the magnetic-like component in PRF is important.

8. Other approaches

In Ref. [4], the case of large spin polarization tensor $\omega_{\mu\nu}$ was considered, however, with two additional conditions[6]

$$\omega_{\mu\nu}\omega^{\mu\nu} = 2(\boldsymbol{b}\cdot\boldsymbol{b} - \boldsymbol{e}\cdot\boldsymbol{e}) \geq 0 , \qquad \omega_{\mu\nu}\tilde{\omega}^{\mu\nu} = -4\boldsymbol{e}\cdot\boldsymbol{b} = 0 . \tag{37}$$

[6] Conditions (37) were relaxed, for example, in Ref. [22].

In this case, one finds

$$f^{\pm}_{rs} = \mathrm{e}^{\pm\xi - p\cdot\beta}\cosh(\xi_s)\left[\delta_{rs} - \frac{\tanh(\xi_s)}{2\xi_s}\,\boldsymbol{b}_*\cdot\boldsymbol{\sigma}_{rs}\right], \tag{38}$$

where

$$\xi_s = \frac{1}{2}\sqrt{\boldsymbol{b}\cdot\boldsymbol{b} - \boldsymbol{e}\cdot\boldsymbol{e}}\,. \tag{39}$$

Thus, the quantity ξ_s (multiplied by T) can be naturally interpreted as a spin potential, as demonstrated in Ref. [4]. The applicability of this approach is restricted, however, to particles with momenta satisfying the condition

$$\left|\tanh(\xi_s)\,\frac{\boldsymbol{b}_*}{2\xi_s}\right| = \left|\frac{\tanh(\xi_s)\,\boldsymbol{b}_*}{\sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}}\right| \leq 1\,, \tag{40}$$

where we used $\sqrt{\boldsymbol{b}\cdot\boldsymbol{b} - \boldsymbol{e}\cdot\boldsymbol{e}} = \sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}$. Condition (40) takes a particularly simple form for particles with $|\boldsymbol{e}_*| \ll |\boldsymbol{b}_*|$. In this case, $\boldsymbol{b}_*/|\boldsymbol{b}_*|$ becomes a unit vector showing the direction of mean polarization, while $\tanh(\xi_s)$ defines its magnitude.

Yet another treatment of spin polarization was introduced in Ref. [10], where (using our notation) the following Ansatz was made for particles

$$\zeta^{\mu}_{+}(x,\boldsymbol{p}) = u^{\mu}n(p)\cdot p - n^{\mu}(p)u\cdot p\,. \tag{41}$$

Here, $n(p)$ is a four-vector that is perpendicular to p. Form (41) does not comply with the requirements discussed above and as such seems to be quite arbitrary. In particular, it is not clear why the flow vector u appears in (41).

9. Insights from models with classical description of spin

Different conditions that appear above for the coefficients of the spin polarization tensor $\omega_{\mu\nu}$ and three-momenta of particles $\boldsymbol{p}$ indicate that the discussed forms of the equilibrium Wigner functions are limited in their physical applications to some definite range of space-time and momentum variables (let us say in LAB frame). Some light can be shed on this limitation if we refer to a kinetic theory with classical description of spin [17]. The classical approach shows that for large spin polarization the systems become anisotropic in momentum space. Such anisotropy has not been addressed yet in present formulations, so this is the work to be done in future studies. Fortunately, the classical description of spin shows also consistency with the forms obtained for small polarization. Consequently, taking together results obtained with the Wigner functions and classical spin description, we obtain a convincing physical picture for sufficiently small $\omega_{\mu\nu}$. In fact, it is not a very much restrictive constraint, since the measured values of the global spin polarization remain at the level of a fraction of 1%.

10. Conclusions

In this paper, we have discussed different concepts and forms of the spin potential entering the formula for the semi-classical equilibrium Wigner function of particles with spin 1/2. Our results suggest using the tensor form of the spin potential that originates from its use as a Lagrange multiplier in the conservation of angular momentum [4, 5]. Moreover, recent forms of the equilibrium Wigner function suggest that the spin potential (scaled by temperature) should be small. In this case, the scalar spin potential can be expressed by the tensor form. Interestingly, the scalar form should be momentum dependent, a feature connected with the fact that the spin polarization is always defined in the particle rest frame.

Several comparisons to other works using various concepts of the spin potential have been made. This can help to relate different results and interpretations. Our results can be useful for further development of hydrodynamics of particles with spin 1/2 and serve to interpret experimental measurements of particle spin polarization.

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Appendix A

Useful formulas and identities

Our conventions for labels and signs of Dirac bispinors are as follows:

$$u_s(p) = \sqrt{E_p + m} \begin{pmatrix} 1 & \varphi_s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m} & \varphi_s \end{pmatrix}, \tag{A.1}$$

$$v_s(p) = \sqrt{E_p + m} \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m} & \chi_s \\ 1 & \chi_s \end{pmatrix} \tag{A.2}$$

with

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \chi_2 = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{A.3}$$

The spin operator $\Sigma^{\mu\nu}$ is defined by the expression

$$\Sigma^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right] , \tag{A.4}$$

which in the Dirac representation gives

$$\Sigma^{0i} = \frac{i}{2}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} , \qquad \Sigma^{ij} = \frac{1}{2}\epsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} , \tag{A.5}$$

with σ^i being the i^{th} Pauli matrix. The γ_5 matrix is defined as $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Using the above definitions of the Dirac bispinors, one can directly derive several useful relations which are listed below. Some of them are well known but the other are rather not popular so we list them all for completeness. With the short-hand notation

$$X_{rs} = \left(\delta_{rs} + \boldsymbol{\zeta}\cdot\boldsymbol{\sigma}_{rs}\right) , \tag{A.6}$$

one obtains:

$$\begin{aligned}
\sum_{r,s}\bar{u}_s(p)u_r(p)X_{rs} &= -\sum_{r,s}\bar{v}_r(p)v_s(p)X_{rs} = 4m\,,\\
\sum_{r,s}\bar{u}_s(p)\gamma_5 u_r(p)X_{rs} &= \sum_{r,s}\bar{v}_r(p)\gamma_5 v_s(p)X_{rs} = 0\,,\\
\sum_{r,s}\bar{u}_s(p)\gamma^{\mu}u_r(p)X_{rs} &= \sum_{r,s}\bar{v}_r(p)\gamma^{\mu}v_s(p)X_{rs} = 4p^{\mu},\\
\sum_{r,s}\bar{u}_s(p)\gamma^0\gamma_5 u_r(p)X_{rs} &= -\sum_{r,s}\bar{v}_r(p)\gamma^0\gamma_5 v_s(p)X_{rs} = 4\boldsymbol{p}\cdot\boldsymbol{\zeta},\\
\sum_{r,s}\bar{u}_s(p)\boldsymbol{\gamma}\gamma_5 u_r(p)X_{rs} &= 4\left(m\boldsymbol{\zeta} + \frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}}{E_p+m}\,\boldsymbol{p}\right) ,\\
\sum_{r,s}\bar{v}_r(p)\boldsymbol{\gamma}v_s(p)X_{rs} &= -4\left(m\boldsymbol{\zeta} + \frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}}{E_p+m}\,\boldsymbol{p}\right) ,\\
\sum_{r,s}\bar{u}_s(p)\Sigma^{0i}u_r(p)X_{rs} &= \sum_{r,s}\bar{v}_r(p)\Sigma^{0i}v_s(p)X_{rs} = -2\epsilon^{ijk}p^j\zeta^k,\\
\sum_{r,s}\bar{u}_s(p)\Sigma^{mn}u_r(p)X_{rs} &= -2\epsilon^{mni}\left(\frac{(\boldsymbol{p}\cdot\boldsymbol{\zeta})\,p^i}{E_p+m} - E_p\zeta^i\right) ,\\
\sum_{r,s}\bar{v}_r(p)\Sigma^{mn}v_s(p)X_{rs} &= -2\epsilon^{mni}\left(\frac{(\boldsymbol{p}\cdot\boldsymbol{\zeta})\,p^i}{E_p+m} - E_p\zeta^i\right) .
\end{aligned} \tag{A.7}$$

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