FRACTIONAL BROWNIAN MOTIONS

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(Received February 10, 2020)

Properties of different models of fractional Brownian motions are discussed in detail. We shall collect here several possible ways of introducing and defining various possible fBms, discuss their properties, find how they are similar, and how they differ. In particular, we shall try to find what mechanisms or details in their definitions make these motions anomalous and whether can the various models be distinguished experimentally. To this aim, the main tool used here will be the autocorrelation function $C(t,s)$, and related to it characteristics: nonmarkovian behaviour and so-called weak ergodicity breaking.

DOI:10.5506/APhysPolB.51.1097

1. Introduction

The prehistory of description of phenomena known today as the Brownian motions (Bms) ranges back 2000 years ago, to Lucretius (*ca.* 99–55 BC) [1], who used his observations of random motions of multitudes of tiny glittering particles in reflected lateral light [2] as the proof of atomistic structure of matter.

Indeed, these tiny motions 2000 years later became the ultimate proof of the atomistic theory. However, the idea needed to wait for about 2000 years to be commonly accepted.

More detailed observations of processes similar to those described by Lucretius were impossible without the construction of microscope, *i.e.*, until the 17^{th} century. It is difficult to assess what indeed was then observed. Sometimes it is claimed that the first observation of a process which could be interpreted as the Brownian motion was reported in 1784 by Jan Ingenhousz [3], but this interpretation seems doubtful [4]. The first observation of a true Brownian motion, easy to verify, is due to Robert Brown (1827) [3, 5].

Nevertheless, these early observations were treated as unimportant curiosities, strange effects without any rational explanation (Brown himself did not even publish his — now seminal — results as a separate paper [5]). Increasing controversy between opponents and adherents of atomistic theory of matter, advances in observational techniques (Zsigmondy's ultramicroscope), and some hard to interpret phenomena such as critical opalescence and the blue colour of the sky, both easily explained by the presence of fluctuations of temperature and density [6, 7] — inspired renewed interest in the theoretical description of mechanisms of Brownian motions [7, 8], and resulted in the common acceptance of atomistic theory. It is worth emphasizing that better microscopes enabled Perrin to record traces (sample paths) of individual Brownian particles with time-step 30 s [9]. Earlier, Zsigmondy noted changing numbers of gold nanosized particles (dissolved in molten glass, about 1000 atoms together), visible (by reflected lateral light) in the field of his ultramicroscope (time resolution of a few minutes!). One ought to mention that this is now considered as first observations of nanoparticles!

The last important contribution to this early stage of the Brownian motion (Bm) story was the formalization of Einstein and Smoluchowski stochastic approach to the form of the stochastic differential equation of motion containing both mechanical and stochastic forces [10]. Nowadays, this *Langevin equation* is still in use (together with its numerous generalizations), and is treated as the standard way of description of motions of Brownian particles.

The importance of all these discoveries was confirmed by three Nobel Prizes: Zsigmondy (chemistry 1925), Svedberg (chemistry 1926), and Perrin (physics 1926).

Noteworthy is Smoluchowski's approach to the theory of Brownian motions, *viz.* substitution of simple stochastic force in the place of extremely complicated mechanical forces between a moving Brownian particle and surrounding molecules of gas or liquid gave birth — even though without intention — to recent theory of stochastic processes [11–13].

Later, Bm story seemed to be closed. The Brownian motions were (and still are) just microscopic mechanisms of macroscopic diffusion, *because the process of diffusion is the superposition of Brownian motions of the molecules of the substantion under consideration* [14]. This mechanism was frequently used in several diverse processes, in particular in sedimentation and coagulation [15] (induced by Smoluchowski [16]), but Bms as such recessed to history.

Renewed interest in Bms themselves is the result of recent progress in experimental methods [17], including the measurements of single-molecule transport processes in living cells or organelles with time-step of the order of one millisecond [18]. This led to the "discovery" of the so-called

anomalous transport: in small dense and nonuniform systems, the Brownian motion (diffusion) is not exactly such as it has been predicted by the Einstein–Smoluchowski theory [19]. Perhaps it is worth noting that anomalous Brownian motion (diffusion, anomalous transport) is typical, especially in micro- and nanosize systems, and can also be easily generated in macroscopic simple systems, by variations of temperature, density, or some other parameters [20].

In more detail, one discerns two main, different types of anomalous transport: so-called continuous time random walk (CTRW) [19, 21, 22] and fractional Brownian motions (fBms) [19] both frequently used in literature as models for measured data. Other models or systems, such as diffusion on fractals [23], Brownian motions on curved surfaces (more generally non-Euclidean spaces) [24], expanding/contracting spaces or systems [20], *etc.*, were considered rarely, usually without applications to experimental data.

The two main models (CTRW and fBm) are easy to recognize visually by the different shapes of registered 2- or 3-dimensional sample paths. Main feature of CTRW is the characteristic groupings of the trajectory in a few regions ("traps") with random jumps between these regions, and with non-exponential distribution of waiting times between jumps and non-Gaussian distribution of lengths of jumps (*cf.* [22]). fBm trajectories are similar to the first pictures of Brownian motions drawn by Perrin [9] being just a collection of lines of random length and random direction connecting the subsequent observed positions of a Brownian particle. Such characteristic pictures are shown in Fig. 1.

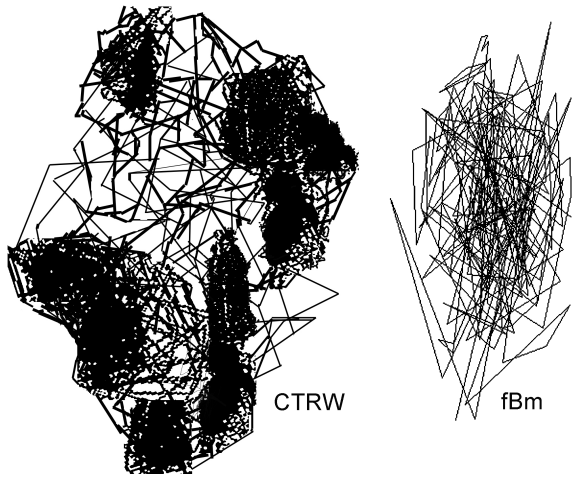

Fig. 1. CTRW: Normal Bm $W(t)$ inside traps, long jumps between traps from Pareto distribution. fBm: Fractional Brownian motion, $H = 0.15$. This picture is drawn to resemble (roughly) visually the experimental data [26, 27].

Common of all fBms is the same behaviour of dispersion $\sigma(t)$ of their trajectories $X(t)$: the second moment $\sigma^2(t) = \langle [X(t) - X(0)]^2 \rangle \sim t^{2H}$, where $\langle \ldots \rangle$ denotes the ensemble average. The main aim of most of the measurements or simulations is the determination of the value of the Hurst exponent H ($H = 1/2$ means that the diffusion/Brownian motion is normal).

We shall collect here several possible ways of introducing and defining various possible models of fBms, discuss their properties, find how they are similar and how they differ, mainly from the point of view of experimental data. The determination of the value of the Hurst exponent says only that the observed process is sub- ($H < 1/2$) or super- ($H > 1/2$) diffusional. More detailed information is needed.

To quote Saxton [25]: "Much work is being done on anomalous subdiffusion in the plasma membrane, cytoplasm, and nucleus of cells, and in model systems. The main experimental questions: Is diffusion anomalous or normal, and what are the parameters describing it? The main theoretical questions: What mechanism makes the diffusion anomalous? The main question linking these: How can the various mechanisms be distinguished experimentally?"

Recent answer to the first of these problems is as follows: mainly Hurst exponent and identification of CTRW *vs.* fBm. In the case of CTRW, of interest is also the distribution of lengths and time intervals between subsequent jumps (details of "albatross search pattern"). The case of fBms will be discussed in detail in this paper.

Aim of this paper: an attempt to find some partial answers to the second and third problems stated by Saxton, *i.e.*, to examine several other characteristics, to find what measurable (observable) properties can be used to discern between different models and find which model best fits the measured process (transport) in a given environment, *etc.* We must keep in mind that experimentally and computationally accessible are only sample paths $\{X(t_1), X(t_2), \ldots X(t_N)\}$, $t_n = n\Delta t$, Δt being the time step of observation of continuous processes $X(t)$. Moreover, all experimental data are burdened by inevitable errors.

In particular, we shall introduce some generalizations of known models (Kolmogorov, Lévy, scaled) of Bms, analyse the differences introduced in the extensions of these models, and try to find which observables can be used to identify a given model. In the last aspect, most promising seems to be the autocorrelation function $C(t, s)$

$$C(t, s) = \langle X(t)X(s) \rangle \,, \tag{1}$$

($\langle \cdots \rangle$ denotes ensemble average, *i.e.*, average over appropriate probability density or — equivalently — over $M \to \infty$ realizations of process $X(t)$), and related to it such more formal properties as nonmarkovianity (NM) and

the so-called weak ergodicity breaking (WEB). However, let us repeat: the identification, whose model of fBm fits best the observed data is burdened by inevitable experimental errors. As we have seen above, discrimination between CTRW and fBm models is simple. We shall demonstrate below that this is not the case of discrimination between different fBm models. At best, we can ascribe a given set of data to a given set (group) of models. In this respect, all fBms can be divided into subgroups differing both in their characteristics and in the systems (physical or not) in which they were observed.

First such division is between sub- ($H < 1/2$) and superdiffusional ($H > 1/2$) processes. Subdiffusional is, among others, passive transport of molecules within living cells, superdiffusional — active motions in the same environment [28].

Other natural division is into processes exhibiting a kind of memory of their past (a.k.a. nonmarkovian processes) and memoryless (markovian) ones. These differ strongly in several observable properties, but appear in all known systems. In economical processes, where memory of past behaviour of participants (*e.g.*, in stock-exchange) plays substantial role, one would expect that only nonmarkovian description is of any meaning, but in many cases markovian models are in common use.

2. Definitions of fBms

As we have said, our main interest is how to find which model of possible fractional Brownian motions describes best the given set of experimental data. Brownian particles, as described by different fBms, move with different average velocities [29], characterized by their effective diffusion coefficients. However, the measured velocities, apart of experimental errors, are influenced by the properties of environment in which they move (mainly temperature and viscosity), and depend on properties of effective stochastic forces acting on the observed particle. This means, among others, that measured velocities cannot serve as tools for recognition of which kind of (f)Bm is observed. Absolute values of autocorrelation functions $C(t,s)$ also depend on respective effective diffusion coefficients. All these effects lead to some difficulties when of interest is the comparison of and differentiation between models of fractional Brownian motions.

Therefore, *(i)* we shall be solely interested in the shapes of $C(t,s)$, *i.e.*, we shall use only their reduced forms $C_{\mathrm{r}}(t,s) = [C(t,s) - C_{\min}]/[C_{\max} - C_{\min}]$. Note that $C_{\min} < 0$ means anticorrelated process. *(ii)* All results will be presented below in figures in the reduced form: $\phi(X(t)) = \phi_{\mathrm{red}}(X(t)) = [\phi_{\mathrm{raw}} - \min(\phi)]/[\max(\phi) - \min(\phi)]$, where $\phi(X(t))$ is the observable under consideration.

We shall use here the physical notation, less general than mathematical, but more clear physically.

Let us consider the stochastic process $X(t)$ and its pdf $P(x;t)$, defined on the domain $\{\Omega_x, t \in [0,\infty)\}$ (in the most general case, $\Omega_x = \Omega_x(t)$) such that *(i)* it scales as $X_H(at) = a^H X_H(t)$, and *(ii)* at least two of its first moments are finite

$$\begin{aligned} \left\langle [\Delta X(t)]^2 \right\rangle &= \sigma^2(t)\,, \qquad \sigma(t) < \infty\,, \qquad \Delta X(t) = X(t) - \langle X(t)\rangle\,, \\ \langle X^m(t)\rangle &= \int\limits_{\Omega_x} x^m P(x;t)\mathrm{d}x\,. \end{aligned} \tag{2}$$

When $\sigma^2(t) = 2K_\alpha t^\alpha, 0 < \alpha = 2H < 2$, the process $X(t)$ is called "diffusion" ("Brownian motion") ("normal" for $\alpha = 1$, anomalous otherwise).

One more comment seems to be in order at the beginning of this section: all Brownian motions are driven by stochastic forces (*cf.* remarks on the Langevin equation, Sect. 1, and Eq. (3) below). At the beginning, such forces were always modelled by the thermal noise (Gaussian-distributed, and δ-correlated GWN). This is not necessary, any "proper" stochastic force can be used (*cf.* below), though in physical (and some other, biological in particular) systems thermal noise is (almost) ever-present.

Normal Brownian motion (a.k.a. Wiener process $W(t)$)

First analytic definition is due to Langevin [10]

$$\begin{aligned} \mathrm{d}W(t) &= \xi_0(t)\mathrm{d}t\,, \qquad W(t) = \int\limits_0^t \mathrm{d}u\xi_0(u)\,, \\ \langle \xi_0(t)\rangle &= 0\,, \qquad \langle \xi_0(t)\xi_0(s)\rangle = \sigma_0^2\delta(t-s)\,, \end{aligned} \tag{3}$$

with the autocorrelation function

$$C_W(t,s) \sim \tau \equiv \min(t,s)\,. \tag{4}$$

$\xi_0(t)$ is the stochastic force, physically simulating the collisions of Brownian particle with particles of the medium, with Maxwell distribution of momenta, and approximated by white (uncorrelated) Gaussian noise (GWN). $\langle \ldots \rangle$ denotes the ensemble average. Mathematically $W(t)$ is known as the Wiener process. When $\xi_0(t)$ represents thermal noise, then its dispersion $\sigma_0(t)$ depends on temperature, and usually on some other physical parameters.

Lévy Brownian motion (L-fBm, $L_H(t)$, ${}_L X_H(t)$)

The oldest fractional Bm is the Lévy one [30]

$$L_H(t) = \frac{1}{\Gamma(H+1/2)} \int_0^t \mathrm{d}u\, (t-u)^{H-1/2}\, \xi_0(u)\,, \tag{5}$$

with the autocorrelation function

$$C_L(t,s) \sim \int_0^\tau \mathrm{d}u\, (|t-u||s-u|)^{H-1/2}\,, \tag{6}$$

which can be easily computed numerically.

This process has been for a long time overlooked by physicists, and still is of little use. According to Mandelbrot and Van Ness [31], the (Holmgren–Riemann–Liouville) integral (5) "puts too great an importance on the origin for many applications" (too strong "memory" of the past?).

In fact, the above constatation is true only for $H > 1/2$, (*i.e.*, for superdiffusional process); for $H < 1/2$, (*i.e.*, for subdiffusion) the kernel $(t-u)^{H-1/2}$ dampens the influence of the beginning (*i.e.*, memory of the past behaviour) of the process.

The Lévy fBm correlation function, Eq. (6), visually is very similar to that defined by Eq. (7) (*cf.* Figs. 2 and 3 below).

Kolmogorov Brownian motion (K-fBm, $B_H(t)$, $K_H(t)$, ${}_K X_H(t)$)

The best-known, in fact even treated frequently as THE fractional Brownian motion, is the Kolmogorov fBm $B_H(t)$ (we use here traditional name $B_H(t)$ which should, in fact, denote just any Brownian motion) defined [32] as the Gaussian stochastic process with the correlation function

$$C_B(t,s;H) \sim t^{2H} + s^{2H} - |t-s|^{2H}\,. \tag{7}$$

Note that the definition of K-fBm is given not by the kinetic (Langevin) equation but by the autocorrelation function. We will return to this fact later. The K-fBm process $B_H(t)$ can be also defined as Brownian motion driven by fractional (anticorrelated, antipersistent) Gaussian noise (fGn) $\xi_H(t)$ [33, 34]

$$\dot{B}_H(t) = \xi_H(t)\,, \qquad B_H(t) = \int_0^t \mathrm{d}u\, \xi_H(u)\,, \qquad \langle \xi_H(t) \rangle = 0\,,$$

$$\langle \xi_H(t)\xi_H(s) \rangle = 2H(2H-1)\kappa_H |t-s|^{2H-2} + 4H\kappa_H |t-s|^{2H-1} \delta(t-s)\,, \tag{8}$$

with appropriate initial conditions. Note that for $H = 1/2$, $\xi_H(t) = \xi_0$.

Contrary to the Wiener process and Lévy fBm (and scaled Bms — *cf.* below), numerical simulations of $B_H(t)$'s are very difficult, because the only reliable exact method is the Cholesky decomposition, which is very demanding numerically (*cf.* below). In the literature, several integral formulas enabling numerical simulations of sample paths are described, their derivations based on approximations (mostly poor), or on different mathematical devices such as equivalence of different stochastic processes, Lamperti transforms, Volterra representation, *etc.* Most are also demanding numerically (*cf.* an extensive survey by Coutin [35]). The numerical simulations of fGn are still more demanding.

The K-fBm process (or rather its many approximations) is, among others, very popular for creations of fractal landscapes, *etc.* [36] where approximations are not so important.

Comparison of K- and L-fBms is shown in Figs. 2 and 3. In all these figures, there are shown three versions of drawings of reduced $C(t,s)$: 3D (first row), its projection on 2D plane (second row), both with heights coded by colour, and five cuts through 3D picture at constant values of t (third row).

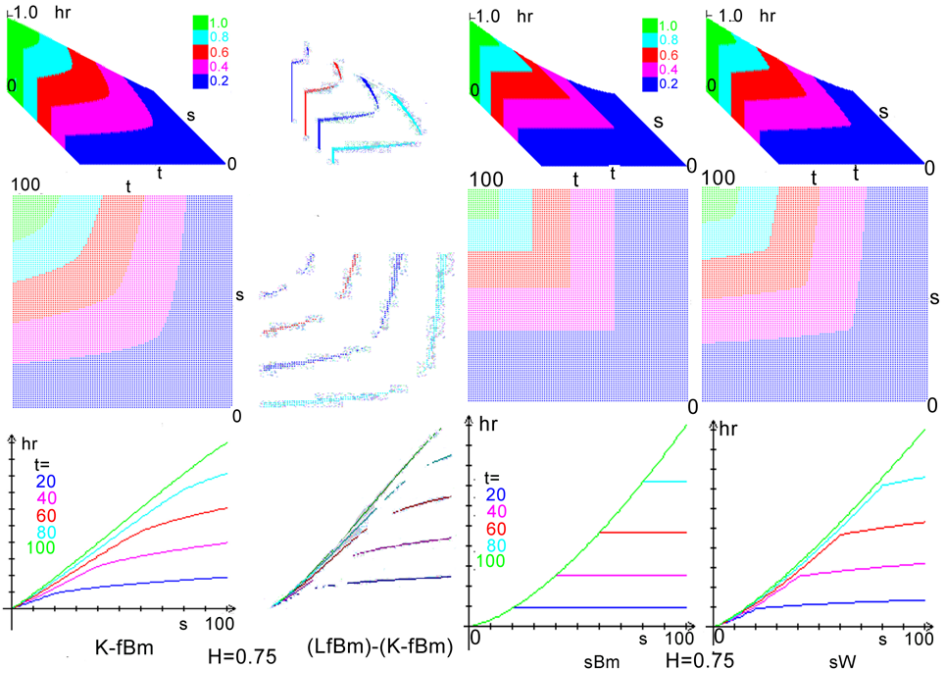

Fig. 2. (Colour on-line) Comparison of reduced autocorrelation functions of subdiffusional Kolmogorow and Lévy fractional Brownian motions (left) and of scaled Brownian motion and scaled Wiener process (right). $H = 0.25$.

For $H = 0.75$, reduced autocorrelation functions of K-fBm and L-fBm are visually almost identical, except that values of $C_{\min}$ and $C_{\max}$ are different. Figure 3 shows therefore $C_{\mathrm{K,r}}(t,s)$ and difference $\Delta_{\mathrm{LK}} = C_{\mathrm{L,r}}(t,s) - C_{\mathrm{K,r}}(t,s)$. These very small differences are augmented in the nonmarkovian characteristics $R(u)$ — *cf.* Sect. 4.

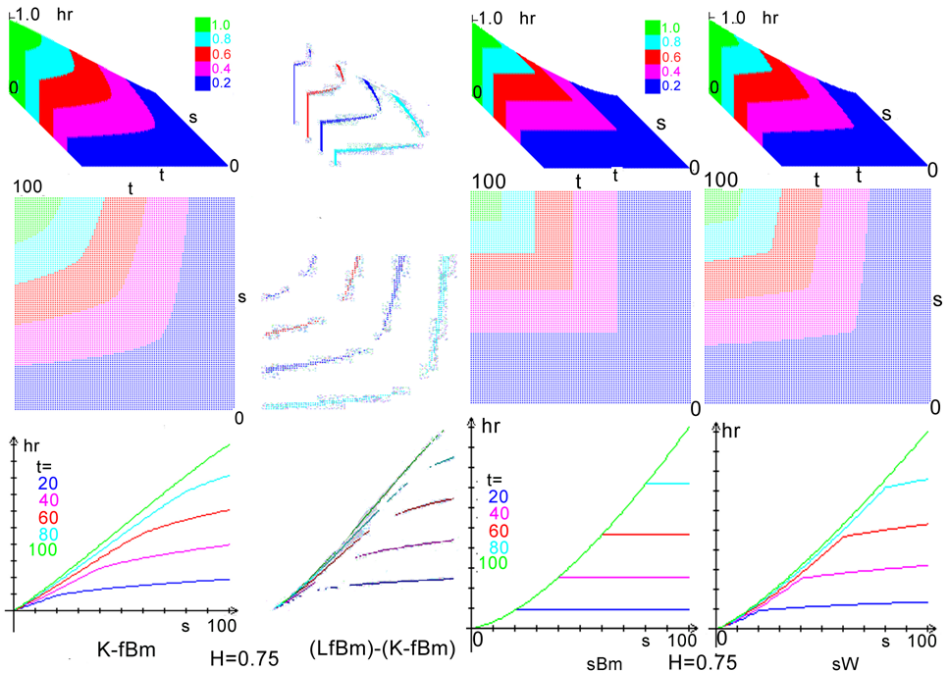

Fig. 3. (Colour on-line) Comparison of reduced autocorrelation functions of subdiffusional Kolmogorow and Lévy fractional Brownian motions (left) and of scaled Brownian motion and scaled Wiener process (right). $H = 0.75$.

Both L-fBm and K-fBm processes are nonmarkovian (*cf.* below, Sect. 4). Comparison with markovian sBm and sW fBms (defined below) is also shown in Figs. 2 and 3, which enables the direct visual observation of different behaviour of processes belonging to these two different subclasses of fBms. These markovian fractional Brownian motions differ significantly from K- and L-fBms but are similar to each other. Still, similarities are much stronger for $H = 0.75$ (superdiffusional processes) than for $H = 0.25$ (subdiffusional ones).

Mandelbrot–Van Ness fBm (MVN-fBm, ${}_M\boldsymbol{X}_H(t)$)

According to Mandelbrot and Van Ness [31], the (Holmgren–Riemann–Liouville) integral (5) "puts too great an importance on the origin for many

applications", therefore (as the remedy for the fault of L-fBm?), they proposed another, now best-known integral definition of a fractional Brownian motion via the Weyl integral

$$
\begin{aligned}
{}_M X_H(t) &= L_H(t) + T_H(t)\,, \\
T_H(t) &= \frac{1}{\Gamma(H+1/2)} \int\limits_{-\infty}^{0} \mathrm{d}u \left[(t-u)^{H-1/2} - (-u)^{H-1/2}\right] \xi_0(u)\,. \quad (9)
\end{aligned}
$$

It seems that in the opinion of the Authors, this process represents better a balanced memory of the past behaviour.

Autocorrelation function of ${}_M X_H(t)$ is equal to that of $B_H(t)$ [38, 39], *i.e.*, the process ${}_M X_H(t)$ fulfils the definition of Kolmogorov's fBm. Therefore, Eq. (9) can be treated as an integral representation of K-fBm. This implies, among others, that Kolmogorov's diffusion coefficient is equal to that of Mandelbrot–Van Ness' one (*cf.* [29]).

The numerical simulations of ${}_M X_H(t)$ are practically impossible with sufficient accuracy because of the extremely slow convergence of integral $T_H(t)$ at its lower limit $u \to -\infty$. Thus, several cut-off MVN-fBms with finite lower limit were proposed. Note that in fact the L-fBm is also a cut-off MVN-fBm.

Scaled Brownian motions (sBms, ${}_S X_H(t)$)

Several authors considered diffusion-like processes with time-dependent (more generally, variable) diffusion coefficient, including the applications to experimental data (*cf. e.g.* [37]). The corresponding Langevin equation for this problem could be written in the form of

$$\mathrm{d}X(t) = f(t)\xi(t) \quad (10)$$

with the autocorrelation correlation function (for any δ-correlated noise $\xi(t)$)

$$C_X(t,s) \sim f^2(\tau)\,. \quad (11)$$

However, only the processes with algebraic $f(t)$ fulfil our definition of the fractional Brownian motion. We shall call such processes the scaled Brownian motions (sBm)

$${}_S X_H(t) \sim \int\limits_0^t \mathrm{d}u\, u^{H-1/2}\, \xi_0(u)\,, \quad (12)$$

with the autocorrelation function [29]

$$C_S(t,s) \sim \tau^{2H}\,. \quad (13)$$

If the L-fBm "puts too great importance on the origin" (for superdiffusion), or "on the end of the process" (for subdiffusion), then this process behaves contrariwise, and all above remarks about true behaviour of the L-fBm hold in inverse: for $H < 1/2$, (*i.e.*, for subdiffusion) the kernel $u^{H-1/2}$ increases, whereas for $H > 1/2$, (*i.e.*, for superdiffusional process) dampens the influence of the beginning (a kind of "aging", quasi-"memory" of the past behaviour) of the process.

We may also define another model of fBm, *viz.* the scaled Wiener process (SW)

$$_{\mathrm{SW}}X_H(t) = W_H(t) \sim \exp(\gamma \log(t)) \int_0^t \mathrm{d}u\, \xi_0(u) \sim t^\gamma\, W(t)\,, \tag{14}$$

with the autocorrelation function

$$C_{\mathrm{SW}}(t,s) \sim \tau^{1+2\gamma}\,, \qquad \sigma^2(t) \sim t^{1+2\gamma}\,. \tag{15}$$

For $\gamma \in (-1/2, +1/2)$, SW fulfils the definition of diffusional processes, described at the beginning of this section. In particular, its first two moments, and its correlation function scale properly. Its main observables will behave identically as those of sBms with $H = \gamma - 1/2$, therefore, operationally SW is indiscernible from sBm. However, whereas physical interpretation of sBms is simple, the same cannot be said of SW — its kinetic (Langevin) equation reads

$$_{\mathrm{SW}}\dot{X}_H(t) = \gamma \mathrm{e}^{(\gamma-1)t} W(t) + \mathrm{e}^{\gamma t}\xi_0(t)\,, \tag{16}$$

being a kind of scaled Ornstein–Uhlenbeck process driven by scaled white noise. We might say that this process "puts most strong importance on its end".

Comparison of sBm and sW fBms is shown also in Figs. 2 and 3. These fractional Brownian motions are similar to each other, but differ significantly from K- and L-fBms. Moreover, similarities are much stronger for $H = 0.75$ (superdiffusional processes) than for $H = 0.25$ (subdiffusional ones).

L-family of fractional Brownian motions (μ-fBm, $_\mu L_H(t)$)

Let us define the family of models being the generalization of the Lévy process, called here μ-fBm [29]

$$_\mu L_H(t) = \frac{1}{\Gamma(H+1/2)} \int_0^t \mathrm{d}u\, |\mu t - \nu u|^{H-1/2}\, \xi_0(u)\,. \tag{17}$$

with obvious generalization of Eq. (6) for the autocorrelation function. Note that Lévy process (5) is obtained by putting $\mu = \nu = 1$. This form enables easy manipulation of the weights of beginning and the end of the process, with rather easy way of generation of sample paths of any (finite) length. Detailed discussion of properties of this L-family fBms can be found in [29].

K-family of fractional Brownian motions

In analogy to the definition of Kolmogorov's fBm, we can define other processes by their correlation functions. For these to be fBms, they must be diffusional, *i.e.*, fulfil the definition in Eq. (2). Moreover, to use the Cholesky decomposition for generation of sample paths $\{X_H(t)\}$, the correlation matrix $C_H(t,s)$ must be *(i)* nonsingular, *(ii)* symmetric, $C(t,s) = C(s,t)$, and *(iii)* its diagonal $C_H(t,t) \sim t^{2H}$.

The simplest change which can be introduced to Kolmogorov's fBm, in analogy to L-family's μ-fBm, is to introduce coefficients being here the weights of parts of that definition

$$C_\mu(t,s;H) \sim \mu_1 \tau^{2H} + \mu_2 \theta^{2H} - \nu|\theta - \tau|^{2H}\,, \tag{18}$$

with $\tau = \min(t,s)$, $\theta = \max(t,s)$. It is easy to see that by putting $\nu = 0$, we get a rather uninteresting process, and by putting $\nu > \mu_1, \mu_2$ — a process which is in part anticorrelated: $C_\mu < 0$ for some values of t, s.

Other simple and obvious possibilities is to use as the first (positive-definite) parts of the whole $C(t,s)$ the autocorrelation function of other known fBms, defined above, and add or subtract $|t-s|^{2H}$. Subtraction of this from the L-fBm results in shifting the whole to anticorrelations. This point will be discussed in more detail in Sect. 4 (*cf.* Eqs. (30), (31), and Figs. 6 and 7). The same can be done with the combinations (linear or otherwise) of such $C(t,s)$. Substitution of $|t-s|^{2H}$ by $|t^q - s^q|^{2H/q}$ results in changes of widths of peaks at $t = s$. However, all these variations of K-L-, S- and SW-fBms lead to similar $C(t,s)$ with main differences between markovian (K- and L-) and nonmarkovian (S- and W-) processes, therefore, changes introduced by additional parameters (μ, ν, *etc.*) seem to be very hard to identify experimentally. On the other hand, artificial introduction of additions as in Eqs. (30), (31) to experimentally determined autocorrelation functions might serve for such identifications.

More interesting seem to be oscillations of the first or second component of $C(t,s)$, which may simulate the behaviour of Bm in layered systems (environments), *e.g.*, in a kind of laminates of thin films with different properties (parameters), such as density, concentrations of components, temperature, or the composite system including inside and outside of the cell, or the cell and organelle, together with separating membranes, and the like.

Fractional noises

There is yet another possibility of creating new fractional Brownian motions: define new fractional (coloured) noise $\xi_f(t)$ either by its $C_\xi(t,s)$, by its Cholesky decomposed "square root" $Q_\xi(t,s)$, or by any other method, and next calculate related fBm, either from Eq. (19), using the standard GWN $\xi_0(t)$, or any other white noise (*e.g.*, some α-stable one), or just from a simple Langevin equation, Eq. (3), with $\xi_f(t)$ in place of $\xi_0(t)$.

However, the physical interpretation of such fBms can be difficult.

2.1. Noises

Computation of sample paths from the definitions of fBms needs the application of appropriate stochastic force — "noise" — acting on the Brownian particle. Moreover, in several other situations, closer specification of the properties of these noises is necessary.

The answer to the second Saxton question (physical mechanism) is in Smoluchowski's approach (*cf.* Sect. 1) of interactions with (mainly transfer of momentum from) neighbour particles of the environment in which the Brownian particle is immersed, *i.e.*, mechanical process (very complicated), modelled by appropriate stochastic force. The key phrase is "appropriate": how to find what force — "noise" — is "appropriate". In macroscopic systems with established local equilibrium (well-defined observationally local temperature), the main role is played by thermal fluctuations, *i.e.*, random distribution of momenta of all particles (Maxwell distribution), well-approximated by Gaussian white noise (GWN). In crowded nonuniform environments, in particular in living cells, where many different processes take place — transport (Bm or other) of other particles, (bio) chemical reactions, changes of electric potentials (*e.g.*, of biomembranes), *etc.*, all practically not connected directly with the process in question; all these more or less accidental interactions can be represented by some additional effective — usually nonwhite — random forces.

Such additional forces, originating from physical processes and treated as "noises", can be, and usually are, correlated, *i.e.* not independent. When originating from several sources, will be even not identically distributed. At best, i.i.d. will be (approximately) their increments. Anyway, all these disturbances from other processes sum up to an effective additional "noise", either coloured, or more or less white.

However, we must add here that *(i)* in many calculations GWN can be substituted by any white (*i.e.*, δ-correlated) process ("noise"). In particular, in all calculations of $\sigma^2(t)$, $C(t,s)$, *etc.* presented in this section, GWN can be substituted by any white "noise", leading to identical results, and *(ii)* GWN (and all white noises as well?) is unphysical and there are serious objections against its application to physical processes [40].

In nonphysical systems, mainly in economy, this mechanism does not apply. Still, in practice, in most cases discussed in literature, the stochastic forces $\{\xi_0\}$ are treated (explicitly or implicitly) as GWN. Closer inspection shows that in most cases, the only property of $\{\xi_0\}$ which is used is that they are i.i.d. and δ-correlated (white). The only serious argument for GWN is that the Gaussian distribution is the easiest for most applications and computations.

In some physical systems dominated or at least accompanying, there are well-recognized noises. When the number of particles is small, more proper than GWN seems to be the Poisson distribution together with several its "descendants" (realizations): Schottky (shot) noise (dominating in semiconductor currents [41]), telegraphers (dichotomous) noise [42] (among others, distribution of currents through biomembrane nanochannels is a dichotomous process [43]), *etc.*

In relation to Brownian motions, processes driven by α-stable noises are the most discussed in literature. Shot noise was found in neurons [45] and recently applied to Langevin dynamics of motion of a Brownian particle moving in a potential field [46], and in anomalous transport of active particles [47].

Integral definition of K-fBm, Eq. (8) is given through the fractional Gaussian noise. All other fBms can also be defined by appropriate fractional noises. In general, one can introduce several other stochastic processes as "noises" simulating mentioned above quasi-random effects of side processes going in the vicinity — this remark applies not only to physical processes, but to social, economic, *etc.* ones as well. Usually, it is required that such noises should be stationary, but it seems that these conditions are not crucial. For example, infinitesimal increments of other fBms, treated as noises, do fulfil these requirements, mentioned nonmarkovian versions of shot and dichotomous noises too, but experimentally accessible finite increments sample of paths usually will not (for more details, *cf.* [29]). Ionic nanochannel processes, which can be treated as noises influencing the cell membrane electric potentials [48] are dichotomous, but not stationary.

Presence of the additional — to GWN — noise can be identified by nonzero imaginary part of the characteristic function $\phi(t)$, but only if such addition introduces asymmetry with respect to time-reversal, *i.e.*, when effective stochastic force $f(t) \neq f(-t)$, more correctly, when the pdf $P(x)$ is asymmetric with respect to $x = 0$.

2.2. Cholesky decomposition

Given a symmetrical nondegenerate $N \times N$ autocorrelation matrix $C(t, s)$, the process $X(t)$ with this correlation function can be recovered exactly as follows:

let $C(t,s) = Q(t,s)Q^T(t,s)$, with $Q(t, s > t) = 0$; then

$$X(t) = \sum_k^N Q(t,k)\xi_k \,, \tag{19}$$

where the set $\{\xi_k\}$ is the white noise driving the process $X(t)$. Usually, in literature the thermal noise (GWN) is used, as *e.g.*, in Kolmogorov's definition but, in general, any random or chaotic [49] δ-correlated (white) process will also lead to similar, if not identical, Brownian motion.

The matrix Q, called sometimes *square root* of matrix C, can be computed from $C(t,s)$ (exactly) by the so-called *Cholesky decomposition* [50] (very demanding numerically!). The inverse operation

$$C(t,s) = \sum_k^N Q(t,k)Q^T(k,s) \tag{20}$$

enables recovering the correlation function from $Q(t,s)$.

The above relations suggest another, not used so far method of generation of various diffusional processes: from prescribed correlation function and its "square root" $Q(t,s)$, with $Q(t, s > t) = 0$. It might be useful, among others, for description of diffusion (Brownian motion) through strongly nonuniform, laminated, chaotic, or changing environments.

Moreover, instead of $C(t,s)$, we may as well prescribe $Q(t,s)$ to define some new process $X(t)$. There is the rub, however: $C(t,s)$ is related to — usually — rather clear (or easy to interpret) physical interpretation, whereas $Q(t,s)$ — is not, though one can try to interpret the resulting $C(t,s)$. On the other hand, the Cholesky decomposition is very demanding numerically (main limitation is the amount of memory needed to store, which increases as N^2 where N is the length of sample paths), so that in practice only short sample paths $\{X(t)\}$ can be obtained. Starting from given $Q(t,s)$ has no such limitations: in fact, one does not need to store the whole matrix — only needed to compute subsequent elements of the sample path are subsequent elements $Q(t,s)$ — *cf.* Eq. (19).

Ballistic motions

The use of autocorrelation function as a tool for defining new fBms enables also introduce a deterministic fBm, *viz.* anomalous (sub- and super-) ballistic motion defined by

$$\begin{aligned} C_b(t,s) &\sim (ts)^H \,, \qquad X_b(t) \sim t^{H-1/2} \,, \\ \dot{X}(t) &\sim t^{H-3/2} \,, \qquad \ddot{X}(t) \sim t^{H-5/2} \end{aligned} \tag{21}$$

describing the damped motion with time-dependent deceleration.

This process can be treated also as the "true" fBm, driven by a coloured fractional noise leading to the same $X_b(t)$ and the same $C_b(t,s)$ through the Langevin equation similar to that for K–fBm, Eq. (8).

The possible uses of such processes are discussed in more detail in Sects. 4 and 6.

3. Observables

For more detailed discussion, *cf.* [51].

By "observable" we shall understand quantity which can be either directly measured or computed from sample paths (measured or simulated). Some quantities can be calculated by analytical formulae from definitions: integrals or $C(t,s)$, which enables determination whether given measured process can be ascribed to a given theoretical (mathematical or otherwise) model (within experimental error).

Most used in practice is the dispersion (second moment) $\sigma_X^2(t)$, Eq. (2), and its dependence on time. When autocorrelation function is known *a priori* (*e.g.*, as definition of a given $X(t)$), $\sigma_X^2(t) = C_X(t,t)$. However, by definition (Sect. 2), all the diffusional processes are characterized by the same form of dispersion $\sigma_X^2(t) = \langle X^2(t)\rangle \sim t^H$, and differ only by the value of the Hurst exponent H. Therefore, some other measurable characteristics are needed. In this section, we shall discuss a few possible characteristics.

Second most frequently discussed in literature is the problem whether the given Brownian motion is nonergodic. This is based on determination of the behaviour of time and ensemble averages of the $\sigma_X^2(t)$ — are these equal, or disparate? This will be discussed in Sect. 5.

In the present author's opinion, the most important, and at the same time rather easy to determine (both experimentally and numerically), is the autocorrelation function $C(t,s)$, discussed in detail, along the definitions of various fBms, in Sect. 2.

Directly related to the autocorrelation functions is the nonmarkovian characteristics of a given fBm. This is discussed in detail in Sect. 4 below.

Other possible observables which can be calculated from a collection of sample paths are mostly useless for the discerning the types of fBms. Worth mention are:

Power spectrum $S(f)$

Power spectrum can be determined directly from measured data (sample paths)

$$S_{X;M;T,N}(f) = \frac{\Delta t}{T}\left|\sum_{n=1}^{N} X_M(t_n)\mathrm{e}^{-2\pi i f n}\right|^2, \qquad T = N\Delta t\,, \tag{22}$$

and

$$S_X(f) = \lim_{T\to\infty, M\to\infty} \frac{1}{M} S_{X;M;T,N}(f)\,, \tag{23}$$

M being the number of different realizations of the measured or computed sample paths of the process $X(t)$. The above relations can be treated as the operational definition of $S(f)$.

It is worth to note that single-trajectory power spectra were recently discussed and experimentally measured [52].

The power spectra do not give much additional information, comparing with other observables. For most cases, $S(f) \sim f^{-2}$ a.k.a. *brown noise*, only for some fBms with very low values of the Hurst exponent, $S(f) \sim f^{-\alpha}$, $\alpha < 2$ a.k.a. *pink noise*.

Probability density, characteristic function, first passage time

The probability distribution function $P(x)$ is sometimes determined from experimental data. Maybe easier for calculation from such data is its Fourier transform, *i.e.*, the characteristic function $\phi(t)$

$$\phi(t) = \int\limits_{\mathcal{D}_X} \mathrm{d}x\, \mathrm{e}^{itx}\, P(x) = \left\langle \mathrm{e}^{itx} \right\rangle . \tag{24}$$

The last expression can also be treated as an operational definition, enabling the computation of $\phi(t)$ directly from experimental data or from simulations of sample paths.

Frequently discussed in literature distribution of first passage times (fpt) to a prescribed target [53] is directly connected to $P(x)$.

Both $P(x)$ and $\phi(t)$ carry information mainly about the stochastic forces (noise) driving the given fBm, therefore, are not very interesting when the main noise is the thermal one, maybe with one exception. In a living cell, alongside with the measured process, there are many other processes, the presence of which can perturb each other. This additional stochastic force adds to the thermal noise, and its presence can be detected in these distributions. In particular, GWN (model of thermal noise, resulting from the Maxwell distribution of momenta of surrounding particles) is symmetric, therefore $\Im\ \phi(t) = 0$. Appearance of nonzero imaginary part gives a direct signal of the presence of an admixture of some nonthermal noise. Strong irregularities in determined $P(x)$ (or $\phi(t)$) may signal the presence of some obstacles in the environment in which Brownian particles move (*cf.* "repelling boundaries" point below).

The shape of distribution of fpt, as being calculated directly from $P(x)$, depends additionally on the effective diffusion coefficient. However, the latter describes not only the trajectory of a Brownian particle (*i.e.*, the model

of fBm) but also the properties of the medium and of the boundaries within which this particle moves, therefore, is also of little value for recognition of the type of measured fBm.

Of course, when Brownian motion of interest models nonphysical situations, *e.g.*, when the economic processes are considered, the distributions $P(x)$ and $\phi(t)$ may bring over some really valuable information, the more that such processes, *e.g.*, the behaviour of stock investors is the process strongly influenced by the memory of the past behaviour of all inventors.

Persistence and restart

The restart method of acceleration of arrival time to prescribed target is in fact nothing but a simplified albatross search pattern with return to the same (starting) point, and in this respect is related to CTRW [22]. The restart process contains intervention from the observer, thus — though recently fashionable — is worthless from the point of view of identification of the model of fBm being restarted.

Interesting enough is the property called persistence. True noises are independent (even if not identically distributed), *i.e.*, subsequent values of ξ_0 are not influenced by any from previous ones. Markovian process remembers the previous value, persistent process — two preceding values. Distribution of persistence times depends mainly on the value of the Hurst exponent.

Again, these results do not seem to be characteristic enough to enable differentiation between models of fBms. At best, they can narrow the class of fBms of similar characteristics, and provide additional arguments for or against assumed classification.

Repelling boundaries

Recent results of investigations of influence of existing boundaries on properties of fBms [54, 55] seem to open the whole new chapter in the Brownian motions story.

Most of models of fBms discussed so far in the literature assume that the motions of Brownian particles are unlimited, "free": $X_B(t) \in (-\infty, +\infty)$. The investigation of the influence of existing boundaries (walls) on the behaviour of fBm began only recently, by considering the motions limited to semi-infinite axis, $X_B(t) \in [0, +\infty)$, with reflecting wall at $x = 0$ [54]. These first results showed that some of properties of so limited fBms change radically. Most characteristic feature is here strongly nonexponential probability density near boundaries, with cumulation of density near the wall for superdiffusional processes, and depletion for subdiffusional ones.

These results were confirmed experimentally by measurements of the motions of Brownian particles inside axons of some specific group of brain neurons [55]. The results show that the details of motion depend not only on the nearness of the reflecting walls (*i.e.*, diameter of the axon), but — even more strongly — on the shape (degree of tortuousity) of the whole channel.

These results suggest, among others, that difficulties with dealing with boundary value problems (*e.g.* in discussion of first passage times) for fBms may be related to the strange behaviour of probability density near boundaries (targets) [56].

4. Nonmarkovianity

All stochastic processes are either markovian or nonmarkovian. Markovian ones are popular, because they are much simpler for use in modelling natural phenomena. On the other hand, most of processes being now of interest are nonmarkovian, or can be at least suspected of being nonmarkovian. In the latter case, the degree of deviation from markovian character can be also of interest.

The most general condition which all markovian processes must fulfil is the Bachelier–Smoluchowski equation [14, 16, 57], known also in literature as the Chapman–Kolmogorov [40] one (*cf. e.g.* [13] for more details)

$$P_{1|1}(x_2,t|x_1,s) = \int\limits_{\mathcal{D}_x} \mathrm{d}x_3 P_{1|1}(x_2,t|x_3,u) P_{1|1}(x_3,u|x_1,s)\,, \tag{25}$$

with $s < u < t$, where $\mathcal{D}_x$ denotes the domain of states x, and $P_{1|1}$ — conditional probability distribution. In the following, we shall call Eq. (25) BSCK. The analogous equation for data in the form of (discrete) sample paths is obvious.

BSCK condition is most general, but at the same time, most difficult to use and most impractical when one has to deal not with general analytic formulae, but with series of data in the form of $\{x(t_1), x(t_2), \ldots, x(t_n)\}$ (sample paths), in many cases nonstationary, either experimental, or obtained from numerical simulations.

Two-point distributions $P_2(x,t;y,s) = P_{1|1}(x,t|y,s)\,P_1(y,s)$ can be determined from experimental data, but the amount of data needed is enormous and practically unavailable, except some rare cases of stationary systems ($P_2(x,t;y,s) = P_2(x,t-s;y,0)$) with a few distinct well-defined states ($x,y \in [a,b,c,\ldots]$) (as *e.g.* in transport of ions through nanochannels [43]).

All Bm processes discussed here are nonstationary. This means that to determine nonmarkovian index directly from BSCK equation, one cannot determine distributions $P_{1|1}(x_1,t_1|x_2,t_2)$, or $P_2(x_1,t_1;x_2,t_2)$ by moving-

window averages as in [43]. In other words, one needs to determine four-dimensional matrices $P_{1|1}$ from experimental data, for all measured values of $X_H(t)$, and average these over sufficiently many realizations of the process. This is practically impossible. Therefore, some simpler measures (indices) of deviations from markovianity are needed.

Sufficiently simple seems to be index based on normalized autocorrelation functions [44]:

Assume that, by analogy to Eq. (25),

$$C_n(t,s) = C_n(t,u)C_n(u,s) \quad \text{for} \quad s < u < t\,, \tag{26}$$

where $C_n(t,s)$ is the normalized autocorrelation function of the process $X(t)$, defined

$$\begin{aligned} C_n(t,s) &= C(t,s)/\sqrt{\langle X^2(t)\rangle\langle X^2(s)\rangle}\,, \\ C(t,s) &= \langle X(t)X(s)\rangle\,. \end{aligned} \tag{27}$$

The proof of equivalence of this relation and BSCK equation is given in [44].

Let us define the following signatures and measures of nonmarkovianity, resulting directly from Eq. (26):

(i) the function (NM signature) $R(u)$

$$R(u) = \frac{C_n(t,u)C_n(u,s)}{C_n(t,s)} = \frac{C(t,u)C(u,s)}{C(t,s)\sigma^2(u)}\,, \tag{28}$$

where $t < u < s$ ($\sigma^2(u) = C(u,u)$). Note: $R(u=t) = R(u=s) = 1$. From these definitions, we get that if $R(u) = 1$ over the whole interval $u \in [t,s]$, then the process $X(t)$ is markovian.

(ii) In order to compare the deviations from markovianity, define two proper (*i.e.*, non-negative-definite) measures

$$\Delta_1 = \left|1 - \frac{1}{t-s}\int_s^t \mathrm{d}u R(u)\right|, \qquad \Delta_2 = \frac{1}{t-s}\int_s^t \mathrm{d}u|1-R(u)|\,. \tag{29}$$

These quantities enable direct comparison of degrees of nonmarkovianity between various processes. Note that in this parametrization, all unimportant for our considerations prefactors, diffusion coefficients, *etc.* cancel.

Signature $R(u)$ gives qualitative visual information quantified by measures Δ_1 and Δ_2, taking into account a kind of "global" and "local" nonmarkovianity. Usually, when $1 - R(u)$ does not change sign over the whole

interval $u \in [t, s]$, $\Delta_2 \leq \Delta_1$. Δ_2 is useful in cases when parts of $1 - R(u)$ are positive, other parts — negative (*e.g.*, $R(u)$ is oscillating). Then information carried by Δ_1 can be misleading. This property can be seen in Fig. 7 where in some range of parameter q, Δ_1 of the superdiffusional L-fbM tends to zero when Δ_2 tends only to minimum.

On the other hand, Δ_2 can be significantly different than zero when $R(u)$ is just fluctuating around zero as the effect of experimental (computational) errors. This implies: in case of doubt, determine not only both Δ_1 and Δ_2, but also $R(u)$ — the latter contains more detailed information.

Of practical interest can be the fact that the signature $R(u)$ enhances the differences between different processes, badly visible in $C(t, s)$ alone. Mentioned in Sect. 2 insignificant differences between superdiffusional autocorrelation functions of K-fBm and L-fBm (*cf.* Fig. 3) become distinct when these functions are presented in the form of signatures $R(u)$. This is shown in Fig. 4 (left panel) where for comparison are shown also $R(u)$ for subdiffusional case, which is more strongly nonmarkovian than superdiffusion. Figure 4 shows also how signatures of both fBms change with the values of Hurst parameter H (right panel). Although this information does not help the identification, it stresses the substantial differences between both fBms, not always visible in other characteristics.

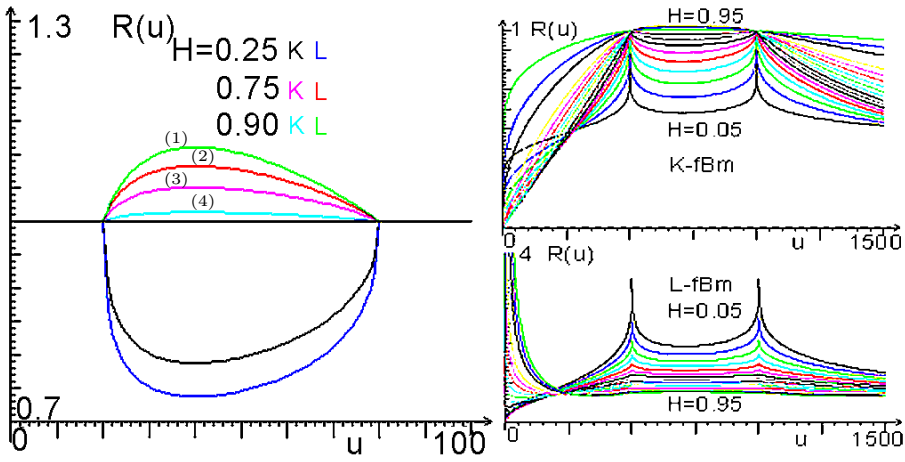

Fig. 4. (Colour on-line) Comparison of signatures $R(u)$ of K- and L-fBm. Left column: $R(u) < 1$: $H = 0.25$, $R(u) > 1$, $H = 0.75$ (red (2) and magenta (3)) and $H = 0.95$ (green (1) and cyan (4)). Differences between both fBms are clearly visible in all cases. Right column: Dependence of $R(u)$ on the Hurst exponent H. Tails $u < t$ and $u > s$ are added for better visibility of different behaviour of L- and K-fBms.

The further arguments in identification of the model of fBm which describes a given set of sample paths can be obtained by calculating from the experimental data the signatures $R(u)$ for increasing ranges of $R(u)$ along the path. This is illustrated in Fig. 5.

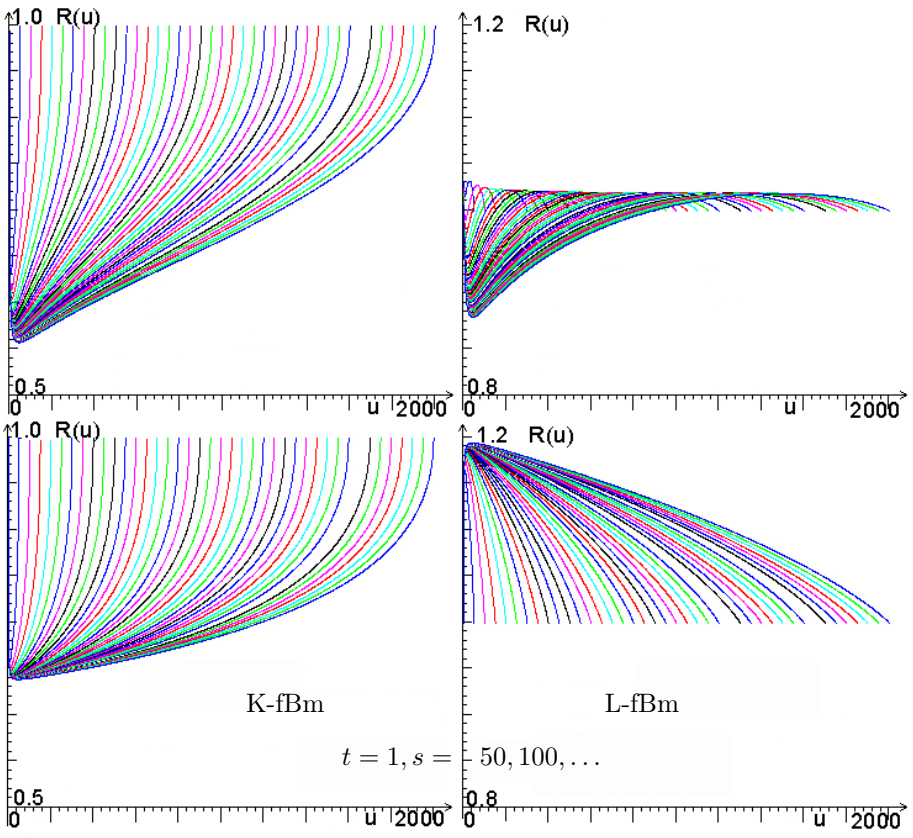

Fig. 5. Local nonmarkovian characteristics $R(u)$ of Kolmogorov (left panels) and Lévy (right panels) fractional Brownian motions for increasing range of $R(u)$; upper panel: $H = 0.25$, lower panel: $H = 0.75$.

If subdiffusional ($H = 0.25$) K-fBm and L-fBm differ between themselves more than the superdiffusional ones, still their "representations" via $R(u)$ differ much more — *cf.* Figs. 4 and 5.

The values of both measures may (and usually do) depend on details: interval $[t, s]$, place of this interval along the whole range of $X(t)$ (in particular whether it is located near the beginning, middle or end of the measured data), roughness of $R(u)$ (this is taken in part into account by Δ_2), *etc.* This is shown in Fig. 5.

Apart from the visualization of the nonmarkovian character of the process under consideration, the signature $R(u)$ enables an easy method of looking for the role of various factors in the definition of considered fBm, to find how the changes of some parameters change the picture, *etc.*

Still other use of the nonmarkovian signatures and measures is that they enable simple detailed investigation of the results of changes of properties of different fBms caused by changes of their definitions via autocorrelation functions. Of obvious interest is what causes the appearance of nonmarkovianity in processes which in their "zero-th" form are markovian,

An elementary example is the Wiener process which is obviously markovian: substitution of its autocorrelation function $C_W(t,s) = \min(t,s)$ into definition of $R(u)$ above results in $R_{W0} = 1$ in the whole interval $u \in [t,s]$. Addition of the factor $q|t-s|^{2H}$ changes this process into nonmarkovian one

$$\begin{aligned} C_{qW}(t,s) &= (1-q)\min(t,s) + q\max(t,s)\,, \\ R_{qW}(u) &= \frac{[(1-q)t+qu][(1-q)u+qs]}{[(1-q)t+qs]2u}\,, \end{aligned} \tag{30}$$

which is obviously nonmarkovian, except for $q = 0$ and $q = 1$. The last case, for $q = 1$, changes the Wiener process to a kind of markovian anti-Wiener with $C(t,s) \sim \max(t,s)$.

The same procedure can be applied to other fBms. The results, in the form of measures Δ_1 and Δ_2 as functions of the strength parameter q is shown in Figs. 6 and 7. The labels there mean

$$\mathrm{K}: K_0 = t^{2H} + s^{2H}\,, \qquad \mathrm{W}: C_0 = [\min(t,s)]^{2H}\,, \qquad \mathrm{B}: C_0 = (ts)^H\,. \tag{31}$$

This secondary representation enables further identification of the model fBm under consideration.

$B_H(t)$ measures are not much different than these for $W(t)$ (implies: $B_H(t)$ is weakly nonmarkovian). Besides, all three model fBms shown in Fig. 6 are rather similar, in particular that for all three $\Delta_1 = \Delta_2$. This is no longer true for the Lévy fBm, which is much stronger nonmarkovian than B_H [44] (*cf.* Fig. 7).

The changes of behaviour of the models of fBms shown here can be also applied to experimental data: add to the measured autocorrelation functions the factor $q|t-s|^{2H}$ (H — measured Hurst exponent) and look for the changes.

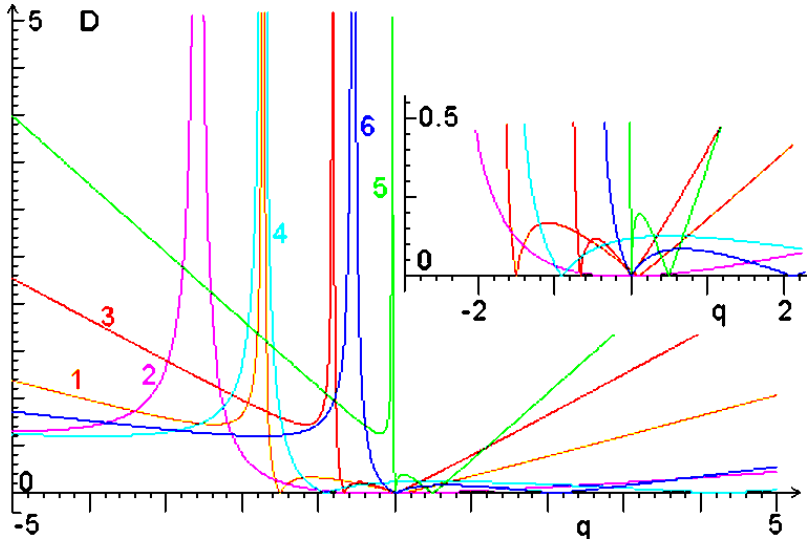

Fig. 6. Comparison of measures $\Delta_1 = \Delta_2$ for the three variants (Eq. (30)) of changes in nonmarkovian versions of Kolmogorow (K), Wiener (W) and ballistic (B) processes (*cf.* Eq. (31)). Labels 1, 2, 3: subdiffusion, $H = 0.25$, 4, 5, 6: superdiffusion, $H = 0.75$; labels 1, 4: K, 2, 5: W, 3, 6: B. Inset: details of nonmarkovian behaviour near $q = 0$.

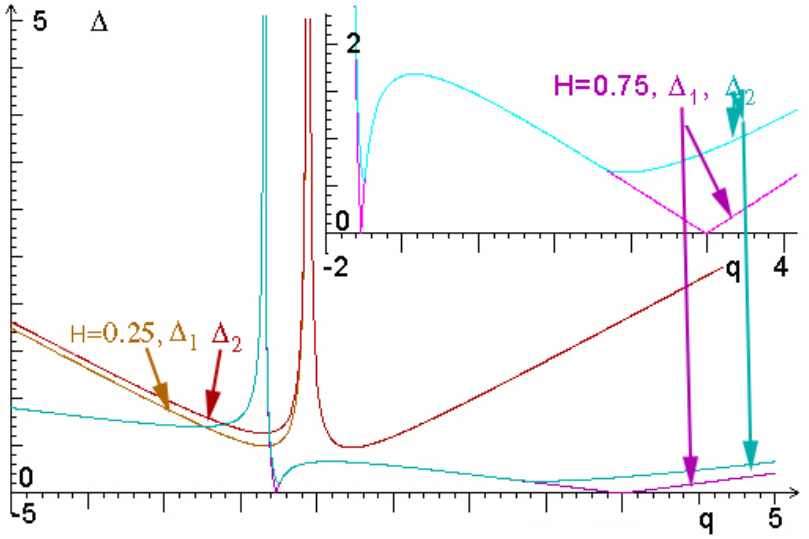

Fig. 7. Measures Δ_1 and Δ_2 of Lévy fBm with weighted addition of the term $q|t-s|^{2H}$. Note differences between Δ_1 and Δ_2. Inset: details of nonmarkovian behaviour near $q = 0$.

5. Weak ergodicity breaking

Another frequently discussed in literature subject, apart from the second moment (dispersion), is the problem whether the given Brownian motion is nonergodic (*cf.* [28, 33, 58, 60]). This problem can be defined as *are the time and ensemble averages disparate?*

Boltzmann hypothesis states that the system/process is ergodic when averages over ensemble and over (long) trajectory (*i.e.*, over time) of observables on a system or process are equal. In recent literature, commonly used is the dispersion of position of Brownian particle

$$\overline{\delta^2 X(t,T)} = \left\langle X^2(t) \right\rangle , \tag{32}$$

where $\langle \cdots \rangle$ denotes averaging over ensemble (over probability density), $\overline{\cdots}$ — over time (along trajectory), with

$$\overline{\delta^2 X(t,T)} = \frac{1}{T-t} \int\limits_0^{T-t} \mathrm{d}s [X(s+t) - X(s)]^2 \tag{33}$$

relating two positions of the process (walker) separated by a time lag t.

In practice, the dispersion of one trajectory is useless in the case of Brownian motions due to too large jumps along sample paths. Therefore, in practice, one uses the slightly weaker measure of WEB

$$\Delta_X(t,T) = \left| \left\langle \overline{\delta^2 X(t,T)} \right\rangle - \left\langle X^2(t) \right\rangle \right| , \tag{34}$$

which is measurable with good enough accuracy.

Extensive discussion can be found in [34, 58]. Ergodic properties of fBms discussed in this paper are described in detail in [60].

Here, let us only add a few remarks, to elucidate some misunderstandings:

(i) Fulfilling the Boltzmann equality by one of observables does not prove anything. In the truly ergodic system (process), all observables on this process must fulfil Eq. (32).

(ii) The Khinchin theorem states that if the autocorrelation function $C_X(t,s)$ of the process $X(t)$ is *(i)* stationary (*i.e.*, $C_X(t,s) = f(|t-s|)$ only), and *(ii)* tends to zero for $|t-s| \to \infty$ (process becomes uncorrelated), then the process $X(t)$ is ergodic. This does not mean (as sometimes is suggested) that if $C_X(t,s)$ is nonstationary, then $X(t)$ is of necessity nonergodic. In the present author's opinion, nonstationarity of $C(t,s)$ gives only the strong suggestion (suspicion) that $X(t)$

is indeed nonergodic. Let us note that autocorrelation functions of all discussed here processes (Brownian motions, and more generally diffusive ones) are nonstationary, and most of them do not vanish in the limit $|t-s| \to \infty$ (*cf.* Sect. 2). Let us also add that (almost) all chaotic processes, with probability 1, are ergodic. In literature, the processes for which the Boltzmann equality holds for dispersion, Eq. (32), are frequently called "weakly ergodic". This property, so-called *weak ergodicity vs.* its counterpart, *weak ergodicity breaking — WEB*, gives indeed good characteristics of Brownian motions.

(iii) Even the averaged (weaker) version of weak ergodicity, Eq. (34), is fully fulfilled by the Wiener process and Kolmogorov fBm. In both these cases, both quantities are indeed proportional to t, and do not depend on the trajectories lengths, T. For other Bms, even simple ones, ensemble average is always $\sim t^{2H}$, whereas averaged time-average is a function of both t and T. Besides, in practice, because experimentally measured (also simulated) sample paths are of finite length (T in above formulae), the time-averaged dispersions will — in some cases — depend not only on proper running time t, but also on the length T of the measured trajectory. Therefore, at present, one speaks of (weak) ergodicity or WEB not when $\Delta_X(t,T) = 0$, but when both averages scale in t identically [28, 58].

(iv) The average dispersion along trajectory can be written with the help of the autocorrelation function of the process [59]

$$\left\langle \overline{\delta^2 X(t,T)} \right\rangle = \frac{1}{T-t} \int_0^{T-t} \mathrm{d}s \left\langle [X(s+t)-X(s)]^2 \right\rangle$$
$$= \frac{1}{T-t} \int_0^{T-t} \mathrm{d}s \left[\sigma_X^2(s+t) - 2C_X(s+t,s) + \sigma_X^2(s) \right] , \quad (35)$$

which is easy to compute for all fBms discussed here.

This result needs the additional commentary: note that dispersions are second moments of (averages over) one-point probability density $P_1(x,t)$, whereas the autocorrelation function is an average over two-point one, $P_2(x,t;y,s)$. These two quantities seem to be slightly disparate. This suggests that the so-defined average over trajectory is related also to the nonmarkovian characteristics of the process $X(t)$, whereas $\langle X^2(t) \rangle$ is insensitive to such properties.

6. Final remarks and conclusions

The main aim of this paper is, as we have stated in the introduction, to verify to what extent various fractional Brownian motions are different. In particular, we tried to find what mechanisms or details in their definitions make these motions anomalous, and whether the models can be distinguished experimentally. In other words, we tried to answer three questions–problems proposed by Saxton [25]. Although Saxton wrote about "anomalous subdiffusion in the plasma membrane, cytoplasm, and nucleus of cells", all these problems are common for all investigations related to Brownian motions.

First problem: The main experimental questions: "Is diffusion anomalous or normal, and what are the parameters describing it" is mainly answered in Sect. 1: *(i)* we know now (both by experimental data and from theoretical models) that there are two main types, CTRW and fBm (plus several practically not exploited — *cf.* Introduction); these can be divided further into several subtypes. *(ii)* their main characteristics, easy to determined, is the Hurst exponent H — for all fBms and for local enclosed traps with Bm inside.

Second problem: The main theoretical questions: "What mechanism makes the diffusion anomalous?" *(i)* it seems that for CTR the main factor is their second characteristics: long jumps between "traps" [22]. *(ii)* For fBms, the situation is more complicated: to answer this question, we need to know the structure of as many models of fBms as possible. Inspection of results presented in Sect. 2.2 shows that the most obvious in this respect is the diagonal of $C(t,s)$: as $\sigma^2(t) = C(t,t)$, and anomalous Bm is defined by $\sigma^2(t) \sim t^{2H}$ with $H \neq 1/2$, we may say that the diagonal of $C(t,s) \sim t$ defines the normal Bm. The answer to the second problem can therefore be formulated: "any mechanism which changes that diagonal".

The change of the behaviour of diagonal is, however, not simple. First, the whole autocorrelation function must fulfil conditions defining the process as diffusion (Brownian motion), Eq. (2); second — the physical interpretation of such a change must be clear (third Saxton's problem!). *E.g.*, a simple change of the type of $C_X(t,s) \to [C_X(t,s)]^{\gamma \neq 1}$ (*e.g.*, change $\min(t,s) \to \min^{\gamma}(t,s)$ in $C_W(t,s)$, resulting in the change of the Wiener process into a one of sBms) will be sufficient. However, such a change seems to be rather difficult for physical interpretation.

One of the possibilities is offered by changes in the environment. There is a kind of precedent: the motion of a potassium channel in a live cell membrane was found to be nonergodic, but after application of some drugs influencing, the properties of membrane became (weakly) ergodic, though remained anomalous of CTRW type [61]. Such changes — addition of some substance, changes of some parameters (temperature, in particular) are possible for experimental realization.

Another quite obvious possibility: presence (influence) of another stochastic force. Again, this is not so simple as looks at first sight: addition of second process associated with $H \neq 1/2$ will result in a nondiffusional process, addition of another normal process, $H = 1/2$, does not change the resulting diagonal. The only other possibility seems to be the coupling with some side process with appropriate characteristics, *i.e.*, in models, multiplication of the original $C(t,s)$ by other one with proper characteristics.

Third, main problem: "How can various mechanisms be distinguished experimentally?" In the present author's opinion, it is worth to estimate as many other observables besides the most popular: the Hurst exponent and (weak) ergodic properties as possible. As we said in Sect. 3, most of observables listed there are not characteristic enough to serve as sole quantity enabling to decide whether given measured Bm is described by a given model, at most they can exclude some of them. Several such exclusions or indications can, at least, narrow the class of models.

There is, however, one observable so far underestimated, or even almost totally neglected in the present literature: the autocorrelation function $C(t,s)$ (Sect. 2), and related to it the nonmarkovian signatures and measures (Sect. 4). The use of this method, Eqs. (28) and (29), *cf.* [44] enables the relatively simple calculations and estimations, sufficient to classification whether a given process is markovian or nonmarkovian. This, in turn, supplies more detailed information about memory characteristics of a given Bm.

The results presented in Sect. 4 suggest also that the good tool for identification of model in question is the subtraction of another term, even the whole model autocorrelation function. Maybe the even better will be the use of addition of the term $q|t-s|^{2H}$ to measured $C(t,s)$, with variable weight q, as in Figs. 6 and 7.

Correlation functions can also be used, together with the dispersions, to evaluate the scaling of weak ergodicity measures (Sect. 5).

Therefore, the partial answer to the third question is, in the present author's opinion: one ought to look for correlations and for nonmarkovian characteristics, particularly to the secondary ones (Figs. 5, 6, and 7), and additionally for as many other observables as is possible to determine from a given set of measured sample paths.

Still, all these attempts will result in classification of a given experimentally investigated Bm into one of subclasses. Moreover, more detailed estimation of the values of parameters within a given subclass, *i.e.*, indication of specific model of this Bm, seems to be at present rather questionable, due to experimental errors.

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[2] In fact, such motions are superpositions of slow, laminar aerodynamic flows within air currents, and superposed small glittering random, ("chaotic") Brownian motions. Sometimes it is doubted whether it is possible to see this small component by naked eye. Comparison of physical conditions in observations of microsize pollen in water, nanosize gold particles in melted glass, and dust in air, in particular nanosize particles of quartz, glass, or like (main factors: size and viscosity, plus reflected light from nanosize particles) implies that it seems possible. Maybe it would be worth verification?

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