PROBABILITY DISTRIBUTION FOR THE FIRST CASIMIR OPERATOR C_1 IN THE QUANTUM COULOMB FIELD

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Probability distribution for the first Casimir operator C_1 in the quantum Coulomb field is calculated from first principles of quantum theory of the Coulomb field formulated by the Author. This is followed by a certain, I hope novel, formulation of probabilistic interpretation of Quantum Mechanics, which allows to avoid lots of "philosophical" talk about Quantum Mechanics. This talk is very voluminous but not necessarily enlightening.

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1. A nice formula

In Ref. [1] I gave the formula for the matrix element $\langle u\left|\exp(-\sigma C_1)\right| u\rangle$, where $|u\rangle$ is the quantum Coulomb field moving with four-velocity u, C_1 is the first Casimir operator of the Lorentz group and $\sigma > 0$

$$\begin{aligned}\langle u\left|\exp(-\sigma C_1)\right| u\rangle &= (1-z)\mathrm{e}^z\mathrm{e}^{-\sigma z(2-z)} \\ &-\frac{2z^2\mathrm{e}^z}{\pi}\int\limits_0^\infty \mathrm{d}\nu\,\nu\mathrm{e}^{-\sigma\left(1+\nu^2\right)}\sum_{n=-\infty}^{\infty}\frac{(\nu+i(2n+1-z))^{n-1}}{(\nu+i(2n+1+z))^{n+2}}\,. \end{aligned} \tag{1}$$

Here, $z=\mathfrak{e}^2/\pi$, $0<z<1$ is the fine structure constant divided by π. Concerning this formula and some of the formulae reported later one should note that in infinite sums and integrals one cannot in general change the order of summation and integration. A simple example is:

$$\int\limits_0^\infty \left\{\sum_{n=1}^{\infty}\left(n\mathrm{e}^{-nx}-(n+1)\,\mathrm{e}^{-(n+1)x}\right)\right\}\mathrm{d}x = 1\,, \tag{2}$$

but

$$\sum_{n=1}^{\infty}\left\{\int_{0}^{\infty}\left(n\mathrm{e}^{-nx}-(n+1)\,\mathrm{e}^{-(n+1)x}\right)\mathrm{d}x\right\}=0\,. \tag{3}$$

The same situation seems to occur in the formula above.

The assumption that $\sigma > 0$, *i.e.*, that we are dealing with the so-called heat kernel, is very useful as it makes the integral in (1) very rapidly convergent. However, the sum in (1) is convergent to a function which vanishes for $\nu \longrightarrow \infty$ at least as quickly as $1/\nu$. Therefore, we can take in (1) σ to be purely imaginary obtaining the autocorrelation function $\langle u\,|\exp(-i\sigma C_1)|\,u\rangle$, $-\infty < \sigma < +\infty$. Fourier transform of the autocorrelation function of a self-adjoint operator is the probability distribution for this operator in the considered state, see the next paragraph. Calculating this Fourier transform by means of the usual trick with Dirac's δ-function, we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\sigma\,\mathrm{e}^{ip\sigma}\,\langle u\,|\exp(-i\sigma C_1)|\,u\rangle \\
&= \text{ probability density that the first Casimir operator } C_1 \\
&\quad\text{has the value } p \text{ in the Coulomb state } |u\rangle \\
&= (1-z)\mathrm{e}^{z}\delta[p-z(2-z)] \\
&- \Theta(p-1)\frac{z^2\mathrm{e}^{z}}{\pi}\sum_{n=-\infty}^{\infty}\frac{\left(\sqrt{p-1}+i(2n+1-z)\right)^{n-1}}{\left(\sqrt{p-1}+i(2n+1+z)\right)^{n+2}}\,,
\end{aligned} \tag{4}$$

$0 < z < 1$. The first term is a bound state described in [1] and belonging to the supplementary series while the sum represents continuous spectrum which belongs to the main series. As probability density, (4) should be non-negative and summable to 1. Numerically, according to Professor Wosiek, this seems to be the case but it would be nice to prove it. This, however, is tricky because of impossibility, mentioned above, of changing order of summation and integration.

2. Kinematical Axioms for Quantum Mechanics

Formulating Axioms for a great Theory, a theory which creates *Weltanschauung*, I will not translate this German word, because of various pitfalls around, we have to distinguish between Kinematics and Dynamics.

For example, in the Newtonian Mechanics, kinematics is the geometry of the Galilean space-time. Galilean geometry is a difficult subject, for the first time described correctly by Elié Cartan at the first half of the 20^{th} century. Before Cartan, there was a certain lack of clarity, however, with Newton, who introduced the notion of "true mathematical time", being the closest to the truth. Probably, he did not go further, because, as Roger Penrose suggests, he already had optics in mind. By the way, the Galilean geometry of Galilean space-time is, up to date, a topic which is not presented the way it deserves. Arnold [2] in his excellent book describes it in a footnote. Landau and Lifshitz in their very good and popular textbook [3] describe it after Lagrange's generalized coordinates and the principle of least action. Some of their statements as, *e.g.*, "relative to arbitrary coordinate system, space is neither homogeneous nor isotropic" should really be punishable. For a correct description of geometry of Galilean space-time, see Kopczyński and Trautman [4], and Penrose [5]. If the Reader is under impression that the subject of Galilean geometry of Galilean space-time is not terribly important I can quote Professor Witkowski, a prominent quantum theoretical chemist, who told me that there are people who are under impression that the Schrödinger wave function is a scalar with respect to general Galilean transformations.

In the Special Theory of Relativity the Kinematics consists of metric geometry of $(3 + 1)$ flat space-time. Paradoxically, here the textbooks are somewhat better; there are Authors who do start with the assumed space-time geometry, though not all. There are many people who continue to write about trains passing each other, observers discussing their clocks *etc.* History of Science, in the case under consideration a science from a century before us, is very important and should be known but not mixed up with teaching, which should be based on modern understanding. As the former associate Editor (1974–1994), and later (1995–2005) Chief Editor of this journal, I had many exchanges with people who did not like the Special Theory of Relativity. I had always had the impression that they were victims of their textbooks; they endlessly analyzed trains, clocks, observers comparing their clocks *etc.* The geometry of space-time is a modern equivalent of Euclid and should be taught as such. The geometry of space-time taught in this way would rise emotions very similar to those generated by Euclid.

In the General Theory of Relativity the Kinematics is the Riemannian Geometry of $(3 + 1)$ dimensional space-time, while Dynamics is the law of motion proposed by Einstein. There is now an industry of inventing laws of motion which are different from those proposed by Einstein. This industry is not very edifying, but can be mentioned to illustrate the difference between Kinematics and Dynamics which might create some difficulty in the General Theory of Relativity.

Quantum Mechanics, when compared with the great theories mentioned above, is a disaster. I mean pedagogical and cultural disaster, because, in purely cognitive terms, it contributed perhaps more to the volume of our understanding that the great theories mentioned previously. The Authors as a rule do not describe the difference between Kinematics and Dynamics. As an example, I will quote Bongaarts [6] who gives Axioms for Quantum Mechanics.

To my mind his Axioms I and II are definitions rather then axioms, because they describe the way we are going to use the words *state* and *observable*. But I will not argue. I take from Bongaarts Axioms I and II:

Axiom I. The state of a quantum system is represented by a unit vector ψ in a Hilbert space $\mathcal{H}$.

Axiom II. An observable of the system is represented by a self-adjoint operator A in $\mathcal{H}$.

I omit Bongaarts' Axiom III because it takes the Author three pages to formulate it and contains, among other things, the Schrödinger equation and the Probabilistic Interpretation of Quantum Mechanics (!), without ever mentioning the geometry of underlying space-time. Instead, having Axioms I and II, I will formulate one Definition and one Theorem. The Theorem actually belongs to the Functional Analysis but, for the Reader's convenience, I will give a simplified proof.

Definition. The matrix element $\left\langle \psi \left| \mathrm{e}^{-i\lambda A} \right| \psi \right\rangle$, where λ is a real number and A is an observable, is called autocorrelation function for the observable A in the state $|\psi\rangle$.

Theorem. The Fourier transform of each autocorrelation function is non-negative and summable to 1.

Proof. We write $|\psi\rangle = \sum_n c_n |\psi_n\rangle$, where $|\psi_n\rangle$ are eigenfunctions of the observable A, with corresponding eigenvalues being A_n. Therefore,

$$\begin{aligned}\left\langle \psi \middle| \mathrm{e}^{-i\lambda A} \middle| \psi \right\rangle &= \left\langle \sum_n c_n \psi_n \middle| \mathrm{e}^{-i\lambda A} \middle| \sum_m c_m \psi_m \right\rangle \\ &= \sum_{n,m} \overline{c_n} c_m \left\langle \psi_n \middle| \mathrm{e}^{-i\lambda A_m} \middle| \psi_m \right\rangle \\ &= \sum_{n,m} \overline{c_n} c_m \delta_{mn} \mathrm{e}^{-i\lambda A_m} = \sum_n \left| c_n \right|^2 \mathrm{e}^{-i\lambda A_n} ,\end{aligned}$$

and the Fourier transform is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\lambda \, \mathrm{e}^{ip\lambda} \sum_n |c_n|^2 \, \mathrm{e}^{-i\lambda A_n} = \sum_n |c_n|^2 \, \delta(A_n - p) ,$$

which is clearly a non-negative and summable to 1 function:

$$\int_{-\infty}^{\infty} \mathrm{d}p \sum_n |c_n|^2 \, \delta(p - A_n) = \sum_n |c_n|^2 = 1 \, .$$

□

Now, a question arises: what can be done about a function which is non-negative and summable to 1? The great idea due to Andrei Kolmogorov is that it should be called a probability distribution. This idea by Kolmogorov introduces clarity to the subject, which, in despite of great efforts by truly great thinkers such as Pascal, Laplace, Maxwell, Boltzmann or Gibbs, was plagued by circular arguments or metaphysics which baffled even such thinkers as Russel and Popper.

In what follows, we call the Fourier transform of the autocorrelation function $\langle \psi | \mathrm{e}^{-i\lambda A} | \psi \rangle$ the probability distribution for the observable A in the state $|\psi\rangle$.

As a simple application we shall derive the Born rule which is thus seen to be a simple theorem and not a postulate, which in the past generated a voluminous discussion in which an obscure metaphysics mixes up with an error and/or inconsistence. The proof is as follows. Let $\psi(x)$ be a square integrable function on the straight line $-\infty < x < \infty$ which describes a state and X be an observable which corresponds to the position on the line. We have

$$\begin{aligned}
&\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\lambda \, \mathrm{e}^{iq\lambda} \left\langle \psi \left| \mathrm{e}^{-i\lambda X} \right| \psi \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\lambda \, \mathrm{e}^{iq\lambda} \int_{-\infty}^{+\infty} \mathrm{d}x \; |\psi(x)|^2 \, \mathrm{e}^{-i\lambda x} \\
&= \int_{-\infty}^{+\infty} \mathrm{d}x \; |\psi(x)|^2 \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\lambda \, \mathrm{e}^{i\lambda(q-x)} \right) = \int_{-\infty}^{+\infty} \mathrm{d}x \; |\psi(x)|^2 \, \delta(q - x) \\
&= |\psi(q)|^2
\end{aligned}$$

which means that the probability of finding the particle in q is $|\psi(q)|^2$.

3. The probability density for the first Casimir operator C_1 in the quantum Coulomb field $|u\rangle$, continued

In accordance with what is above, for $0 < z = \mathfrak{e}^2/\pi < 1$,

$$\begin{aligned}
&(1 - z)\mathrm{e}^z \delta \left[p - z(2 - z) \right] \\
&- \Theta(p - 1) \frac{z^2 \mathrm{e}^z}{\pi} \sum_{n=-\infty}^{\infty} \frac{\left(\sqrt{p - 1} + i(2n + 1 - z) \right)^{n-1}}{\left(\sqrt{p - 1} + i(2n + 1 + z) \right)^{n+2}}
\end{aligned}$$

is the probability distribution for the first Casimir operator C_1 in the quantum Coulomb field $|u\rangle$. Professor Wosiek says that numerically this function seems indeed to be non-negative. But we lack a general proof, mainly due to the difficulty mentioned at the beginning of Section 1. However, for small z which is the case experimentally where $z = \mathfrak{e}^2/\pi \approx 0.0023$, we can argue as follows: we can neglect z under the sum and obtain

$$(1-z)\mathrm{e}^z\delta\left[p-z(2-z)\right] - \Theta\left(p-1\right)\frac{z^2\mathrm{e}^z}{\pi}\sum_{n=-\infty}^{\infty}\left(\sqrt{p-1}+i(2n+1)\right)^{-3}$$
$$= (1-z)\mathrm{e}^z\delta\left[p-z(2-z)\right] + \Theta\left(p-1\right)\frac{\pi^2z^2\mathrm{e}^z}{8}\frac{\sinh\left(\frac{\pi}{2}\sqrt{p-1}\right)}{\cosh^3\left(\frac{\pi}{2}\sqrt{p-1}\right)}$$

which is seen to be everywhere positive by inspection. Summability to 1 is also simple to see, one has only to remember that in the contribution from the bound state one has to keep only the terms up to z^2 in z.

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REFERENCES

[1] A. Staruszkiewicz, «A new proof of existence of a bound state in the quantum Coulomb field III», *Acta Phys. Pol. B* **40**, 3771 (2009).

[2] V.I. Arnold, «Mathematical methods of classical mechanics», *Springer*, New York, NY 1989, https://www.springer.com/gp/book/9780387968902

[3] L.D. Landau, E.M. Lifshitz, «Mechanics», *Pergamon Press, Elsevier*, 1982.

[4] W. Kopczyński, A. Trautman, «Spacetime and gravitation, John Wiley & Sons», 1992, Warszawa: Polish Scientific Publishers (PWN), 1992.

[5] R. Penrose, «The emperor's new mind», *Oxford University Press*, 1989.

[6] P. Bongaarts, «Quantum Theory, a mathematical approach», *Springer International Publishing*, Switzerland 2015, https://www.springer.com/gp/book/9783319095608