

NESTED SOFT–COLLINEAR SUBTRACTIONS IN NNLO QCD COMPUTATIONS*

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Thanks to a remarkable progress in understanding of how to perform NNLO QCD computations, many processes at hadron collisions have been recently computed to that precision. Despite these developments, the search for the optimal subtraction scheme that allows us to handle infrared and collinear singularities in an efficient and general way is still ongoing. In the following, I will review the nested soft–collinear subtraction scheme proposed in F. Caola, K. Melnikov, R. Rötsch, *Eur. Phys. J. C* **77**, 248 (2017). This scheme seems to possess many features of the possible optional scheme; for example, it is analytic, fully local and highly modular. I will describe an application of this scheme to the description of deep inelastic scattering of a proton with an electron (DIS) that, together with results on colour singlet production and decay, completes the set of building blocks that are required for the application of this scheme to arbitrary processes at hadron colliders.

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1. Introduction

A differential partonic cross section at NNLO QCD can be written as

$$d\hat{\sigma}_{\text{nnlo}} = d\hat{\sigma}_{\text{vv}} + d\hat{\sigma}_{\text{rv}} + d\hat{\sigma}_{\text{rr}} + d\hat{\sigma}_{\text{pdf}}, \quad (1)$$

where $d\hat{\sigma}_{\text{vv}}$ describes the two-loop corrected hard process, $d\hat{\sigma}_{\text{rv}}$ describes a one-loop corrected single-real emission, $d\hat{\sigma}_{\text{rr}}$ describes a double-real emissions, and $d\hat{\sigma}_{\text{pdf}}$ originates from the collinear renormalization of the parton distribution functions.

Virtual corrections, present in contributions $d\hat{\sigma}_{\text{vv}}$ and $d\hat{\sigma}_{\text{rv}}$, contain *explicit* poles in the dimensional regularization parameter $\epsilon = (d-4)/2$ that is known to be independent of the hard matrix element [1, 2]. The same holds

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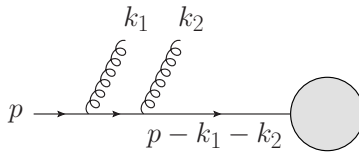
true for collinear renormalization contributions $d\hat{\sigma}_{\text{pdf}}$. In contrast to this, real corrections, present in contributions $d\hat{\sigma}_{\text{rv}}$ and $d\hat{\sigma}_{\text{rr}}$, contain singularities that become poles in $1/\epsilon$ *only* upon phase-space integration. We need to extract these poles without integrating over kinematic features of resolved final-state particles since our goal is to re-write fully-differential cross sections in such a way that computation of arbitrary infrared safe observables becomes possible. This can be achieved using subtraction and slicing methods [3–18]. They allow us to extract poles that originate from real emission contributions *without* integrating over the resolved phase space and thus to keep the cross section fully differential. Moreover, as we will show in what follows, the structure of $1/\epsilon$ poles follows from factorization properties of QCD matrix elements so that it is possible to demonstrate their cancellation for a *generic process*.

2. Singularities of real emissions

Singularities of real emission QCD amplitudes come in two varieties: infrared (soft) singularities which appear when the energy of emitted partons vanishes and collinear singularities which appear when partons are emitted in the same direction as another final-state parton. At NNLO, we need to consider up to two additional partonic emissions relative to the Born process. The corresponding singular double-soft and triple-collinear limits of the amplitudes are known to be independent of the hard process [19]. For instance, the double-soft limit of the matrix element for a process at NNLO QCD with two additional gluon emissions with momenta k_1 and k_2 has the singular structure¹

$$|M(\{p\}, k_1, k_2)|^2 \underset{k_1 \sim k_2 \rightarrow 0}{\approx} \text{Eik}(\{p\}, k_1, k_2) \times |M(\{p\})|^2, \quad (2)$$

where $\text{Eik}(\{p\}, k_1, k_2)$ is a generic function that contains the singularities and that *factorizes* from the matrix element $|M(\{p\})|^2$ that describes the hard process. However, propagators of individual diagrams suggest that there are singular limits beyond purely soft and collinear ones. For instance, looking at the diagram



$$\sim \frac{1}{2p \cdot k_1 + 2p \cdot k_2 - 2k_1 \cdot k_2} \xrightarrow[k_2 \rightarrow 0]{k_1 \parallel p} \infty, \quad (3)$$

¹ For simplicity, ignoring colour correlations.

we observe that an entangled singularity develops if one gluon becomes soft and the other becomes collinear. For a given amplitude, it can be checked explicitly that this entangled limit disappears and that remaining soft–collinear limits can be described by taking soft and collinear limits independently. In Ref. [20], it was pointed out that this result is general thanks to the phenomenon of *colour coherence*. As a result, the known soft and collinear limits [19, 21, 22] *must be* sufficient to describe and regulate all singularities in NNLO QCD scattering amplitudes with real emissions.

Therefore, it is useful to study a subtraction scheme for simple processes with only two external colour charged particles. In the case of the nested soft–collinear subtraction scheme, this was done for colour singlet production [23], colour singlet decay [24], and for DIS [25] covering all possible kinematic configurations with two external QCD partons.

3. Deep inelastic scattering

We will now discuss the construction of the nested soft–collinear subtraction scheme using DIS as an example. Since final-state quarks do not develop soft singularities, we only consider the partonic channel

$$q(p_1) + e^-(p_2) \rightarrow e^-(p_3) + q(p_4) + g(p_5) + g(p_6), \quad (4)$$

which possesses the most complex singular structure. For this channel, we define²

$$2s \, d\hat{\sigma}_{\text{rr}} = \int [dg_5][dg_6] \, \theta(E_5 - E_6) \, F_{\text{LM}}(1, 4, 5, 6) \equiv \langle F_{\text{LM}}(1, 4, 5, 6) \rangle, \quad (5)$$

where³

$$\begin{aligned} F_{\text{LM}}(1, 4, 5, 6) = & \mathcal{N} \int d\text{Lips}(\{p\}) \, (2\pi)^d \delta^{(d)} \left(p_1 + p_2 - \sum_{i=3}^6 p_i \right) \\ & \times |M^{\text{tree}}(\{p\}, p_5, p_6)|^2 \times \hat{\mathcal{O}}(p_3, p_4, p_5, p_6), \end{aligned} \quad (6)$$

and

$$[dg_i] = \frac{d^{d-1}p_i}{(2\pi)^{d-1}2E_i} \, \theta(E_{\text{max}} - E_i), \quad (7)$$

is the phase-space volume of the parton i . E_{max} is a sufficiently large but otherwise arbitrary⁴ parameter that provides an upper bound on energies

² For simplicity, we do not show the momenta labels of the electrons in the F_{LM} function.

³ Now and in the following, we use $\{p\} = \{p_1, p_2, p_3, p_4\}$ as the set of hard momenta describing the hard process.

⁴ More specifically, E_{max} should be greater than or equal to the maximal energy that a final-state parton can have according to the momentum conservation constraint.

of individual partons; its role will become clear later. The two emitted partons are identical. Instead of averaging the amplitude, we found convenient to use this fact to order them in energy by introducing $\theta(E_5 - E_6)$ in Eq. (5). Thereby, the only single-soft singularity that needs to be regularized is $E_6 \rightarrow 0$ since $E_5 \rightarrow 0$ implies that both gluons $g(p_5)$ and $g(p_6)$ become soft. The factor $\mathcal{N}$ in Eq. (6) includes all the relevant symmetry factors, $\text{dLips}(\{p\})$ is the phase space of the hard process, M^{tree} is the matrix element and $\hat{\mathcal{O}}$ is an arbitrary infrared safe observable. We will proceed with the discussion of how infrared and collinear singularities can be extracted from the function $F_{\text{LM}}(1, 4, 5, 6)$ without integration over resolved phase space.

3.1. Soft singularities

We begin by regulating the double-soft singularity. To this end, we introduce an operator $\mathbb{S}$ that extracts the leading singularity by acting on the function $F_{\text{LM}}(1, 4, 5, 6)$. Its action is defined as⁵

$$\begin{aligned} &\langle \mathbb{S} F_{\text{LM}}(1, 4, 5, 6) \rangle \\ &= \int [\text{d}g_5][\text{d}g_6] \theta(E_5 - E_6) \text{Eik}(\{p_1, p_4\}, p_5, p_6) \times \langle F_{\text{LM}}(1, 4) \rangle, \end{aligned} \quad (8)$$

where $\langle F_{\text{LM}}(1, 4) \rangle$ is the fully-differential cross section of the hard process. The soft gluons factorize from the matrix element [19], *cf.* Eq. (2), the infrared safe observable and the energy-momentum conserving δ function. We insert the identity operator $I = [I - \mathbb{S}] + \mathbb{S}$ into the phase space and obtain

$$\langle F_{\text{LM}}(1, 4, 5, 6) \rangle = \langle [I - \mathbb{S}] F_{\text{LM}}(1, 4, 5, 6) \rangle + \langle \mathbb{S} F_{\text{LM}}(1, 4, 5, 6) \rangle. \quad (9)$$

In the first term on the right-hand side, the double-soft singularity is regulated. In the second term on the right-hand side (subtraction term), we require the fully-differential cross $F_{\text{LM}}(1, 4, 5, 6)$ in the double-soft limit Eq. (8). Since the gluons decouple entirely from the hard process, we can integrate analytically over the phase space of the two emitted gluons and the $1/\epsilon$ poles can be extracted explicitly [26] *independent* of the hard process. Since energies of soft gluons are no more bound by energy conservation, we introduced an explicit upper bound E_{max} on their energies, *cf.* Eqs. (5), (7)⁶. The double-soft regulated term still contains unregulated single-soft and collinear singularities. We will now discuss how to regularize them.

⁵ For simplicity, ignoring colour correlations. A comprehensive definition is given in [20].

⁶ Since the left-hand side of Eq. (9) is E_{max} -independent, the explicit E_{max} dependence in the analytic subtraction term needs to cancel with an implicit dependence in the regulated term; the possibility to vary this parameter provides a useful check on the implementation of the subtraction scheme.

We continue with the single-soft limit. Due to the energy ordering $\theta(E_5 > E_6)$, there is only one single-soft singularity $E_6 \rightarrow 0$. To regularize it, we introduce an operator S_6 [26] that extracts the leading single-soft singularity and insert the identity operator $I = [I - S_6] + S_6$ into the phase space. We obtain

$$\begin{aligned} & \langle [I - S] F_{\text{LM}}(1, 4, 5, 6) \rangle \\ &= \langle [I - S_6] [I - S] F_{\text{LM}}(1, 4, 5, 6) \rangle + \langle S_6 [I - S] F_{\text{LM}}(1, 4, 5, 6) \rangle. \end{aligned} \quad (10)$$

The first term on the right-hand side is free of soft singularities. However, it still contains collinear singularities. In the subtraction term, gluon p_6 decouples from the function $F_{\text{LM}}(1, 4, 5, 6)$. Hence, we can again integrate analytically over its phase space and extract the $1/\epsilon$ poles *independent* of the hard process. What remains reduces to an NLO correction to DIS whose remaining singularities are treated in the usual FKS approach [27, 28] for NLO QCD computations.

3.2. Collinear singularities

In the collinear limits, many different singular configurations exist. However, working in the physical gauge, collinear singularities factorize on external legs and we can decompose the matrix element squared as

$$\begin{aligned} & |M^{\text{tree}}(\{p\}, p_5, p_6)|^2 \\ &= \left| \begin{array}{c} 5 \quad 6 \\ \diagup \quad \diagdown \\ \text{---} \bullet \text{---} \bullet \text{---} \\ | \quad | \\ 1 \quad 4 \end{array} \right|^2 + \left| \begin{array}{c} 5 \quad 6 \\ \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \bullet \text{---} \\ | \quad | \\ 1 \quad 4 \end{array} \right|^2 + \left| \begin{array}{c} 5 \quad 6 \\ \text{---} \bullet \text{---} \bullet \text{---} \\ | \quad | \\ 1 \quad 4 \end{array} \right|^2 + \left| \begin{array}{c} 6 \quad 5 \\ \text{---} \bullet \text{---} \bullet \text{---} \\ | \quad | \\ 1 \quad 4 \end{array} \right|^2 \\ &+ (\text{finite in any collinear limit}), \end{aligned} \quad (11)$$

where either three definite partons become collinear (first two) or two pairs of partons become collinear at once (last two). To control which partons these are, we follow the FKS approach [27, 28] and its NNLO extension [4], and introduce partition functions

$$1 = w^{51,61} + w^{54,64} + w^{51,64} + w^{54,61} \quad (12)$$

into the first term on the right-hand side of Eq. (10). The partition functions $w^{5i,6j}$ are designed to dampen all but a few collinear singularities⁷

$$\lim_{5 \parallel i} w^{5j,6k} \sim \delta_{ij}, \quad \lim_{5 \parallel i} w^{5j,6k} \sim \delta_{ik} \quad \text{for } i, j, k \in \{1, 4\}, \quad (13)$$

and, therefore, project on the different singular contributions on the right-hand side of Eq. (11).

⁷ Explicit formulas for our choice of the partition functions $w^{5i,6j}$ and detailed properties can be found in Ref. [25].

The last two singular contributions on the right-hand side of Eq. (11) that correspond to the double-collinear partitions $w^{51,64}$ and $w^{54,61}$ contain collinear singularities that are effectively NLO-like. However, in the first two that describe triple-collinear partitions $w^{51,61}$ and $w^{54,64}$ still different kinematic configurations are present. For example, looking at the partition $w^{51,61}$

$$\left| \begin{array}{c} 5 \quad 6 \\ \diagup \quad \diagdown \\ 1 \rightarrow \text{blob} \end{array} \right|^2 = \left| \begin{array}{c} 5 \quad 6 \\ \diagup \quad \diagdown \\ 1 \rightarrow \text{blob} \end{array} \right|^2 + \left| \begin{array}{c} 6 \quad 5 \\ \diagup \quad \diagdown \\ 1 \rightarrow \text{blob} \end{array} \right|^2 + \left| \begin{array}{c} 6 \quad 5 \\ \diagdown \quad \diagup \\ 1 \rightarrow \text{blob} \end{array} \right|^2, \quad (14)$$

we find contributions to the amplitude squared that are still singular when $(p_5 \parallel p_1)$, $(p_6 \parallel p_1)$ and $(p_5 \parallel p_6 \parallel p_1)$. They are isolated in the phase space and can be separated by splitting the angular phase space of the two emissions into different regions. We, therefore, introduce yet another partition of unity

$$\begin{aligned} 1 &= \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) \\ &\quad + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right) \\ &\equiv \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}, \end{aligned} \quad (15)$$

with $\eta_{ij} = (1 - \cos \theta_{ij})/2$, where θ_{ij} is the angle between the directions of particle i and j . In each partition and sector, the structure of collinear singularities is now fully defined and it is straightforward to write down the fully-regulated double real contribution. As an example, we consider the triple-collinear partition $w^{51,61}$ and the sector $\theta^{(a)}$. By construction, there are two collinear singularities: a double-collinear when $(p_6 \parallel p_1)$ and a triple-collinear when $(p_5 \parallel p_6 \parallel p_1)$. Introducing operators $\mathbb{C}_1$ and C_{61} [26] that extract the corresponding leading triple-collinear and double-collinear singularities, and inserting them iteratively into the phase space, we obtain

$$\begin{aligned} &\left\langle [I - S_6] [I - \mathbb{S}] w^{51,61} \theta^{(a)} F_{\text{LM}}(1, 4, 5, 6) \right\rangle \\ &= \left\langle [I - C_{61}] [I - \mathbb{C}_1] [I - S_6] [I - \mathbb{S}] w^{51,61} \theta^{(a)} F_{\text{LM}}(1, 4, 5, 6) \right\rangle \\ &\quad + \left\langle C_{61} [I - \mathbb{C}_1] [I - S_6] [I - \mathbb{S}] w^{51,61} \theta^{(a)} F_{\text{LM}}(1, 4, 5, 6) \right\rangle \\ &\quad + \left\langle \mathbb{C}_1 [I - S_6] [I - \mathbb{S}] w^{51,61} \theta^{(a)} F_{\text{LM}}(1, 4, 5, 6) \right\rangle. \end{aligned} \quad (16)$$

In partition $w^{51,61}$ and sector $\theta^{(a)}$, all singularities are now regulated. We proceed in the same way for the remaining partitions and sectors.

3.3. Fully-regulated differential cross section

To write down a formula for the fully-regulated differential cross section, we need to introduce additional operators that extract various soft and collinear singularities. The complete list of such operators is presented below. It includes the soft operators

$$\begin{aligned} \mathbb{S} \quad & \text{Double-soft : } E_5, E_6 \rightarrow 0, \\ S_6 \quad & \text{Single-soft : } E_6 \rightarrow 0, \end{aligned} \tag{17}$$

and the collinear operators⁸

$$\begin{aligned} \mathbb{C}_i \quad & \text{Triple-collinear : } p_5 \parallel p_6 \parallel p_i, \\ C_{5i}, C_{6i} \quad & \text{Double-collinear : } p_5 \parallel p_i, p_6 \parallel p_i, \\ C_{56} \quad & \text{Double-collinear : } p_5 \parallel p_6, \end{aligned} \tag{18}$$

with $i, j \in \{1, 4\}$. Using these operators, the fully-regulated contribution for the double-real emission is written as

$$\begin{aligned} & 2s \, d\hat{\sigma}_{\text{rr}} \\ &= \sum_{\substack{i,j=1,4 \\ i \neq j}} \langle [I - \mathbb{S}] [I - S_6] [I - C_{6j}] [I - C_{5i}] [\text{dg}_5] [\text{dg}_6] w^{i5,j6} F_{\text{LM}}(1, 4, 5, 6) \rangle \\ &+ \sum_{i=1,4} \left\langle [I - \mathbb{S}] [I - S_6] \left[\theta^{(a)} [I - \mathbb{C}_i] [I - C_{6i}] + \theta^{(b)} [I - \mathbb{C}_i] [I - C_{56}] \right. \right. \\ &\quad \left. \left. + \theta^{(c)} [I - \mathbb{C}_i] [I - C_{5i}] + \theta^{(d)} [I - \mathbb{C}_i] [I - C_{56}] \right] \right. \\ &\quad \left. \times [\text{dg}_5] [\text{dg}_6] w^{i5,i6} F_{\text{LM}}(1, 4, 5, 6) \right\rangle. \end{aligned} \tag{19}$$

Details about the analytic integration of the subtraction terms, including explicit formulas for the $1/\epsilon$ poles, can be found in Ref. [25] and references therein. After combining with remaining contributions Eq. (1) (and channels), all $1/\epsilon$ poles cancel and we arrive at a finite formula that can be used to compute arbitrary infrared safe observables in $d = 4$ dimensions numerically.

4. Conclusion and outlook

We have analytically computed all the subtraction terms for NNLO QCD corrections to deep inelastic scattering within the nested soft–collinear subtraction scheme in Ref. [25]. We implemented the fully-differential cross

⁸ A complete definition of the action of this operators on the cross section and phase space can be found in Ref. [25].

section in a computer code that allows us to compute arbitrary observables with NNLO QCD precision. We used it to validate the analytic formulas for the subtraction terms through numerical checks against predictions obtained from a direct integration of analytic DIS coefficient functions [7, 29–33]. We found that our formalism performed well and allowed us to check individual NNLO *coefficients* to a few permille precision. In general, we found that we obtained permille precision on the NNLO total cross section, corresponding to a few percent precision on the NNLO coefficient, already after running for a few hours on an 8-core machine.

The derived analytic results for NNLO QCD corrections to deep inelastic scattering allow us to extend the nested soft–collinear subtraction scheme to processes involving partons both in the initial and in the final state. We are currently using these results as building blocks to design subtractions for more complicated LHC processes.

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