ON THE UNIVERSALITY OF THE KRK FACTORIZATION SCHEME* **

Stanisław Jadach

Institute of Nuclear Physics Polish Academy of Sciences
Radzikowskiego 152, 31-342 Kraków, Poland

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The factorization scheme (FS) abbreviated as KRK FS including a new definition of the PDFs for initial hadrons was formulated while developing KrkNLO scheme of matching QCD NLO corrections for the hard process with the parton shower heavy-boson production in hadron–hadron collision and for deep inelastic lepton–hadron scattering. KRK FS (originally called Monte Carlo FS) can be regarded as a variant of the $\overline{\mathrm{MS}}$ system. It is, therefore, trivially universal, that is process-independent. The question of its universality is formulated differently: As the basic role of KRK FS is to drastically simplify NLO corrections, the question is now whether the same *single* variant of PDFs in the KRK FS is able to achieve the same maximal simplification of the NLO corrections for *all processes* with one or two initial hadrons and any number of the final hadrons? Our answer is positive and the proof is elaborated in the present note within the Catani–Seymour subtraction methodology. KRK FS is mandatory in the KrkNLO method of matching NLO calculation and parton shower — a much simpler alternative of POWHEG and/or MC@NLO. However, the use of KRK FS and the corresponding PDFs simplifies NLO calculations for any other method of calculating NLO corrections and for arbitrary processes as well.

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1. Introduction

The first idea of the KRK factorization scheme (KRK FS) of the KrkNLO method of upgrading hard process of the parton shower Monte Carlo (MC) to NLO level was formulated for the Drell–Yan (DY) process in Ref. [1]. Later on, in Ref. [2], the KrkNLO method was elaborated in a quite detail

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for the DY and the deep inelastic *ep* scattering (DIS) processes with parton distribution functions (PDFs) defined in the KRK FS. The first practical implementation of KrkNLO methodology for the DY process on top of Sherpa and Herwig parton shower MCs was presented in Ref. [3], including comparisons with the NLO and NNLO fixed order calculations, and also comparing with the calculation in the MC@NLO [4] and POWHEG [5] matching schemes.

Later on, in Refs. [6], the use of PDFs in the KRK factorization scheme was formulated for the DY and Higgs production processes and, finally, applied for the MC simulations of the Higgs boson production at the LHC within the KrkNLO method in Ref. [7].

Universality of PDFs (process independence) is of paramount practical importance, because it allows to determine them in one process (typically DIS) and then use them as an input in order to obtain precise theoretical predictions in any other process, with one or two incoming hadrons. PDFs in the $\overline{\mathrm{MS}}$ scheme are universal, as we know both from experimental tests and also from theoretical arguments.

In most the above-mentioned works, PDFs in the KRK FS were defined in the context of the DY-like processes such as Z-boson or Higgs-boson production in the *pp* colliders, sometimes also for the DIS process. Hence, the question of the universality (process independence) of PDFs in the KRK FS was not a burning issue but was waiting for answer. In the present note, we are going to argue that one can answer this question in a systematic way within the framework of the Catani–Seymour subtraction scheme [8] of NLO calculations for any scattering process with any number of leptons and coloured partons in the initial and final state.

A master formula for NLO calculation for m partons within the Catani–Seymour (CS) scheme [8] reads schematically as follows:

$$\begin{aligned}
&\sigma^{\mathrm{NLO}}(p)=\sigma^{\mathrm{B}}(p)\\
&+\int\limits_m \left[\mathrm{d}\sigma^{\mathrm{V}}(p)+\mathrm{d}\sigma^{\mathrm{B}}(p)\otimes\ \boldsymbol{I}\right]_{\varepsilon=0}+\int \mathrm{d}z\int\limits_m \left[\mathrm{d}\sigma^{\mathrm{B}}(zp)\otimes\ (\mathbf{P}+\mathbf{K})(z)\right]_{\varepsilon=0}\\
&+\int\limits_{m+1}\left[\mathrm{d}\sigma^{\mathrm{R}}(p)_{\varepsilon=0}-\left(\sum_{\mathrm{dipoles}}\mathrm{d}\sigma^{\mathrm{B}}(p)\otimes\ \mathrm{d}V_{\mathrm{dipole}}\right)_{\varepsilon=0}\right],
\end{aligned}\tag{1.1}$$

where p stands for an initial parton(s) embedded in PDF(s), symbol $\otimes$ denotes phase-space convolution, colour and spin summations. The counterterm $\mathrm{d}\sigma^{\mathrm{B}}(p)\otimes\mathrm{d}V_{\mathrm{dipole}}$ defined in $m+1$-particle phase space encapsulates all soft and collinear singularities — it is added and subtracted. Thanks to clever kinematic mapping, it factorizes off and is integrable analytically in $d=4+2\varepsilon$ dimensions, $\boldsymbol{I}=\sum_{\mathrm{dipoles}}\int_1\mathrm{d}V_{\mathrm{dipole}}$ over the entire NLO phase space.

In Refs. [2, 6], it was shown that thanks to transformation of PDFs from $\overline{\mathrm{MS}}$ to MC FS, one can get rid of the annoying third term in Eq. (1.1) with $(\mathbf{P}+\mathbf{K})$ matrix for the DY-type process and DIS process. The eliminated term collects technical artifacts of the dimensional regularization (collinear remnants), which can be regarded as *unphysical*. The resulting NLO formula reads as follows:

$$\sigma^{\mathrm{NLO}}(p) \;=\; \sigma^{\mathrm{B}}(p) + \int\limits_{m} \left[\mathrm{d}\sigma^{\mathrm{V}}(p) + \mathrm{d}\sigma^{\mathrm{B}}(p)\; I(\varepsilon)\right]_{\varepsilon=0} + \int\limits_{m+1} \left[\mathrm{d}\sigma^{\mathrm{R}}(p)_{\varepsilon=0} - \left(\sum_{\mathrm{dipoles}} \mathrm{d}\sigma^{\mathrm{B}}(p) \otimes \mathrm{d}V_{\mathrm{dipole}}\right)_{\varepsilon=0}\right] . \tag{1.2}$$

The KrkNLO method of matching NLO calculation with PS MC relies vitally on the validity of the above simplified formula.

The question addressed in the following will be at the two levels: Is the above simplification restricted to processes with only two coloured legs, such as heavy-boson(s) production in pp collision or ep scattering? Or can it be achieved for any process with arbitrary number of coloured legs? In case the simplification is feasible for any process, then the second question is: is this *the same* set of PDFs in new KRK FS, which provides for the simplification of Eq. (1.2) for any process, without the need of adjusting the definition of PDFs in the KRK FS process by process? Full universality of the PDFs in the KRK FS requires a positive answer to both the above questions.

Let us illustrate the main points of the proposed factorization scheme and explain its role in the KrkNLO method using examples of the production of any heavy boson such as Z, γ, W, H in quark–antiquark annihilation with kinematics depicted in Fig. 1. For the sake of simplicity, let us focus on the gluonstrahlung subprocesses, *i.e.* $a = q,\; b = \bar{q},\; k = G, c = q$ in Fig. 1. We are going to show why KRK FS is mandatory for KrkNLO scheme and what is the relation between CS dipoles and transformation between PDFs in KRK and $\overline{\mathrm{MS}}$ schemes.

In the KrkNLO matching, the NLO corrected differential cross section in the CS subtraction scheme is compared (matched) with the same distribution in the parton shower with NLO corrected hard process. Identifying and matching the same elements in both distributions can only be successful if both of them are brought to the same form. Following closely Ref. [3], let us compare both distributions in the formulation without any resummation (always present in the parton shower) and with subtraction like it is in the final CS formula in $d = 4$ dimensions.

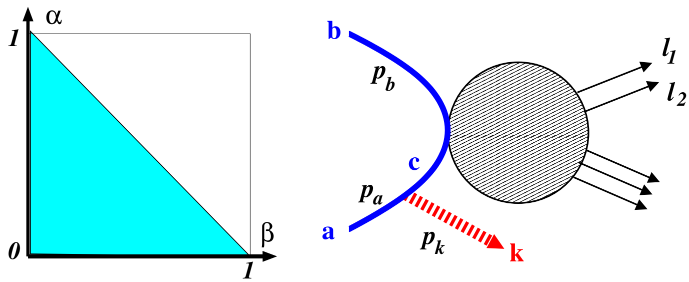

Fig. 1. Kinematics and Sudakov plane for DY-like processes.

The final formula for the NLO cross section with CS dipole subtractions in $d=4$ dimensions reads in the notation of Ref. [3][1] as follows:

$$\begin{aligned}
\sigma_{\rm NLO}^{\overline{\rm MS}} \;=\; & \int {\rm d}x_{\rm F}{\rm d}x_{\rm B}{\rm d}z\; {\rm d}x\;\delta_{x=zx_{\rm F}x_{\rm B}} \Big\{ \Big[\delta_{1=z}(1+\Delta_{\rm VS}) \\
& +2\frac{\alpha_{\rm s}}{2\pi} P_{qq}(z) \ln\frac{\hat{s}}{\mu_{\rm F}^2} + \Sigma_q(z) \Big] {\rm d}\sigma_0\left(szx,\hat{\theta}\right) J_{\rm LO} \\
& + \left({\rm d}^5\sigma_1^{\rm NLO}\,(sx,\alpha,\beta,\Omega) J_{\rm NLO} - \left({\rm d}^5\sigma_1^{\rm F} + {\rm d}^5\sigma_1^{\rm B}\right) J_{\rm LO} \right) \delta_{1-z=\alpha+\beta} \Big\} \\
& \times f_q^{\overline{\rm MS}}(sx,x_{\rm F}) f_{\bar{q}}^{\overline{\rm MS}}(sx,x_{\rm B})\,, \qquad (1.3)
\end{aligned}$$

where $J_{\rm NLO} \equiv J(x_{\rm F},x_{\rm B},z,k_1^{\rm T})$ and $J_{\rm LO} \equiv J(x_{\rm F},x_{\rm B},1,0)$ are explicit experimental event selection functions. Two CS dipoles with initial-state emitter and initial-state spectator are[2]

$$ {\rm d}^5\sigma_{q\bar{q}}^{\rm F} = {\rm d}^5\sigma_{q\bar{q}}^{\rm LO} \frac{\alpha}{\alpha+\beta}\,, \qquad {\rm d}^5\sigma_{q\bar{q}}^{\rm B} = {\rm d}^5\sigma_{q\bar{q}}^{\rm LO} \frac{\beta}{\alpha+\beta}\,, \qquad (1.4)$$

where

$$ {\rm d}^5\sigma_{q\bar{q}}^{\rm LO}(sx,\alpha,\beta,\Omega) = \frac{C_F\alpha_{\rm s}}{\pi}\,\frac{{\rm d}\alpha{\rm d}\beta}{\alpha\beta}\,\frac{{\rm d}\varphi}{2\pi}{\rm d}\Omega\, \frac{1+(1-\alpha-\beta)^2}{2}\frac{{\rm d}\sigma_0}{{\rm d}\Omega}\left(sx,\hat{\theta}\right). \qquad (1.5)$$

Finally, the NLO 1-real gluon emission distribution ${\rm d}^5\sigma_1^{\rm NLO}$ is that of Eq. (3.3) in Ref. [3] and $\Sigma_q(z)$, see Eq. (B.5) therein, reads

$$ 2\Sigma_q(z) = \frac{2C_F\alpha_{\rm s}}{\pi} \left\{ \frac{1+z^2}{2(1-z)} \ln\frac{(1-z)^2}{z} + \frac{1+z^2}{2(1-z)}\ln\frac{\hat{s}}{\mu^2} + \frac{1-z}{2} \right\}_+ . \qquad (1.6)$$

[1] See formula of Eq. (B.7) in the notation introduced in Eqs. (3.1)–(3.7) in Ref. [3].

[2] These are $d=4$ versions. It is essential to define CS dipoles in $d=4+2\varepsilon$ as well.

In the KrkNLO method, upgrade of the hard process to NLO level is done by means of reweighting each MC event of the parton shower (PS) with the single finite positive correcting weight

$$W^{(1)}_{\rm NLO}(k_1)\,,$$

where k_1 is momentum of gluon with the highest transverse momentum $k_{\rm T}$, even if the PS is actually not based on the $k_{\rm T}$ ordering algorithm. The actual form of $W^{(1)}_{\rm NLO}(k_1)$ will result from the matching procedure. Bringing NLO corrected parton shower distribution to exactly the same analytical formula as in Eq. (1.4) is a quite nontrivial task. It was done quite carefully and explicitly in Section 3.4 in Ref. [3]. The resulting formula, see Eq. (3.39) in Ref. [3], reads as follows:

$$\begin{aligned}\sigma^{\overline{\rm MS}}_{\rm NLO} \;=\; & \int {\rm d}x_{\rm F}{\rm d}x_{\rm B}{\rm d}z\;{\rm d}x\;\delta_{x=zx_{\rm F}x_{\rm B}}\left\{\delta_{1=z}W^{(1)}_{\rm NLO}\Big|_{k_1=0}\,{\rm d}\sigma_0\left(szx,\hat\theta\right)J_{\rm LO}\right.\\ & \left.+\left(W^{(1)}_{\rm NLO}\left(sx,\alpha,\beta,\varOmega\right)J_{\rm NLO}-J_{\rm LO}\right)\left({\rm d}^3\rho^{\rm F}_1+{\rm d}^3\rho^{\rm B}_1\right)\delta_{1-z=\alpha+\beta}\right\}\\ & \times f^{\sf KRK}_q\left(sx,x_{\rm F}\right)f^{\sf KRK}_{\bar q}\left(sx,x_{\rm B}\right)\,. \end{aligned}\tag{1.7}$$

The matching between Eq. (1.3) and Eq. (1.7) results in fixing the form of the MC correcting weight

$$W^{(1)}_{\rm NLO}(k_1)=(1+\varDelta_{\rm VS})\,\frac{{\rm d}^5\sigma^{\rm NLO}_1\left(sx,\alpha,\beta,\varOmega\right)}{{\rm d}^5\sigma^{\rm F}_1+{\rm d}^5\sigma^{\rm B}_1}\,.\tag{1.8}$$

The same matching also provides the unambiguous relation between PDFs in the $\overline{\rm MS}$ and KRK. In the KRK scheme, the entire $\sim\delta(k^2_{1{\rm T}})\varSigma_q(z)$ is eliminated (modulo $\mathcal{O}(\alpha_{\rm s}^2)$ terms) thanks to the assignment $\hat s\equiv sx_{\rm F}x_{\rm B}=\mu^2$ and redefinition of the PDFs

$$f^{\sf KRK}_{q,\bar q}\left(\mu^2,x\right)=\int{\rm d}z{\rm d}x'\delta(x-zx')\left[\delta(1-z)+\varSigma_q(z)\right]_{\hat s=\mu^2}\,f^{\overline{\rm MS}}_{q,\bar q}\left(\mu^2,x'\right)\,.\tag{1.9}$$

A few remarks are in order: The term similar to the $\varSigma_q(z)$ function is completely absent in distribution (1.7) for any kind of parton shower with the NLO corrected hard process. In the KrkNLO method, it is absorbed in the redefined PDF. In other matching schemes such as MC@NLO [4] and POWHEG [9], this term is incorporated into PDFs by the "in-flight" transformation done on the PDFs *inside* the MC program during the event generation. In the KrkNLO method, the same transformation is performed on PDFs outside the MC program. Consequently, the process independence of the $\varSigma_q(z)$ function is very important for the KrkNLO method and not so important for the other matching methods[3]. In the above, it was assumed that LO MC was identical with the sum of two CS dipoles. In a more general case, the denominator of Eq. (1.8) is $\mathrm{d}^5\sigma^{\mathrm{LO}}_{q\bar{q}}$ generated in the PS MC (not necessarily equal to the sum of two CS dipoles). However, the finitness of $W^{(1)}_{\mathrm{NLO}}(k_1)$ requires that this $\mathrm{d}^5\sigma^{\mathrm{LO}}_{q\bar{q}}$ has exactly the same soft and collinear limits as the sum of two CS dipoles.

Having shown the critical role of the $\Sigma_q(z)$ function in the KrkNLO matching scheme, before analysing its process independence (universality), let us look more precisely where from it came in our particular DY case. It is born out from partial integration over the distribution of the sum of two CS dipoles in $d = 4+\varepsilon$ dimensions

$$\rho^{\mathrm{CS}}_{q\bar{q}\to V}(k_1,\epsilon) = \frac{2C_F\alpha_{\mathrm{s}}}{\pi}\frac{(4\pi)^{-\epsilon}}{\Gamma(1+\epsilon)}\left(\frac{s_1\alpha\beta}{\mu^2}\right)^{\epsilon}\frac{1+z^2+\varepsilon(1-z)^2}{2\alpha\beta}\frac{\mathrm{d}\sigma_0}{\mathrm{d}\Omega}(zs_1,\theta)\,, \tag{1.10}$$

where $z = 1-\alpha-\beta$. In the CS subtraction scheme, this distribution *is added* in the integrated form in $d = 4+\varepsilon$ dimensions to NLO virtual corrections and *subtracted* in $d = 4$ dimensions from the real NLO distributions. As it is well-known in the NLO the real+virtual distribution in the dimensional regularization remains uncancelled single pole term times LO kernel, which in our particular case is

$$2\Lambda^{\overline{\mathrm{MS}}}_{q\leftarrow q}(\varepsilon,z) = \frac{\alpha_{\mathrm{s}}}{\pi}\frac{(4\pi)^{-\varepsilon}}{\Gamma(1+\varepsilon)}\frac{1}{\varepsilon}C_F\frac{1+z^2+\varepsilon(1-z)^2}{1-z}\,. \tag{1.11}$$

In the $\overline{\mathrm{MS}}$ scheme, this kind of terms, soft collinear counterterms (SCTs), are simply subtracted[4]. It makes sense to combine CS dipoles with SCTs into a single object, which is upon (partial) phase-space integration in $d = 4+\varepsilon$ dimensions combined with standard virtual corrections. In our case, the above combination is

$$\begin{aligned}\mathcal{R}_q(z,\varepsilon) &= \int \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\Omega\,\delta_{1-z-\alpha-\beta}\,\rho^{\mathrm{CS}}_{q\bar{q}\to V}(k_1,\epsilon) - 2\Lambda^{\overline{\mathrm{MS}}}_{q\leftarrow q}(\varepsilon,z)\\ &= S_q(\varepsilon)\delta(1-z) + \Sigma_q(z)\,.\end{aligned} \tag{1.12}$$

The above explains clearly the origin of the $\Sigma_q(z)$ function in the final NLO result in the $\overline{\mathrm{MS}}$ scheme and its relation to the CS dipoles. The split between two parts of $\mathcal{R}_q(z,\varepsilon)$ is unambiguous due to the requirement that Σ_q-like part obeys momentum sum rule — so, in fact, there is a one-to-one

[3] However, keeping this transformation *outside* the MC makes sense, because "in-flight" transformation of PDFs complicates significantly MC program and also might be the source of the annoying negative MC weights.

[4] And are replaced by the PDFs in the $\overline{\mathrm{MS}}$ scheme.

correspondence between $S_q(\varepsilon)$ and $\Sigma_q(z)$ functions and CS dipoles. *Nota bene*, the cancellation of ε poles occurs entirely in one place, that is between $S_q(\varepsilon)$ and virtual loop corrections from the Feynman diagrams.

Let us stress again that the minimal requirements of the KrkNLO scheme to work is that single real parton emission distribution in $d = 4$ dimensions for the sum of CS dipoles on the one hand and for the same distribution of any modern LO PS on the another hand, has the same correct soft collinear limit. In view of that, in our quest for process independence of the **K**-matrix, *we are going to focus on the freedom in the choice of CS dipoles*, because it translates into the shape of the Σ-like functions and $\mathbb{K}$-matrix elements.

Generalising Eq. (1.12) to an arbitrary process, for each NLO splitting $K{\leftarrow}I,\ K, I = q, \bar{q}, G$ in the NLO process, the following component is present in the final CS NLO distributions:

$$
\begin{aligned}
\mathcal{R}_{K\leftarrow I}(z,\varepsilon) &= \int \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\Omega\, \delta_{1-z-\alpha-\beta} \sum_S \rho^S_{K\leftarrow I}(k_1,\epsilon) - \Lambda^{\overline{\mathrm{MS}}}_{K\leftarrow I}(\varepsilon,z) \\
&= S_{K\leftarrow I}(\varepsilon)\delta(1-z) + \Sigma_{K\leftarrow I}(z,\mu_{\mathrm{F}})\,, \\
\Lambda^{\overline{\mathrm{MS}}}_{K\leftarrow I}(\varepsilon,z) &= \frac{\alpha_{\mathrm{s}}}{\pi}\frac{(4\pi)^{-\varepsilon}}{\Gamma(1+\varepsilon)}\frac{1}{\varepsilon}P_{K\leftarrow I}(z,\varepsilon)\,,
\end{aligned}
\tag{1.13}
$$

where I is the emitter, K results from the splitting and S is the spectator[5].

Our reasoning will be now the following:

— First of all, the case when both I and K are in the final state ($\mathcal{FF}$) is for us uninterestingly trivial. The integration over dipole for fixed $z \neq 0$ gives $\Sigma_{K\leftarrow I}(z) = 0$. $S_{K\leftarrow I}(\varepsilon)$ gets combined with virtual corrections, such that CS dipoles do not need any modification.

— Then, the most important modification of the CS scheme is needed in the case of the final-state emitter I and initial-state spectator K ($\mathcal{FI}$)[6]. In the original CS scheme, $\Sigma_{K\leftarrow I}(z)$ gets convoluted with PDFs and the LO process, and the z integration cannot be separated. Clever modification of the kinematic mappings in these dipoles will make the z integration to decouple from PDFs and the LO process, as in the $\mathcal{FF}$ case.

— Next, we are left only with dipoles with the emitter I in the initial state and spectator S either in the initial or final state ($\mathcal{II}$ or $\mathcal{IF}$). We will modify CS dipoles such that $\Sigma_{K\leftarrow I}(z)$ is exactly the same in both cases.

[5] The S-dependent colour factor is temporarily omitted. We shall show that it cancels out due to colour conservation and spectator independence of the modified dipoles.

[6] This case is already present in the DIS process.

— Finally, $\Sigma_{K\leftarrow I}(z)$ depends also on the combination of $\ln(2p_I\, p_S/\mu_{\rm F}^2)$ with nontrivial colour coefficients. We are going to show how to choose $\mu_{\rm F}^2 = \hat{\mu}_{\rm F}^2$ in order to eliminate this component for an arbitrary process.

Once all the above is done, the transformation matrix for PDFs from $\overline{\rm MS}$ to KRK scheme is given by

$$\mathbb{K}_{K\leftarrow I}(z) = \Sigma_{K\leftarrow I}(z,\mu_{\rm F})|_{\mu_{\rm F}^2=\hat{\mu}_{\rm F}^2} \tag{1.14}$$

and is process-independent.

Finally, let us remind the reader that the physical meaning of $\Sigma_q(z)$ has been known since pioneering works of Alterelli *et al.* [10], where it was traced back to the difference between the upper phase-space limit (factorization scale) being the maximum transverse momentum in PDFs of the $\overline{\rm MS}$ and the total available energy in the real world of the hard process. Obviously, the PDFs of the KRK scheme represent the second, physical, case.

2. Dipoles with final-state emitter and initial-state spectator

It is natural to expect that in the $\mathcal{FI}$-type dipoles, with the final-state emitter and initial-stated spectator, the integration over dipole internal (Sudakov) variables decouples from the factorised LO differential cross section and PDFs, as it is the case of $\mathcal{FF}$-type dipoles with both emitter and spectator in the final state. However, it is not the case for the $\mathcal{FI}$-dipoles in the CS work [8]. This is the most sticky issue preventing universality of the $\mathbb{K}$ transformation, hence in the following, we are going to indicate how to solve this problem, while fine details will be presented in Ref. [11].

Figure 2 illustrates the kinematics of the $\mathcal{FI}$ dipole. The Sudakov variables for the dipole phase space are introduced as follows[7]:

$$\begin{aligned}
p_k &= \bar{\alpha}\, p_a + \bar{\beta}\, p_b + p_k^{\rm T}\,, \qquad \bar{\alpha} = \frac{p_k\cdot p_b}{p_a\cdot p_b}\,, \qquad \bar{\beta} = \frac{p_k\cdot p_a}{p_a\cdot p_b}\,,\\
\alpha &= \frac{\bar{\alpha}}{1+\bar{\beta}}\,, \qquad \beta = \frac{\bar{\beta}}{1+\bar{\beta}}\,, \qquad \max(\alpha,\beta)\le 1\,,\\
Q &= p_b + p_k - p_a\,, \qquad |Q^2| = 2p_a p_b\,, \qquad \frac{1-\alpha}{1-\beta}\,.
\end{aligned} \tag{2.1}$$

The corresponding differential cross section with clear factorization into the LO process and the dipole radiation parts reads[8]:

[7] This is parametrisation of the CS work [8]. However, it was known earlier, see Ref. [12].

[8] The colour correlation factor is omitted for the sake of simplicity.

$$\begin{aligned}
\mathrm{d}\sigma^a_{bk} &= \mathrm{d}\varPhi_{4+2\varepsilon}(p_k)\,\frac{1}{2p_b p_k} 8\pi\mu^{-2\varepsilon}\alpha_{\rm s} P^*_{b\leftarrow c}(\alpha,\beta)\,\frac{p_a\tilde{p}_b}{p_a(\tilde{p}_b-p_k)}\\
&\times\left\{\frac{1}{s}\mathrm{d}\varPhi\left(l_1+\tilde{p}_a;\tilde{p}_b,l_2,X\right)\,\left|\mathcal{M}\left(l_1,\tilde{p}_a;\tilde{p}_b,l_2,X\right)\right|^2\right\}_{d=4+2\varepsilon}\\
&=\frac{\alpha_{\rm s}}{2\pi}\,\left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon}\frac{1}{\varGamma(1+\varepsilon)}\frac{\mathrm{d}\varOmega^{n-3}\left(p^{\rm T}_k\right)}{\varOmega^{n-3}}\,H_{bc}(\alpha,\beta,\varepsilon)\left\{\mathrm{d}\sigma^{\rm LO}(l_1,\tilde{p}_a;\tilde{p}_b,l_2,X)\right\}\,,\\
H_{bc}(\alpha,\beta,\varepsilon) &= \left(\frac{\alpha\beta(1-\beta)}{(1-\alpha)}\right)^{\varepsilon}\,\frac{P^*_{b\leftarrow c}(\alpha,\beta,\varepsilon)}{\alpha}\,. \qquad (2.2)
\end{aligned}$$

The above distribution is defined in the entire NLO phase space $p_a + l_1 \to p_b + p_k + l_2 + X$. However, in the LO part $\{\ldots\}$, the momentum p_k is eliminated and effective momenta $\tilde{p}_a = (1-\alpha)p_a$, $\tilde{p}_b = Q - \tilde{p}_a$, $\tilde{p}_a^2 = \tilde{p}_b^2 = 0$ are used. We denote the 1-particle Lorentz invariant phase-space integration element as $\mathrm{d}\varPhi_{4+2\varepsilon}(p)$ and $\mathrm{d}\varPhi(l_1+\tilde{p}_a;\tilde{p}_b,l_2,X)$ is the multi-particle phase-space element. $P^*_{b\leftarrow c}(\alpha,\beta,\varepsilon)$ is an extrapolation of the spin factor of the splitting kernel over the entire Sudakov phase space, which has to coincide with the standard splitting kernel in the collinear limit. It will be defined in the next section. In the diagonal case $b=c$, it must exclude the initial-state $1/\beta$ singularity. Otherwise, it can be freely adjusted to our needs.

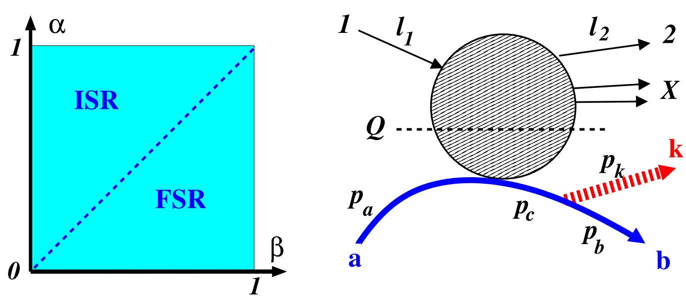

Fig. 2. Kinematics and Sudakov plane for $\mathcal{FI}$ dipole.

The above formula clearly illustrates the problem with the $\mathcal{FI}$ dipole, namely the effective centre-of-mass energy in the LO part $s' = 2l_1\cdot\tilde{p}_a = (1-\alpha)s$ depends on the Bjorken variable $z_B = 1-\alpha$. (It will also enter into the x argument of the PDF.)

Our alternative solution is that instead of the keeping z_B factor in the effective beam momentum $\tilde{p}_a$ of the LO part, it is just "boosted out". Let us explain how it works. A boost has a nice property of the Jacobian being equal to one. One may also profit from Lorentz invariance of the LO matrix element. In Fig. 2, particles are divided into two groups, the dipole part (a,b,k) and the LO rest (l_1,l_2,X). Two groups are connected by the spacelike exchange 4-momentum $Q = b+k-a = l_l - l_2 - X$. There is an entire family of the reference frames, in which $Q = (0, 0, |Q^2|^{1/2}, |Q^2|^{1/2})$ is pointing along z-axis and has a zero energy component. All these frames are connected with boosts in the x–y plane perpendicular to Q. Such a frame becomes uniquely defined (modulo azimuthal rotation) using an additional lightlike momentum, and requiring that it is along the z-axis. Two such frames are important, QMS_a with p_a along z-axis and QMS_1 with l_1 along minus z-axis.

Now, in the frame QMS_a, using the (a, b, k) subset, we construct the $\tilde{a}, \tilde{b}$ effective spectator and emitter. Then, we go to the QMS_1 frame (with l_1 along z-axis) and perform the active boost Λ in the x–y plane perpendicular to Q on the momenta of the $\tilde{a}, \tilde{b}$, such that[9]

$$2l_1 \cdot \Lambda\tilde{p}_a = s\,.$$

The momenta of the (b, l_k, X) are unchanged. Conservation of the 4-momenta

$$Q = \tilde{p}_b - \tilde{p}_a = \Lambda\tilde{p}_b - \Lambda\tilde{p}_a = l_l - l_2 - X$$

holds, because the Λ transformation does not change Q. The resulting momenta $\Lambda\tilde{p}_a, \Lambda\tilde{p}_b, l_l, l_2, X$ are now ready to be plugged into the LO matrix element. (Of course, one may finally transform them to the CMS.) The explicit dependence on α in the LO part of the factorization formula is removed! In the phase-space integration of Eq. (2.2), we introduce a change of the variables

$$l_1 = \Lambda l'_1\,, \qquad l_2 = \Lambda l'_2\,, \qquad X = \Lambda X'$$

and using phase-space invariance under Lorentz transformation Eq. (2.2) turns into

$$\begin{aligned} \mathrm{d}\sigma^a_{bk} &= \frac{\alpha_\mathrm{s}}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon}\frac{1}{\Gamma(1+\varepsilon)}\frac{\mathrm{d}\Omega^{n-3}\left(p^\mathrm{T}_k\right)}{\Omega^{n-3}} \\ &\quad\times H_{bc}(\alpha,\beta,\varepsilon)\left\{\mathrm{d}\sigma^\mathrm{LO}\left(l'_1,\tilde{p}_a;\tilde{p}_b,l'_2,X'\right)\right\}\,, \end{aligned} \tag{2.3}$$

where the condition $2l'_1 \cdot \tilde{p}_a = s = 2l_1 \cdot p_a$ holds, hence the dipole part decouples from the LO differential cross section and can be integrated over analytically, the same way as for $\mathcal{FF}$ dipole. Our goal is achieved.

The following remarks are in order: We were elaborating on the $\mathcal{FI}$ dipole distribution, which is added and subtracted in the NLO calculation, hence it does not change the NLO results. It is arbitrary to a certain degree and this freedom we have exploited. In the complete NLO differential cross section, the effective rescaling of the beam energy by the z_B factor is always present. What we have achieved is that this rescaling is entirely encapsulated in the $\mathcal{IF}$ dipole and completely absent in the $\mathcal{FI}$ dipole.

[9] Using a toy Monte Carlo exercise, it was checked that such a boost always exists.

3. Initial-state emitter and final state spectator

The kinematics of the dipole with the initial-state emitter and final-state spectator $\mathcal{IF}$ is the same as in Fig. 2 and Eq. (2.1) except that the splitting $a \to ck$ is now on the initial leg. Let us consider separately the diagonal splittings $a = b$ with gluon emission and nondiagonal splitting $a \neq b$, with the quark–gluon transition.

3.1. Diagonal splittings

The cases of diagonal splittings $a = b$, $a = q, G$ are special, because of the presence of the soft singularity in the form of the standard eikonal factor[10] $\frac{p_a p_b}{(p_k p_a)(p_k p_b)} \sim \frac{1}{\alpha\beta}$. In the CS technique, such a singularity is split into two parts using "soft partition functions" (SPFs) $m_+ + m_- = 1$, $m_\pm \geq 0$

$$\frac{1}{\alpha\beta} = \frac{1}{\alpha+\beta}\frac{1}{\beta} + \frac{1}{\alpha+\beta}\frac{1}{\alpha} = m_+(\alpha,\beta)\frac{1}{\alpha\beta} + m_-(\alpha,\beta)\frac{1}{\alpha\beta}\,.$$

The $m_+/(\alpha\beta)$ part of the eikonal factor is incorporated into the $\mathcal{IF}$ dipole and $m_-/(\alpha\beta)$ part into the $\mathcal{FI}$ dipole. SPFs are not unique and we are going to examine three choices[11]

$$m_+^{(a)}(\alpha,\beta) = \theta_{\beta<\alpha}\,, \qquad m_+^{(b)}(\alpha,\beta) = \frac{\alpha}{\alpha+\beta}\,, \qquad m_+^{(c)}(\alpha,\beta) = \frac{\alpha-\alpha\beta}{\alpha+\beta-\alpha\beta}\,. \tag{3.1}$$

The important point is that, because the $\mathcal{FI}$ dipole (thanks to kinematic mapping of the previous section) does not contribute to the Σ-function by means of manipulating SPFs, we may adjust the Σ-function from the diagonal $\mathcal{IF}$ dipole to be the same as from the $\mathcal{II}$ dipole (our ultimate goal!).

Since the $\mathcal{FI}$ and $\mathcal{IF}$ dipoles are strongly entangled through the $m_\pm$-functions, let us write common expression for both of them, similar to that of Eq. (2.2)

$$\begin{aligned}
\mathrm{d}\sigma_{ak}^{b\pm} &= \frac{\alpha_\mathrm{s}}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon}\frac{1}{\Gamma(1+\varepsilon)}\frac{\mathrm{d}\Omega^{n-3}\left(p_k^\mathrm{T}\right)}{\Omega^{n-3}} \\
&\quad\times H_{aa}^{\pm}(\alpha,\beta,\varepsilon)\left\{\mathrm{d}\sigma^{\mathrm{LO}}(l_1,\tilde{p}_a;\tilde{p}_b,l_2,X)\right\}, \\
H_{aa}^{\pm}(\alpha,\beta,\varepsilon) &= \left(\frac{\alpha\beta(1-\beta)}{(1-\alpha)}\right)^{\varepsilon}\frac{m_\pm(\alpha,\beta)\bar{P}_{a\leftarrow a}(z(\alpha,\beta),\varepsilon)}{\alpha\beta}\,,
\end{aligned} \tag{3.2}$$

where the spin numerators of the unregularised diagonal kernels are

[10] Omitting for simplicity colour structure.
[11] Here, we always use $m_- = 1 - m_+$.

$$\bar{P}_{qq}(z,\varepsilon) = (1-z)\hat{P}_{qq}(z,\varepsilon) = C_F\left[1+z^2+\varepsilon(1-z)^2\right], \tag{3.3}$$

$$\bar{P}_{GG}(z,\varepsilon) = (1-z)\hat{P}_{GG}(z,\varepsilon) = 2C_A\left(\frac{1}{z}-2(1-z)+z(1-z)^2\right), \tag{3.4}$$

and $z(\alpha,\beta)$ must obey the correct collinear limits: $z(\alpha,0)=1-\alpha$ and $z(0,\beta)=1-\beta$. In the present works (in the past as well), we consider three choices

$$z_A(\alpha,\beta) = 1-\max(\alpha,\beta), \quad z_B(\alpha,\beta) = 1-\alpha, \quad z_C(\alpha,\beta) = (1-\alpha)(1-\beta). \tag{3.5}$$

The upper kinematic limit of the dipole phase space $\max(\alpha,\beta)\le 1$ is always compatible with $z\le 1$.

In Eq. (3.2), it is always assumed that in the $\mathcal{FI}$ case, the mapping $l_1\to l_1'$, $l_2\to l_2'$, $X\to X'$ in order to get $l_1'\cdot\tilde{p}_a = s$ is still to be done, while for the $\mathcal{IF}$ case it is "ready to go" with $l_1\cdot\tilde{p}_a = z_B s$. However, if we choose z_A or z_C, it is then understood that also for the $\mathcal{IF}$ case a similar mapping is done to achieve[12] $l_1'\cdot\tilde{p}_a = z_A s$ or $l_1'\cdot\tilde{p}_a = z_C s$.

We have investigated all nine choices of $m_\pm$ and $z(\alpha,\beta)$ and good choices (compatible with $\mathcal{II}$) were found to be Aa, Ac, Ca and Cc[13], hence we conclude that for diagonal splitting, it is rather easy to achieve that $\mathcal{IF}$ dipoles and $\mathcal{II}$ dipoles contribute the same to $\Sigma_{I\leftarrow I}(z,\mu_{\rm F})$ and $\mathbb{K}_{I\leftarrow I}(z)$. On the other hand, the singular term $S(\varepsilon)$ in Eq. (1.13), to be combined virtual corrections, may vary freely with the type of the dipole.

3.2. Nondiagonal $\mathcal{IF}$ dipoles — the problem and workaround

In the $\mathcal{IF}$ CS dipoles for nondiagonal splittings $a\neq b$, $a=q,G$ (quark–gluon transitions) the soft singularity is absent — only the collinear singularity is present — the use of SPFs is in principle not needed.

Unfortunately, from the straightforward analytical calculations, we get slightly different $\Sigma_{K\leftarrow I}(z,\mu_{\rm F})|_{z\neq 1}$, $K\neq I$ for $\mathcal{IF}$ dipoles than for $\mathcal{II}$ dipoles for all choices of $z=z(\alpha,\beta)$ defined in the previous subsection. The difference can be traced back to the upper phase-space limit: $\max(\alpha,\beta)\le 1$ *versus* $\alpha+\beta\le 1$[14].

The simplest workaround is to split $\mathcal{IF}$ nondiagonal dipoles into two parts using again SPFs as in the diagonal cases

$$H^{\pm}_{c\leftarrow a}(\alpha,\beta,\varepsilon) = m_\pm(\alpha,\beta)\frac{1}{\beta}P_{ca}(z,\varepsilon)\big|_{z=z(\alpha,\beta)}, \qquad c\neq a,$$

and treat $H^{-}_{c\leftarrow a}$ as additional (nonsingular) dipoles in the $\mathcal{FI}$ class, decoupled from the LO part and PDFs and not contributing to $\Sigma_{K\leftarrow I}$.

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[12] This makes easy the integration over the dipole phase space.

[13] Details of the calculations will be reported elsewhere [11].

[14] One may map $(\alpha,\beta)\to(\alpha',\beta')$ such that $\alpha'+\beta'\le 1$, however, the Jacobian in d-dimension will cause that the problem is back.

We have checked that using the above workaround, the compatibility of $\mathcal{IF}$ and $\mathcal{II}$ dipoles is obtained for $q{\leftarrow}G$ and for $G{\leftarrow}q$ dipoles for $m_{\pm}^{(a)}$ and z_A. Moreover, the same positive conclusion was obtained for the combined use of z_C and yet another SPF $m_{+}^{(d)} = 1-\beta$.

Altogether, we find that at the expense of introducing additional nonsingular $\mathcal{FI}$ dipoles, one can obtain equality of $\Sigma_{K\leftarrow I}(z,\mu_{\rm F})|_{z\neq 1}$ also for nondiagonal splittings $K \neq I$.

In this way, we have shown that thanks to judicious choice of the dipole distributions, we are much closer to the claim the $\mathbb{K}_{K\leftarrow I}(z)$ matrix is the same, independent of whether it was obtained from $\mathcal{II}$ or $\mathcal{IF}$ dipole.

4. Zeroing the collinear remnant P

The role of the term

$$P_{ij}(z)\ln\frac{\hat{s}}{\mu_{\rm F}^2}$$

present in the Σ-function[15] of Eq. (1.1) of our introductory DY example is to keep the factorization scale in PDF to be equal $\hat{s}$. Any variation of $\mu_{\rm F}$ in PDFs is compensated by this term, such that overall dependence on $\mu_{\rm F}$ in NLO expression cancels up to $\mathcal{O}(\alpha^2)$. It is, therefore, logical and convenient to set $\mu_{\rm F} = \hat{s}$ both in the PDF and in the above term, eliminating it completely. The absence of the above term is also mandatory for the KrkNLO method with a single multiplicative MC weight to work.

The above method of eliminating the troublemaking term works well in DY or DIS process with only two coloured legs. In the general case, the **P**-matrix collinear remnant term in the NLO final result of the CS method reads

$$\begin{aligned}\sigma_{ab}^{\rm col\,rem} &= \int {\rm d}x_a\,{\rm d}x_b\; f_b(\mu_{\rm F},x_b)\; f_a(\mu_{\rm F},x_a)\,\Bigg\{{\rm d}\sigma_{a,b}^{\rm Born}(p_a,p_b)\\ &\quad+\sum_{a'}\int {\rm d}x\,\Bigg\langle\frac{\alpha_{\rm S}}{2\pi}P_{aa'}(x)\left[\sum_i\frac{T_i\cdot T_{a'}}{T_{a'}^2}\ln\frac{\mu_{\rm F}^2}{2xs_{ai}}+\frac{T_b\cdot T_{a'}}{T_{a'}^2}\ln\frac{\mu_{\rm F}^2}{2xs_{ab}}\right]\\ &\quad\times{\rm d}\sigma_{a',b}^{\rm Born}(xp_a,p_b)\Bigg\rangle_{\rm colour}+\dots\Bigg\}\,,\end{aligned}\tag{4.1}$$

where the summation over i and b is the summation over spectators and it collects all such logs of many variables $s_{ab}=2p_ap_b$. Obviously, it is not possible to kill all of them at once by equating $\mu_{\rm F}^2$ to one of them.

[15] Sandwiched between the PDF and the LO cross section.

However, there is a possibility of finding out at each point of LO phase space (with all s_{ab} defined) a unique value of $\hat{\mu}_{\rm F}$ which renders the above entire **P**-matrix equal zero. Let us show how to achieve that.

Using colour conservation $\langle T_{a'} + T_b + \sum_i T_i \rangle_{\rm colour} = 0$ and evolution equations for $f_a(\mu, x)$, we obtain easily the following identity:

$$\sigma_{ab}^{\rm col\ rem} = \int {\rm d}x_a\, {\rm d}x_b\, f_b(\mu_{\rm F}, x_b)\, f_a(\hat{\mu}_{\rm F}, x_a) \Bigg\{ {\rm d}\sigma^{\rm Born}_{a,b}(p_a, p_b)$$
$$+ \sum_{a'} \int {\rm d}x\, \frac{\alpha_{\rm S}}{2\pi} P_{aa'}(x) \Bigg\langle \left[\sum_i \frac{T_i \cdot T_{a'}}{T_{a'}^2} \ln \frac{\mu_{\rm F}^2}{2x s_{ai}} + \frac{T_b \cdot T_{a'}}{T_{a'}^2} \ln \frac{\mu_{\rm F}^2}{2x s_{ab}} + \ln \frac{\hat{\mu}_{\rm F}^2}{\mu_{\rm F}^2} \right]$$
$$\times {\rm d}\sigma^{\rm Born}_{a',b}(x x_a p_1, x_b p_2) \Bigg\rangle_{\rm colour} + \ldots \Bigg\}\,. \tag{4.2}$$

Since $\mu_{\rm F}^2$ is a local dummy parameter in the above expression (colour conservation!), we may substitute $\mu_{\rm F}^2 = 2x s_{ab}$, and solve for $\hat{\mu}_{\rm F}$ the following equation:

$$\sum_{a'} \int_0^1 {\rm d}z P_{aa'}(z) \sum_i \ln \frac{s_{ab}}{s_{ai}} \left\langle \frac{T_i \cdot T_{a'}}{T_{a'}^2} {\rm d}\sigma^{\rm Born}_{a',b}(z p_a, p_b) \right\rangle_{\rm colour}$$
$$+ \sum_{a'} \int_0^1 {\rm d}z P_{aa'}(z) {\rm d}\sigma^{\rm Born}_{a',b}(z p_a, p_b) \ln \frac{\hat{\mu}_{\rm F}^2}{2 z s_{ab}} \equiv 0\,. \tag{4.3}$$

The effective scale $\hat{\mu}_{\rm F}$ to be inserted in the PDF in the KRK scheme can be calculated numerically (1-dim. integral over z) at each point of the Born phase space, $h_1 + h_2 \to p_a + p_b \to 1 + 2 + \ldots m$, or even analytically in some simpler cases. Of course, for the other PDF f_b, a similar independent equation has to be solved and the resulting $\hat{\mu}_{\rm F}$ will be inserted into f_b.

In the construction of all new CS dipoles in the previous sections, we have ignored the role of the colour factors. They enter for a given $a \to a'$ splitting within the summation over all spectators

$$\sum_{S=i,b} \left\langle \frac{T_S \cdot T_{a'}}{T_{a'}^2} \ldots \right\rangle_{\rm colour}\,,$$

in a similar way as Eqs. (4.1)–(4.3). Now, thanks to the achieved independence of the partly integrated[16] modified dipoles on the type of spectator

[16] The integrated contribution for fixed $z \neq 0$.

$S = i, b$ and using colour conservation, we see that the above colour factor factorizes out and gets reduced to unity. This is yet another important profit from our modification of the CS dipoles!

Eliminating the collinear remnant, $\mathbf{P}$, in the NLO differential distribution was the last obstacle on the way to making the $\mathbb{K}_{K\leftarrow I}(z)$ matrix process independent (universal).

We did not provide in this paper explicit expressions for the transition matrix $\mathbb{K}_{K\leftarrow I}(z)$ for transforming PDFs from the $\overline{\mathrm{MS}}$ to the KRK scheme because they are the same as in Eq. (4.3) of Ref. [6], where they were calculated for the Drell–Yan process and now are applicable to any process.

5. Summary

In our analysis, we have exploited the machinery of the Catani–Seymour subtraction scheme to examine the question of universality of the PDFs in the KRK factorization scheme, originally defined and used for the Drell–Yan-type production of heavy colourless bosons. The transition matrix $\mathbb{K}_{K\leftarrow I}(z)$ for transforming PDFs from the $\overline{\mathrm{MS}}$ to the KRK scheme is closely related to partially integrated CS dipoles, while the MC weight of the KrkNLO matching scheme also reflects the shape and normalization of the CS dipoles. The original dipoles of the CS work do not lead to universality of $\mathbb{K}_{K\leftarrow I}(z)$. However, we have shown that one may modify CS dipoles in such a way that they provide a process independent of $\mathbb{K}_{K\leftarrow I}(z)$. The key features of the new CS dipoles are that dipoles with final emitter and initial spectators decouple kinetically from PDFs and LO differential distributions (thanks to a new mapping of the dipole kinematics) and that the remaining dipoles with an initial emitter yield the same contribution to $\mathbb{K}_{K\leftarrow I}(z)$ for spectators in the initial and final state. Full details of the calculations related to new CS dipoles will be reported elsewhere [11].

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