INTRODUCING THE CHIRALITY-FLOW FORMALISM*

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(Received May 5, 2020)

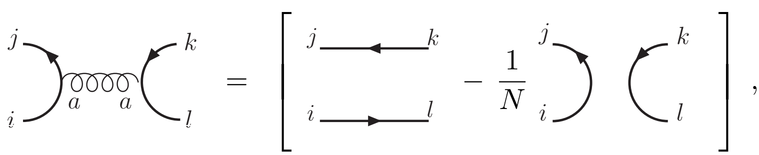
In QCD, we are used to describing the SU(3) color space in terms of a flow of color. At the algebra level, the Lorentz group consists of two copies of the (complexified) $\mathfrak{su}(2)$ algebra, so one may anticipate that a similar way of thinking about the spacetime structure of scattering amplitudes should exist. In this article, we argue that this is indeed the case, and introduce the chirality-flow formalism for massless tree-level QED and QCD. Within the chirality-flow formalism, scattering amplitudes can directly be written down in terms of Lorentz-invariant spinor inner products, similar to how the color structure can be described in terms of a color flow.

DOI:10.5506/APhysPolB.51.1547

1. Introduction

A standing challenge within particle physics is the accurate and efficient calculation of particle scattering amplitudes. The standard method for calculating scattering amplitudes for particle collisions has for a long time been based on Feynman diagrams (although recursive approaches are growing in popularity). During the past decades, strategies for managing the quantum numbers of color in Quantum ChromoDynamics (QCD) [1–36] and helicity, associated with spacetime [26–51], have played a major role in simplifying these calculations.

On the color structure side, the Fierz identity



(1)

* Presented by M. Sjödahl at XXVI Cracow Epiphany Conference on LHC Physics: Standard Model and Beyond, Kraków, Poland, January 7–10, 2020.

(where we use the normalization $\mathrm{Tr}[t^a t^b] = \delta^{ab}$ for the generators) can be applied to remove the adjoint indices. (Here, the vertices represent the color structure only, *i.e.*, t^a_{ij} or t^a_{kl}, and the lines on the right-hand side represent Kronecker delta functions in the quark color indices, *i.e.*, a flow of color.)

At the algebra level, the Lorentz group consists of two copies of $\mathfrak{su}(2)$, and the Dirac spinor structure transforms under the direct sum representation $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$, in the chiral/Weyl basis

$$\begin{pmatrix} u_\mathrm{L} \\ u_\mathrm{R} \end{pmatrix} \to \begin{pmatrix} \mathrm{e}^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}+\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} & 0 \\ 0 & \mathrm{e}^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}-\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} \end{pmatrix} \begin{pmatrix} u_\mathrm{L} \\ u_\mathrm{R} \end{pmatrix},$$

where the upper components correspond to the left chiral spinor, and the lower correspond to the right chiral spinor, *i.e.*, we actually have two copies of SL(2,$\mathbb{C}$), generated by the complexified $\mathfrak{su}(2)$ algebra. In view of this, one may wonder if a similar flow-like description as for SU(3) color can be found for the Lorentz structure of scattering amplitudes.

For $m = 0$, the u and v spinors, expressed in terms of dotted and undotted spinors, have a similar form

$$u(p) = \begin{pmatrix} \tilde{\lambda}_p^{\dot{\alpha}} \\ \lambda_{p,\beta} \end{pmatrix}, \quad \bar{u}(p) = \left(\tilde{\lambda}_{p,\dot{\beta}}\,,\, \lambda_p^{\alpha}\right),$$

$$v(p) = \begin{pmatrix} \tilde{\lambda}_p^{\dot{\alpha}} \\ \lambda_{p,\beta} \end{pmatrix}, \quad \bar{v}(p) = \left(\tilde{\lambda}_{p,\dot{\beta}}\,,\, \lambda_p^{\alpha}\right),$$

and amplitudes can be built up from Lorentz-invariant inner products

$$\underbrace{\epsilon^{\alpha\beta}\lambda_{i,\beta}}_{\equiv\lambda_i^\alpha}\lambda_{j,\alpha} = \lambda_i^\alpha\lambda_{j,\alpha} = \langle ij\rangle\,, \qquad \underbrace{\epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_i^{\dot{\beta}}}_{\equiv\tilde{\lambda}_{i,\dot{\alpha}}}\tilde{\lambda}_j^{\dot{\alpha}} = \tilde{\lambda}_{i,\dot{\alpha}}\tilde{\lambda}_j^{\dot{\alpha}} = [ij]\,,$$

where $\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1$, implying that the inner products are antisymmetric.

After requiring local gauge invariance, the photon couples to

$$\bar{u}(p_1)\gamma^\mu u(p_2) = \underbrace{\left(\tilde{\lambda}_{1,\dot{\alpha}}\,,\, \lambda_1^{\alpha}\right)}_{\bar{u}(p_1)} \underbrace{\begin{pmatrix} 0 & \sqrt{2}\tau^{\mu,\dot{\alpha}\beta} \\ \sqrt{2}\bar{\tau}^{\mu}_{\alpha\dot{\beta}} & 0 \end{pmatrix}}_{\gamma^\mu} \underbrace{\begin{pmatrix} \tilde{\lambda}_2^{\dot{\beta}} \\ \lambda_{2,\beta} \end{pmatrix}}_{u(p_2)},$$

where $\sqrt{2}\tau^\mu = (1,\vec{\sigma})$, $\sqrt{2}\bar{\tau}^\mu = (1,-\vec{\sigma})$ and $\mathrm{Tr}(\tau^\mu\bar{\tau}^\nu) = g^{\mu\nu}$, giving two different vertices, of the form

$$\tilde{\lambda}_{1,\dot{\alpha}}\tau^{\mu,\dot{\alpha}\beta}\lambda_{2,\beta} \quad \text{and} \quad \lambda_1^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}_2^{\dot{\beta}}$$

respectively. Lorentz four-vectors transform under a direct product representation and are mapped to

$$p^{\dot{\alpha}\beta} \equiv p_\mu \tau^{\mu,\dot{\alpha}\beta} = \frac{1}{\sqrt{2}} p_\mu \sigma^{\mu,\dot{\alpha}\beta} = \frac{1}{\sqrt{2}} \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} ,$$

$$\bar{p}_{\alpha\dot{\beta}} \equiv p_\mu \bar{\tau}^\mu_{\alpha\dot{\beta}} = \frac{1}{\sqrt{2}} p_\mu \bar{\sigma}^\mu_{\alpha\dot{\beta}} = \frac{1}{\sqrt{2}} \begin{pmatrix} p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_0 + p_3 \end{pmatrix} ,$$

and it can be proved that transforming the spinor indices in $p^{\dot{\alpha}\beta}$ or $\bar{p}_{\alpha\dot{\beta}}$ using the direct product transformation gives the Lorentz four-vector transformation.

For light-like momenta, $p^2 = 0$, this implies

$$p^2 = \det\left[p^{\dot{\alpha}\beta}\right] = 0 \overset{\text{Dirac}}{\Rightarrow} \not{p} \equiv \sqrt{2} p^{\dot{\alpha}\beta} = \tilde{\lambda}^{\dot{\alpha}}_p \lambda^\beta_p$$

and similarly $\bar{\not{p}} \equiv \sqrt{2} p_\mu \bar{\tau}^\mu_{\alpha\dot{\beta}} \overset{p^2=0}{=} \lambda_{p,\alpha} \tilde{\lambda}_{p,\dot{\beta}}$. Multiplying this with $\tau^{\nu,\dot{\beta}\alpha}$, summing over indices, and using $\mathrm{Tr}(\tau^\mu \bar{\tau}^\nu) = g^{\mu\nu}$, we get

$$\underbrace{\sqrt{2} p_\mu \bar{\tau}^\mu_{\alpha\dot{\beta}}}_{\lambda_{p,\alpha}\tilde{\lambda}_{p,\dot{\beta}}} \tau^{\nu,\dot{\beta}\alpha} = \sqrt{2} p_\mu g^{\mu\nu} = \sqrt{2} p^\nu \implies p^\nu \overset{p^2=0}{=} \frac{1}{\sqrt{2}} \tilde{\lambda}_{p,\dot{\beta}} \tau^{\nu,\dot{\beta}\alpha} \lambda_{p,\alpha}$$

meaning that a light-like fourvector has the same spinor structure as a vertex *i.e.*, it can be viewed as a "pseudo-vertex".

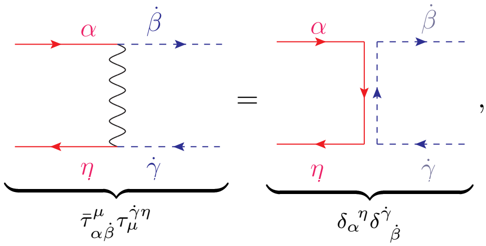
We also need the polarization vectors for (outgoing) external photons

$$\epsilon^\mu_+(p,r) = \frac{\tilde{\lambda}_{p,\dot{\alpha}} \tau^{\mu,\dot{\alpha}\beta} \lambda_{r,\beta}}{\langle rp \rangle} , \qquad \epsilon^\mu_-(p,r) = \frac{\lambda^\alpha_p \bar{\tau}^\mu_{\alpha\dot{\beta}} \tilde{\lambda}^{\dot{\beta}}_r}{[pr]} \tag{2}$$

which, similar to momenta, have the same spinor structure as a vertex *i.e.*, they can be seen as pseudo-vertices. For further conventions, details and proofs, we refer to [52].

2. Building the flow picture

We now have all the ingredients needed to try to build a flow picture, similar to the color-flow picture in QCD. Recalling the QCD Fierz identity, (1), we may analogously write the spinor Fierz identity in a flow form

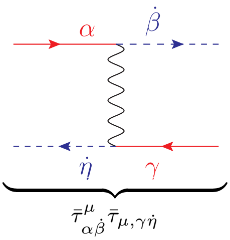


where we always read the spinor indices in matrices and Kronecker deltas along the arrow direction, and where left (right) chiral states are highlighted in dashed/blue (solid/red). Note that compared to the color Fierz identity, the $1/N$-suppressed term is missing so, in this sense, the flow picture works even better than for color.

For the photon exchange above, we had a "flow", coming from the term $\bar\tau^{\mu}_{\alpha\dot\beta}\tau_{\mu}^{\dot\gamma\eta}$, but photon exchange may also contract two factors of τ or $\bar\tau$, for example,

$$\bar\tau^{\mu}_{\alpha\dot\beta}\bar\tau_{\mu,\gamma\dot\eta} = \epsilon_{\dot\beta\dot\eta}\epsilon_{\alpha\gamma}$$

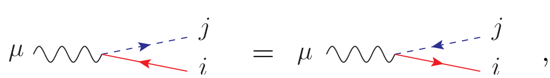
which does not create a flow

$$\bar\tau^{\mu}_{\alpha\dot\beta}\bar\tau_{\mu,\gamma\dot\eta} =$$  $$= ? \qquad (3)$$

However, it turns out that also in the above situation, the expression can be recast into a flow-like picture. Using charge conjugation of a current, we have

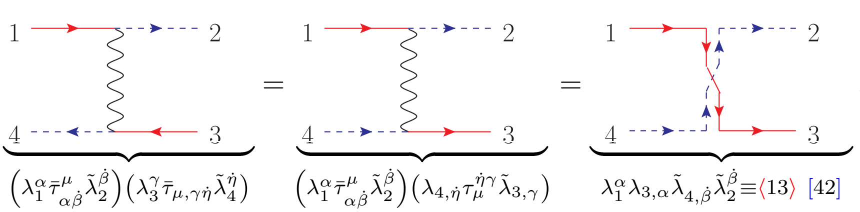
$$\lambda_i^{\alpha}\bar\tau^{\mu}_{\alpha\dot\beta}\tilde\lambda_j^{\dot\beta} = \tilde\lambda_{j,\dot\alpha}\tau^{\mu,\dot\alpha\beta}\lambda_{i,\beta}\,,$$

or graphically



(4)

which can be applied in (3), while used between external spinors, to give



where we have used the graphical representations

$$\lambda_i^{\alpha}\lambda_{j,\alpha} = \langle ij\rangle = $$  and $$\tilde{\lambda}_{i,\dot{\beta}}\tilde{\lambda}_j^{\dot{\beta}} = [ij] = $$ 

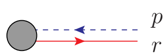
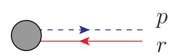
for contraction of external spinors.

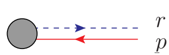
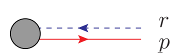
In general, we denote the external spinors

$$\lambda_{j,\alpha} = $$  , $$\lambda_i^{\alpha} = $$  ,

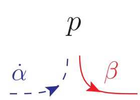
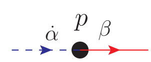
$$\tilde{\lambda}_{i,\dot{\alpha}} = $$  , $$\tilde{\lambda}_j^{\dot{\alpha}} = $$  . (5)

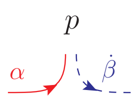
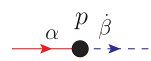
Similarly, the polarization vectors of external photons in (2) may be represented by

$$\epsilon_+^{\mu}(p,r) \to \frac{1}{\langle rp\rangle}$$  or $$\epsilon_+^{\mu}(p,r) \to \frac{1}{\langle rp\rangle}$$  ,

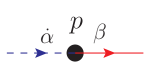
$$\epsilon_-^{\mu}(p,r) \to -\frac{1}{[rp]}$$  or $$\epsilon_-^{\mu}(p,r) \to -\frac{1}{[rp]}$$  . (6)

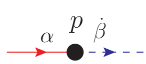
Fermion propagators have a factor $\not{p}_{4\times4} \equiv p_\mu\gamma^\mu$ in the numerator. To treat this, we note that light-like momenta can be split into two terms

$$\not{p} = \sqrt{2}p^{\mu}\tau_{\mu}^{\dot{\alpha}\beta} \stackrel{p^2=0}{=} \tilde{\lambda}_p^{\dot{\alpha}}\lambda_p^{\beta} = $$  $\equiv$  ,

$$\bar{\not{p}} = \sqrt{2}p_{\mu}\bar{\tau}^{\mu}_{\alpha\dot{\beta}} \stackrel{p^2=0}{=} \lambda_{p,\alpha}\tilde{\lambda}_{p,\dot{\beta}} = $$  $\equiv$  .

Since at tree-level, all momenta in all propagators are linear combinations of on-shell massless momenta, p_i, we may use

$$\not{p} = $$  $$= \sum_i \tilde{\lambda}_i^{\dot{\alpha}}\lambda_i^{\beta} \quad \text{for} \quad p_i^2 = 0\,,$$

$$\bar{\not{p}} = $$  $$= \sum_i \lambda_{i,\alpha}\tilde{\lambda}_{i,\dot{\beta}} \quad \text{for} \quad p_i^2 = 0$$

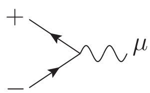
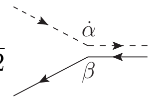
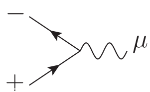
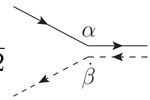
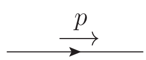
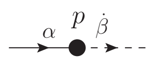
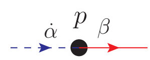
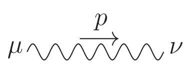
for the matrix structure of Fermion propagators.

3. QED Feynman rules and examples

Using the ingredients above, and proving that the charge conjugation relation (4) is applicable also when more than one photon is attached to the fermion line (see [52] for details), we are ready to write down the massless tree-level QED Feynman rules in Table I.

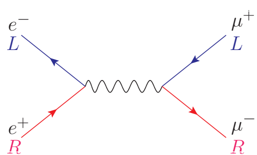
TABLE I

Chirality-flow Feynman rules for massless QED (in Feynman gauge), to be used together with the external spinors in (5) and the external photon polarization vectors in (6).

Feynman	Chirality-flow
	$ieQ_f\sqrt{2}$ 
	$ieQ_f\sqrt{2}$ 
Feynman	**Chirality-flow**
	$\frac{i}{p^2}$  or $\frac{i}{p^2}$ 
	$-\frac{i}{p^2}$  or $-\frac{i}{p^2}$ 

To illustrate the simplicity of the chirality-flow method, we now give a few examples. First, we consider $e^+e^- \to \mu^+\mu^-$, cross all particles to be outgoing, and assign helicities (chiralities).

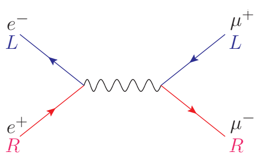
Within the ordinary spinor-helicity formalism, the calculations may be performed in a few steps



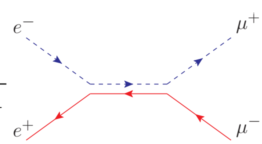
$$= \frac{2ie^2}{s_{e^+e^-}}\left(\tilde\lambda_{e^-,\dot\alpha}\tau_\mu^{\dot\alpha\beta}\lambda_{e^+,\beta}\right)\left(\lambda^{\alpha}_{\mu^-}\bar\tau^{\mu}_{\alpha\dot\beta}\tilde\lambda^{\dot\beta}_{\mu^+}\right)$$

$$= \frac{2ie^2}{s_{e^+e^-}}\tilde\lambda_{e^-,\dot\alpha}\tilde\lambda^{\dot\alpha}_{\mu^+}\lambda^{\beta}_{\mu^-}\lambda_{e^+,\beta} \equiv \frac{2ie^2}{s_{e^+e^-}}[e^-\mu^+]\langle\mu^-e^+\rangle\,,$$

which can be compared to the chirality-flow formalism where we immediately write down

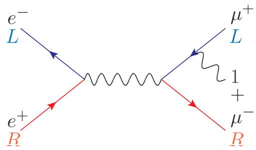


$$= \frac{2ie^2}{s_{e^+e^-}}$$



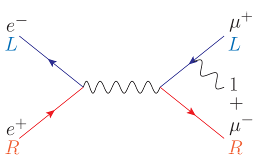
$$= \frac{2ie^2}{s_{e^+e^-}}[e^-\mu^+]\langle\mu^-e^+\rangle\,.$$

Adding to the complexity by emitting a photon, we have with the standard version of the spinor-helicity formalism

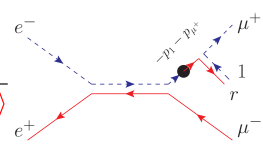


$$
\begin{aligned}
&= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_\mu^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right) \\
&\quad \times\left(\lambda^\alpha_{\mu^-}\bar{\tau}^\mu_{\alpha\dot{\beta}}\underbrace{\left(-\not{p}_1-\not{p}_{\mu^+}\right)^{\dot{\beta}\eta}}_{\not{p}_i=\tilde{\lambda}_i^{\dot{\beta}}\lambda_i^\eta}\underbrace{\bar{\epsilon}^+_{\eta\dot{\gamma}}(1,r)}_{\frac{\lambda_{r,\eta}\tilde{\lambda}_{1,\dot{\gamma}}}{\langle r1\rangle}}\tilde{\lambda}^{\dot{\gamma}}_{\mu^+}\right) \\
&= \frac{i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_\mu^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right) \\
&\quad \times\left(\lambda^\alpha_{\mu^-}\bar{\tau}^\mu_{\alpha\dot{\beta}}\tilde{\lambda}_1^{\dot{\beta}}\lambda_1^\eta\lambda_{r,\eta}+\lambda^\alpha_{\mu^-}\bar{\tau}^\mu_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}_{\mu^+}\lambda^\eta_{\mu^+}\lambda_{r,\eta}\right)\tilde{\lambda}_{1,\dot{\gamma}}\tilde{\lambda}^{\dot{\gamma}}_{\mu^+} \\
&= \frac{i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}_1^{\dot{\alpha}}\lambda_1^\eta\lambda_{r,\eta}+\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{\mu^+}\lambda^\eta_{\mu^+}\lambda_{r,\eta}\right) \\
&\quad \times\lambda^\beta_{\mu^-}\lambda_{e^+,\beta}\tilde{\lambda}_{1,\dot{\gamma}}\tilde{\lambda}^{\dot{\gamma}}_{\mu^+} \\
&= \frac{i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left([e^-1]\langle 1r\rangle+[e^-\mu^+]\langle\mu^+r\rangle\right) \\
&\quad \times\langle\mu^-e^+\rangle[1\mu^+]\,, \qquad (7)
\end{aligned}
$$

whereas, with the chirality-flow method, this can be written down in one step



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}$$



, (8)

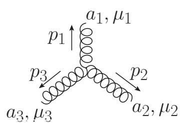
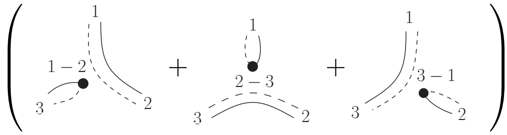
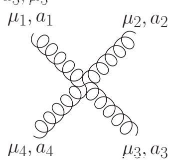
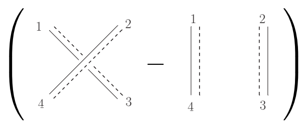
directly corresponding to the final line above.

4. QCD Feynman rules and examples

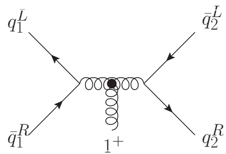
For QCD, the Lorentz structure of the fermion–boson vertex, as well as the propagators and external particles are similar to those in QED, but we also have to deal with the three- and four-gluon vertices which are collected in Table II.

TABLE II

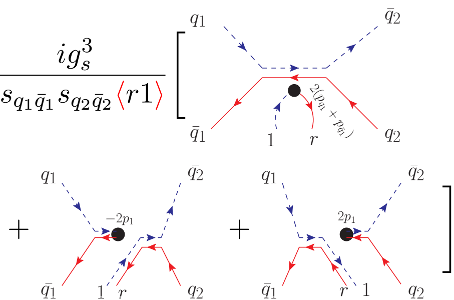
The Lorentz structure of the additional non-Abelian vertices in QCD. Here, f^{abc} are the SU(3) structure constants, defined by $[t^a, t^b] = if^{abc}t^c$ (and using the normalization $\mathrm{Tr}[t^a t^b] = \delta^{ab}$).

Feynman	Chirality-flow
	$i\frac{g_s}{\sqrt{2}}(if^{a_1a_2a_3})\frac{1}{\sqrt{2}}$ 
	$i\left(\frac{g_s}{\sqrt{2}}\right)^2 \sum_{Z(2,3,4)} (if^{a_1a_2b})(if^{ba_3a_4})$ 

As an example of a QCD diagram with a non-Abelian vertex, we consider



$$= \frac{ig_s^3}{2s_{q_1\bar{q}_1}s_{q_2\bar{q}_2}\langle r1\rangle}$$



where we have again immediately written down the result. If desired, the spinor expression in the bracket can also be written out in terms of spinor inner products

$$\Big[\ldots\Big] \equiv \Big\{2[q_1\bar{q}_2]\langle q_2\bar{q}_1\rangle\big([1q_1]\langle q_1 r\rangle + [1\bar{q}_1]\langle\bar{q}_1 r\rangle\big)$$
$$- 2[q_1 1]\langle 1\bar{q}_1\rangle\langle q_2 r\rangle[1\bar{q}_2] + 2[q_1 1]\langle r\bar{q}_1\rangle\langle q_2 1\rangle[1\bar{q}_2]\Big\}.$$

5. Conclusion and outlook

In this paper, we have introduced the chirality-flow formalism which gives a transparent and intuitive way of understanding the Lorentz inner products appearing in scattering amplitudes, similar to how color structure can be thought about in terms of a color flow.

In essence, our method can be seen as a further simplification of the spinor-helicity method, where Dirac spinors are replaced by their left- and right-chiral components, and Dirac matrices by Pauli matrices. In our formalism, we manage to remove the Pauli matrices as well, and instead express all internal structure with Kronecker delta functions, which give rise to Lorentz spinor inner products when contracted with external spinors.

Using this formalism, we are able to directly — without non-trivial steps — write down the value of Feynman diagrams with assigned helicities for massless QED and QCD [52]. We anticipate this to be an advantage for event generators based on Feynman diagrams.

The extension to the full Standard Model is on its way.

This work was supported by the Swedish Research Council (contract number 2016-05996), as well as the European Union's Horizon 2020 research and innovation programme (grant agreement No. 668679). This work has also received funding from the European Union's Horizon 2020 research and innovation programmeas part of the Marie Skłodowska-Curie Innovative Training Network MCnetITN3 (grant agreement No. 722104).

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