ON THE DISTRIBUTION OF THE $\ell^{\underline{\text{th}}}$ LARGEST EIGENVALUE OF SPIKED COMPLEX WISHART MATRICES*

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The study of the statistical distribution of the eigenvalues of Wishart matrices finds application in many fields of physics and engineering. Here, we consider a special case of finite dimensions correlated complex central Wishart matrices, characterized by the fact that the covariance matrix has all eigenvalues equal, except for one which is the largest. Starting from the knowledge of the joint probability distribution function (p.d.f.) of this kind of Wishart matrices, we focus on the evaluation of a tractable form for the distribution of each individual eigenvalue. In particular, we derive an expression for the p.d.f. of the $\ell^{\underline{\text{th}}}$ largest eigenvalue as a sum of terms of the type $x^{\beta}\mathrm{e}^{-x\delta}$, which allows us to write a large class of statistical averages involving functions of eigenvalues in closed form.

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1. Introduction

The analysis of the distribution of the eigenvalues of random matrices represents one of the most important fields of research in multivariate statistics, with particular focus on the statistical analysis of large data sets and on principal component analysis [1–7]. The characteristics of the eigenvalues of random matrices play an important role also in several branches of physics, which include, for instance, the statistical analysis of nuclear spectra, atomic physics, quantum theory, kinetic theory of gases, and cosmology [8–11]. Random matrix analysis is also applied in communication the-

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ory for signal processing applications, multiple antenna systems, and compressed sensing [12–24]. Another field of application is the complex system theory [25, 26]. Among the various kinds of random matrices, Wishart random matrices play a very important role in statistics, physics, and engineering. To give some example, Wishart are considered in Bayesian inference, as they often represent the prior precision matrix of multivariate Gaussian data sets. They can also be used to model the propagation channel of multiple-input–multiple-output systems in wireless communications [12–14]. Wishart matrices have also appeared in several areas of physics: for instance, in nuclear physics, in low energy QCD and gauge theories, quantum gravity, in statistical physics, including directed polymers in a disordered medium, nonintersecting Brownian excursions and fluctuating nonintersecting interfaces over a solid substrate [27–36]. For Wishart matrices with finite dimensions, the derivation of the eigenvalues distribution is generally not trivial. Expressions for the cumulative distribution function (c.d.f.) of the $\ell^{\underline{\text{th}}}$ largest eigenvalue of a complex Wishart matrix have been obtained in [37, 38]; however, the direct computation of the corresponding probability distribution function (p.d.f.) from the cumulative distribution function (c.d.f.) is rather complicated. The distribution of the largest eigenvalue and the probability that all eigenvalues are within an interval, as well as recursive methods for their numerical computation, were studied for real and complex Wishart, multivariate Beta (also known as double Wishart), for the Gaussian orthogonal ensemble (GOE) and for the Gaussian unitary ensemble (GUE) [39–41]. Expressions for the joint p.d.f. of subsets of unordered eigenvalues of uncorrelated noncentral Wishart matrices were given in [42]. Closed form expressions for the marginal c.d.f.s and p.d.f.s of some Hermitian random matrices, which also include Wishart matrices, were given in [43]. The moment generating function (MGF) of the largest eigenvalue for both uncorrelated and correlated central Wishart cases was given in [44]. Expressions for the marginal distribution, joint distribution, and moments of a subset of eigenvalues have been obtained in [45] for a general class of random matrices, including GUE and correlated central Wishart.

Among the class of Wishart matrices, a particular importance has the *spiked* model, which represents a special case of the Wishart ensemble, and a natural generalization of the uncorrelated Wishart case. In the spiked model, the Wishart matrix has a covariance matrix where all eigenvalues are equal except the largest one [7, 46–48]. In this paper, we focus on Wishart matrices with spiked covariance matrix and look for *tractable* expressions for the distribution of the $\ell^{\underline{\text{th}}}$ eigenvalue. In particular, the aim of this paper is to obtain the expressions for the marginal p.d.f. as a sum of terms taking the form $x^{\beta}\mathrm{e}^{-x\delta}$. This expression is suited for further uses of the distribution, such as in the case of evaluation of statistical moments or expectations of functions of eigenvalues. This work extends the results in [49], valid for the uncorrelated Wishart case and for some examples of correlated Wishart, to the spiked covariance Wishart model with finite size. The results of this paper can be summarized as follows:

1. We obtain the p.d.f. for the $\ell^{\underline{\text{th}}}$ largest eigenvalue in the form of a sum of terms $x^\beta e^{-x\delta}$ for the spiked covariance Wishart model.
2. We derive bounds on the number of terms $x^\beta e^{-x\delta}$ and on the values assumed by the coefficients β and δ.
3. As an example, we explicitly write for some small matrices the p.d.f. of the eigenvalues as a mixture of gamma distributions.

Throughout the paper we indicate with $\Gamma(\cdot)$ the gamma function, with $\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt$ the upper incomplete gamma function [50, Ch. 6], with $()^\dagger$ transposition and complex conjugation, and with $|\cdot|$ or $\det(\cdot)$ the determinant. When possible, we use capital letters for random variables, and bold for vectors and matrices. We say that a random variable Z has a standard complex Gaussian distribution (denoted $\mathcal{CN}(0,1)$) if $Z = Z_1 + iZ_2$, where Z_1 and Z_2 are independent, identically distributed (i.i.d.) real Gaussian $\mathcal{N}(0,1/2)$. A complex random vector $\boldsymbol{X}$ is Gaussian circularly symmetric if its p.d.f. has the form $f(\boldsymbol{x}) \propto \exp\left(-\boldsymbol{x}^\dagger \boldsymbol{\Sigma}^{-1} \boldsymbol{x}\right)$, where $\boldsymbol{\Sigma}$ is the covariance matrix. When $\boldsymbol{\Sigma} = \boldsymbol{I}$ the entries of $\boldsymbol{X}$ are i.i.d. $\mathcal{CN}(0,1)$.

2. Joint p.d.f. of the eigenvalues of Wishart matrices with spiked covariance matrix

To obtain the expression for the joint p.d.f. of the eigenvalues of complex central Wishart matrices we can use the following Lemma, which has been proved in [51].

Lemma 1. *Denote by $\boldsymbol{X}$ a $(p \times n)$ random matrix with complex Gaussian, zero mean, unit variance, i.i.d. entries, and by $\boldsymbol{\Sigma}$ an $(n \times n)$ positive definite matrix. The joint p.d.f. of the (real) nonzero ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M \geq 0$, with $M = \min(n,p)$, of the $(p \times p)$ quadratic form $\boldsymbol{W} = \boldsymbol{X}\boldsymbol{\Sigma}\boldsymbol{X}^\dagger$ is*

$$f_{\boldsymbol{\lambda}}^{(\text{ordered})}(x_1, \ldots, x_M) = K\, |\boldsymbol{V}(\boldsymbol{x})| \cdot |\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\phi})| \prod_{i=1}^{M} \xi(x_i)\,, \tag{1}$$

where $\xi(x) = x^{p-M}$, and $\boldsymbol{V}(\boldsymbol{x})$ is the $(M \times M)$ Vandermonde matrix with elements $v_{i,j} = x_j^{i-1}$. The constant K is given by

$$K = \frac{(-1)^{p(n-M)}}{\Gamma_{(M)}(p)} \frac{\prod_{i=1}^{L} \phi_{(i)}^{m_i p}}{\prod_{i=1}^{L} \Gamma_{(m_i)}(m_i) \prod_{i<j} \left(\phi_{(i)} - \phi_{(j)}\right)^{m_i m_j}}\,, \tag{2}$$

where $\Gamma_{(m)}(n) \triangleq \prod_{i=1}^{m}(n-i)!$ *and* $\phi_{(1)} > \phi_{(2)} \ldots > \phi_{(L)}$ *are the* L *distinct eigenvalues of* $\boldsymbol{\Sigma}^{-1}$*, with associated multiplicities* $m_1, \ldots, m_L$ *such that* $\sum_{i=1}^{L} m_i = n$*.*

The $(n \times n)$ *matrix* $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\phi})$ *has elements*

$$g_{i,j} = \begin{cases} g_i(x_j) & = (-x_j)^{d(i)}\, \mathrm{e}^{-\phi_{(e(i))} x_j}\,, \qquad j = 1, \ldots, M\,, \\ \bar{g}_{i,j} & = [n-j]_{d(i)}\, \phi_{(e(i))}^{n-j-d(i)}\,, \qquad j = M+1, \ldots, n\,, \end{cases} \tag{3}$$

where $[a]_n \triangleq a(a-1)\cdots(a-n+1)$*,* $e(i)$ *denotes the unique integer such that*

$$m_1 + \ldots + m_{e(i)-1} < i \leq m_1 + \ldots + m_{e(i)}$$

and

$$d(i) = \sum_{k=1}^{e(i)} m_k - i\,.$$

We remark that the distribution depends only on the eigenvalues of $\boldsymbol{\Sigma}$. A very interesting special case is when $\boldsymbol{\Sigma}$ has a *spiked* shape, *i.e.*, when its eigenvalues are $\sigma_1 > \sigma_2 = \sigma_3 = \sigma_4 = \cdots = \sigma_n$. For this particular correlation model, *i.e.* the spiked correlated Wishart case, we obtain the following result [51].

Lemma 2. *Let* $\boldsymbol{W} \sim \mathcal{CW}_M(n, \boldsymbol{\Sigma})$ *be a complex Wishart matrix,* $n \geq M$*. Denote* $\sigma_1 > \sigma_2 = \ldots = \sigma_M > 0$ *the ordered eigenvalues of* $\boldsymbol{\Sigma}$ *(spiked covariance matrix). Then, the joint p.d.f. of the ordered eigenvalues of* $\boldsymbol{W}$ *is*

$$f_{\boldsymbol{\lambda}}^{(\text{ordered})}(x_1, \ldots, x_M) = K \left|\boldsymbol{\Upsilon}(\boldsymbol{x})\right| \cdot \left|\boldsymbol{E}(\boldsymbol{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i=1}^{M} x_i^{n-M}\,, \tag{4}$$

where $\boldsymbol{\Upsilon}(\boldsymbol{x}) = \{\upsilon_i(x_j)\}_{i,j} = \left\{-x_j^{i-1}\right\}_{i,j}$ *for* $i, j = 1, \ldots, M$*, the matrix* $\boldsymbol{E}(\boldsymbol{x}, \boldsymbol{\sigma})$ *has elements*

$$e_{i,j} = \begin{cases} \mathrm{e}^{-x_i/\sigma_1}\,, & j = 1\,, \\ x_i^{M-j} \mathrm{e}^{-x_i/\sigma_2}\,, & j = 2, \ldots, M \end{cases}$$

and the normalization constant is

$$K = \left[\sigma_1^{n-M+1} \sigma_2^{(n-1)(M-1)} (\sigma_1 - \sigma_2)^{M-1} \prod_{i=1}^{M}(n-i)! \prod_{\ell=2}^{M-2} \ell!\right]^{-1}.$$

Proof. This is a particular case of Lemma 1 [45]. □

3. Individual eigenvalues distribution

An expression for the p.d.f. of the $\ell^{\underline{th}}$ largest eigenvalue of a Wishart matrix with spiked covariance matrix has been obtained in [45] in terms of the pseudo-determinant of a rank 3 tensor. That expression is very compact but not easy to use for further processing, *e.g.*, to perform averages with respect to the distribution of one eigenvalue. To obtain a friendly expression for the p.d.f. of the largest eigenvalue of $\boldsymbol{W}$ we start by the following lemma.

Lemma 3. *Let $X_1, X_2, \ldots, X_M$ be i.i.d. or exchangeable r.v.s, then the marginal p.d.f. of the $\ell^{\underline{th}}$ largest r.v. can be written as [52, pag. 99, Eq. (5.3.1)]*

$$f_\ell(x) = \sum_{s=\ell}^{M} (-1)^{s-\ell} \binom{s-1}{\ell-1} \binom{M}{s} f_{\min:s:M}(x) \,, \qquad (5)$$

where $f_{\min:s:M}(x)$ denotes the p.d.f. of the smallest r.v. considered in any arbitrary subset of s random variables (r.v.s) over the set of M r.v.s $X_1, \ldots, X_M$.

Using Lemma 3 the problem of the evaluation of the $f_\ell(x)$ translates into the computation of $f_{\min:s:M}(x)$, which can be obtained by means of the following Theorem.

Theorem 1. *Let $\lambda_1, \lambda_2, \ldots, \lambda_M$ be the nonzero ordered eigenvalues of a Wishart matrix with spiked covariance matrix, whose joint p.d.f. is described by (4). Then, the p.d.f. of the smallest eigenvalue of a subset of s unordered eigenvalues, $f_{\min:s:M}(x)$, can be written as*

$$\begin{aligned} & f_{\min:s:M}(x) \\ & = \frac{sK}{M!} \sum_{\boldsymbol{\alpha},\boldsymbol{\mu}} \mathcal{C}(\boldsymbol{\alpha},\boldsymbol{\mu}) (-1)^{\alpha_s - 1} \mathrm{e}^{-x \sum_{k=1}^{s} \hat{\sigma}_k^{-1}} \sum_{t_1=0}^{\alpha_1+n-a_1-1} \cdots \sum_{t_{s-1}=0}^{\alpha_{s-1}+n-a_{s-1}-1} \\ & \times x^{\alpha_s - 1 + n - a_s + \sum_{k=1}^{s-1} t_k} \prod_{k=1}^{s-1} \frac{\hat{\sigma}_k^{\alpha_k+n-a_k-t_k} (\alpha_k + n - a_k - 1)!}{t_k!} \,, \end{aligned} \qquad (6)$$

where $\boldsymbol{\alpha}, \boldsymbol{\mu}$ are permutations of the integers $1, 2, \ldots, M$ and

$$\mathcal{C}(\boldsymbol{\alpha},\boldsymbol{\mu}) \triangleq \mathrm{sgn}(\boldsymbol{\alpha}) \mathrm{sgn}(\boldsymbol{\mu}) \prod_{k=s+1}^{M} (-1)^{\alpha_k - 1} \hat{\sigma}_k^{\alpha_k + n - a_k} (\alpha_k + n - a_k - 1)! \qquad (7)$$

with

$$\hat{\sigma}_i \triangleq \begin{cases} \sigma_1 \,, & \mu_i = 1 \,, \\ \sigma_2 \,, & \mu_i > 1 \,, \end{cases} \qquad (8)$$

and

$$a_i \triangleq \begin{cases} M \,, & \mu_i = 1 \,, \\ \mu_i \,, & \mu_i > 1 \,. \end{cases} \qquad (9)$$

Proof. The proof is in Appendix. □

4. The p.d.f. of the $\ell^{\underline{\text{th}}}$ largest eigenvalue as a mixture of Gamma

The results of Lemma 3 and Theorem 1 allow us to obtain an expression for the p.d.f. of the $\ell^{\underline{\text{th}}}$ largest eigenvalue of a Wishart matrix with spiked covariance matrix as the sum of terms having the form $x^{\beta}\mathrm{e}^{-x\delta}$, *i.e.*, as a mixture of Gamma r.v.s. We can also derive the following propositions.

Proposition 1. *The expression for the p.d.f. of the* $\ell^{\underline{\text{th}}}$ *largest eigenvalue of a Wishart matrix with spiked covariance matrix can be written in compact form as*

$$f_\ell(x) = \mathcal{K} \sum_{\boldsymbol{\nu}} \varphi(\boldsymbol{\nu}) \mathrm{e}^{-x\delta(\boldsymbol{\nu})} x^{\beta(\boldsymbol{\nu})} \,, \tag{10}$$

where the sum in (10) *is defined as*

$$\sum_{\boldsymbol{\nu}} \triangleq \sum_{s} \sum_{\boldsymbol{\alpha}} \sum_{\boldsymbol{\mu}} \sum_{t_1} \cdots \sum_{t_{s-1}} , \tag{11}$$

and $\mathcal{K}$, $\varphi(\boldsymbol{\nu})$, $\beta(\boldsymbol{\nu})$ *and* $\delta(\boldsymbol{\nu})$ *are given by*

$$\mathcal{K} = \frac{sK}{M!} \,, \tag{12}$$

$$\begin{aligned} \varphi(\boldsymbol{\nu}) &= \mathrm{sgn}(\boldsymbol{\alpha})\mathrm{sgn}(\boldsymbol{\mu})(-1)^{s-\ell} \binom{s-1}{\ell-1} \binom{M}{s} \\ &\times \left(\prod_{k=s+1}^{M} (-1)^{\alpha_s+\alpha_k-2} \hat{\sigma}_k^{\alpha_k+n-a_k} (\alpha_k+n-a_k-1)! \right) \\ &\times \prod_{k=1}^{s-1} \frac{\hat{\sigma}_k^{\alpha_k+n-a_k-t_k} (\alpha_k+n-a_k-1)!}{t_k!} \,, \end{aligned} \tag{13}$$

$$\delta(\boldsymbol{\nu}) = \sum_{k=1}^{s} \hat{\sigma}_k^{-1} \,, \tag{14}$$

$$\beta(\boldsymbol{\nu}) = \alpha_s - 1 + n - a_s + \sum_{k=1}^{s-1} t_k \,. \tag{15}$$

Proof. By direct substitution of (5) and (6) in (10). □

Proposition 2. *The number of terms* $\mathrm{e}^{-x\delta(\boldsymbol{\nu})} x^{\beta(\boldsymbol{\nu})}$ *in* (10) *is upperbounded by* $sM + 2M! + (s-1)(n-2)$.

Proof. By examination of (10). □

Proposition 3. *The coefficients $\delta(\boldsymbol{\nu})$ in (10) can take only two values*

$$\delta(\boldsymbol{\nu}) \in \left\{ \frac{1}{\sigma_1} + \frac{s-1}{\sigma_2}, \frac{s}{\sigma_2} \right\} .$$

Proof. For the proof we recall that $\hat{\sigma}_k = \sigma_1$ only if $\mu_k = 1$ and, therefore, we have only two cases: (1) one coefficient $\hat{\sigma}_k$ is σ_1 and the remaining are σ_2; (2) all the coefficients $\hat{\sigma}_k$ are σ_2. □

Proposition 4. *The coefficient $\beta(\boldsymbol{\nu})$ in (10) is bounded as follows:*

$$s^2 + s(n-M-2) + 1 \le \beta(\boldsymbol{\nu}) \le \frac{s(2M+2n-s-1)}{2} . \tag{16}$$

Proof. For the proof of the upper bound, we start from the r.h.s. of (15) and rewrite it as

$$\alpha_s - 1 + n - a_s + \sum_{k=1}^{s-1}(\alpha_k + n - a_k - 1) = \sum_{k=1}^{s}(\alpha_k - a_k) + s(n-1) . \tag{17}$$

To obtain an upper bound for $\delta(\boldsymbol{\nu})$, we recall that both α_k and a_k range from $1, \ldots, M$ and, therefore,

$$\sum_{k=1}^{s} \alpha_k \ \le \ M + (M-1) + \cdots + (M-s+1) = \frac{s(2M+1-s)}{2} , \tag{18}$$

$$\sum_{k=1}^{s} a_k \ \ge \ 1 + 2 + \cdots + s = \frac{s(s+1)}{2} \tag{19}$$

and

$$\sum_{k=1}^{s}(\alpha_k - a_k) \le \frac{s(2M-s)}{2} . \tag{20}$$

By applying inequality (20) in (17), we obtain the upper bound in (16). To prove the lower bound, we start from (17) and recall that $\boldsymbol{\mu}$ is a permutation of the set $\{1, 2, \ldots, M\}$, therefore,

$$\sum_{k=1}^{s} \alpha_k \ge 1 + 2 + \cdots + s = \frac{s(s+1)}{2} . \tag{21}$$

Furthermore, the values assumed by a_k are given by (9) and, as a consequence,

$$\sum_{k=1}^{s} a_k \le (M-s+2) + (M-s+3) + \cdots + M + M \tag{22}$$

$$= M + \frac{M(M+1)}{2} - 1 - \left[\frac{(M-s+1)(M-s+2)}{2} - 1 \right] . \tag{23}$$

By substituting (21) and (22) in (17), we finally get the lower bound in (16). □

Proposition 5. *The expression for* $f_\ell(x)$ *can be reorganized as*

$$f_\ell(x) = \sum_{s=\ell}^{M} \sum_{k=\beta_{s,\min}}^{\beta_{s,\max}} \Psi_{s,k}^{(1)} \mathrm{e}^{-x\left(\frac{1}{\sigma_1}+\frac{s-1}{\hat{\sigma}_2}\right)} x^k + \sum_{s=\ell}^{M} \sum_{k=\beta_{s,\min}}^{\beta_{s,\max}} \Psi_{s,k}^{(2)} \mathrm{e}^{-x\frac{s}{\sigma_2}} x^k \,, \qquad (24)$$

where $\Psi_{s,k}^{(1)}$ *and* $\Psi_{s,k}^{(2)}$ *are suitable constants, and*

$$\beta_{s,\min} = s^2 + s(n - M - 2) + 1 \qquad (25)$$

with

$$\beta_{s,\max} = \frac{s(2M + 2n - s - 1)}{2} \,. \qquad (26)$$

The number of distinct terms of $f_\ell(x)$ *does not exceed* $(\ell - M - 1)(\ell^2 - M(1 + M) - \ell(2 + M))$.

Proof. The proof of (24) is a straigthforward applications of Propositions 2 and 3. The second proof comes from the fact that the number of the inner sums in (24) is

$$\beta_{s,\max} - \beta_{s,\min} + 1 = \frac{(3 + 4M)s - 3s^2 - 2}{2} + 1 \qquad (27)$$

and

$$\sum_{s=\ell}^{M} 2(\beta_{s,\max} - \beta_{s,\min} + 1) = (\ell - M - 1)\left(\ell^2 - M(1 + M) - \ell(2 + M)\right) \,. \qquad (28)$$

□

TABLE I

Examples of p.d.f. expressions for $M = 2, 3$.

M	n	ℓ	$f_\ell(x)$
2	2	1	$e^{-x\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)}\left(-e^{x/\sigma_1}(x-\sigma_1)\sigma_1-{\sigma_1}^2+e^{x/\sigma_2}(x-\sigma_2)\sigma_2+{\sigma_2}^2\right)\Big/(\sigma_1\sigma_2(\sigma_2-\sigma_1))$
2	2	2	$e^{-x\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)}\frac{\sigma_1+\sigma_2}{\sigma_1\sigma_2}$
2	3	1	$e^{-x\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)}\frac{x\left(x{\sigma_1}^2\left(e^{x/\sigma_1}-1\right)+{\sigma_2}^2\left(e^{x/\sigma_2}+1\right)+2\left({\sigma_1}^3\left(e^{x/\sigma_1}-1\right)-{\sigma_2}^3\left(e^{x/\sigma_2}-1\right)\right)\right)}{2{\sigma_1}^2(\sigma_1-\sigma_2){\sigma_2}^2}$
2	3	2	$e^{-x\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)}\frac{x\left(x(\sigma_1+\sigma_2)+2\left({\sigma_1}^2+\sigma_1\sigma_2+{\sigma_2}^2\right)\right)}{2{\sigma_1}^2{\sigma_2}^2}$
3	3	1	$\frac{e^{-x\left(\frac{1}{\sigma_1}+\frac{2}{\sigma_2}\right)}}{2\sigma_1(\sigma_1-\sigma_2)^2{\sigma_2}^3}\Big(2(\sigma_1-\sigma_2)^2{\sigma_2}^2(2\sigma_1+\sigma_2)-2e^{x/\sigma_1}\sigma_1{\sigma_2}^2\left(x^2-2x\sigma_1+2{\sigma_1}^2+3x\sigma_2-3\sigma_1\sigma_2\right)$ $+e^{2x/\sigma_2}{\sigma_2}^3\left(x^2-4x\sigma_2+2{\sigma_2}^2\right)+e^{x\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)}\sigma_1\Big(2\sigma_1(2\sigma_1-3\sigma_2){\sigma_2}^3+x^3(-\sigma_1+\sigma_2)$ $+x^2\left(2{\sigma_1}^2+\sigma_1\sigma_2-4{\sigma_2}^2\right)+2x\sigma_2\left(-2{\sigma_1}^2+\sigma_1\sigma_2+3{\sigma_2}^2\right)\Big)$ $+e^{x/\sigma_2}\Big(x^3\left(-{\sigma_1}^2+{\sigma_2}^2\right)+x^2\left(-2{\sigma_1}^3+3{\sigma_1}^2\sigma_2+{\sigma_2}^3\right)+2x\sigma_2\left(2{\sigma_1}^3-3{\sigma_1}^2\sigma_2+2{\sigma_2}^3\right)$ $-2{\sigma_2}^2\left(2{\sigma_1}^3-3{\sigma_1}^2\sigma_2+s{\sigma_2}^3\right)\Big)\Big)$
3	3	2	$\frac{e^{-x\left(\frac{1}{\sigma_1}+\frac{2}{\sigma_2}\right)}}{2\sigma_1(\sigma_1-\sigma_2)^2{\sigma_2}^3}\Big(-4(\sigma_1-\sigma_2)^2{\sigma_2}^3(2\sigma_1+\sigma_2)+2e^{x/\sigma_1}\sigma_1{\sigma_2}^3\left(x^2-2x\sigma_1+2{\sigma_1}^2+3x\sigma_2-3\sigma_1\sigma_2\right)$ $-e^{x/\sigma_2}\sigma_2\Big(x^3\left(-{\sigma_1}^2+{\sigma_2}^2\right)+x^2\left(-2{\sigma_1}^3+3{\sigma_1}^2\sigma_2+{\sigma_2}^3\right)+2x\sigma_2\left(2{\sigma_1}^3-3{\sigma_1}^2\sigma_2+2{\sigma_2}^3\right)$ $-2{\sigma_2}^2\left(2{\sigma_1}^3-3{\sigma_1}^2\sigma_2+2{\sigma_2}^3\right)\Big)\Big)$
3	3	3	$e^{-x\left(\frac{1}{\sigma_1}+\frac{2}{\sigma_2}\right)}\frac{2\sigma_1+\sigma_2}{\sigma_1\sigma_2}$

Proposition 6. *For $M = n$ the distribution of the smallest eigenvalue becomes*

$$f_M(x) = \left(\frac{1}{\sigma_1} + \frac{M-1}{\sigma_2}\right) \mathrm{e}^{-x\left(\frac{1}{\sigma_1}+\frac{M-1}{\sigma_2}\right)} . \tag{29}$$

Proof. Can be obtained by following the proof in Appendix for $M = n$. □

Note that this is also the specialization to the spiked case of the result given in [53, Th. 2].

The previous propositions prove that the eigenvalues can be interpreted as mixture of gamma distributions, which can be evaluated numerically or symbolically. Some explicit examples of p.d.f.s are shown in Table I for $M = 2$ and $M = 3$.

5. Expectation of functions of the eigenvalues

From (10), we can easily evaluate closed-form expressions for the following cases of interest.

— Mean value of λ_ℓ

$$\mathbb{E}\left\{\lambda_\ell\right\} = \mathcal{K} \sum_{\boldsymbol{\nu}} \varphi(\boldsymbol{\nu}) \frac{(\beta(\boldsymbol{\nu})+1)!}{\delta(\boldsymbol{\nu})^{\beta(\boldsymbol{\nu})+2}} . \tag{30}$$

— Mean value of $1/\lambda_\ell$

$$\mathbb{E}\left\{\frac{1}{\lambda_\ell}\right\} = \mathcal{K} \sum_{\boldsymbol{\nu}} \varphi(\boldsymbol{\nu}) \frac{(\beta(\boldsymbol{\nu})-1)!}{\delta(\boldsymbol{\nu})^{\beta(\boldsymbol{\nu})}} . \tag{31}$$

— Mean value of $\frac{\lambda_\ell}{A+B\lambda_\ell}$

$$\mathbb{E}\left\{\frac{\lambda_\ell}{A+\lambda_\ell B}\right\} = \mathcal{K} \sum_{\boldsymbol{\nu}} \varphi(\boldsymbol{\nu}) \frac{A^{\beta(\boldsymbol{\nu})+1}}{B^{\beta(\boldsymbol{\nu})+2}} \mathrm{e}^{\frac{\delta(\boldsymbol{\nu})A}{B}} \times \Gamma\left(-1-\beta(\boldsymbol{\nu}), \frac{\delta A}{B}\right)(\beta(\boldsymbol{\nu})+1)! \tag{32}$$

Note that, to derive (32), we used the following identity [54, Eq. (3.383.10)]

$$\int_0^\infty \frac{x^{a-1}\mathrm{e}^{-\mathrm{d}x}}{x+c} \mathrm{d}x = c^{a-1}\mathrm{e}^{c\,d}\,(a-1)!\;\Gamma(1-a, c\,d) \tag{33}$$

valid for $a, c, d \in \mathbb{R}$ with $a, c, d > 0$.

— Mean value of $\ln(1 + A\lambda_\ell)$

$$\mathbb{E}\{\ln(1 + A\lambda_\ell)\} = \mathcal{K} \sum_{\boldsymbol{\nu}} \varphi(\boldsymbol{\nu}) \frac{\beta(\boldsymbol{\nu})!}{A^{\beta(\boldsymbol{\nu})+1}} \mathrm{e}^{\delta(\boldsymbol{\nu})/A}$$
$$\times \sum_{k=1}^{\beta(\boldsymbol{\nu})+1} \left[\left(\frac{A}{\delta(\boldsymbol{\nu})} \right)^k \Gamma\left(k - 1 - \beta(\boldsymbol{\nu}), \frac{\delta(\boldsymbol{\nu})}{A}\right) \right]. \tag{34}$$

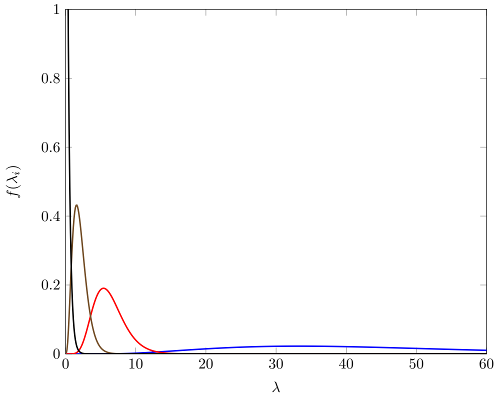
To derive (34), we have used the following identity [55, Eq. (78)]:

$$\int_0^\infty \ln(1 + x) x^{a-1} \mathrm{e}^{-\mathrm{d}x} \mathrm{d}x = (a - 1)!\, \mathrm{e}^d \sum_{k=1}^{a} \frac{\Gamma(-a + k, d)}{d^k} \tag{35}$$

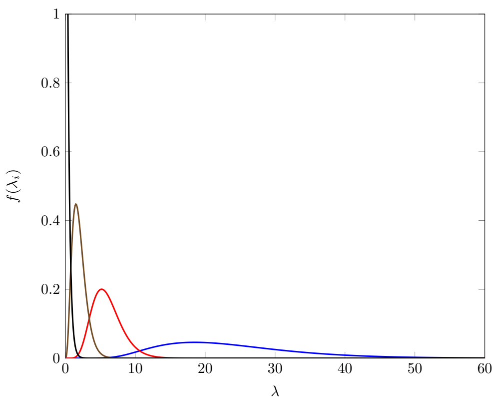
valid for $a, d \in \mathbb{R}$ with $a, d > 0$.

6. Numerical examples

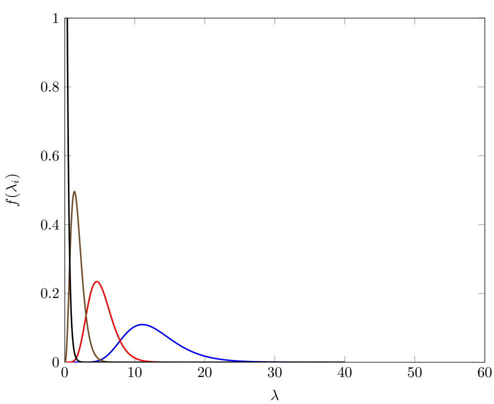
In this section, we present some numerical examples obtained using the expressions derived previously. More specifically, we will show the p.d.f. behavior of all the eigenvalues of a complex Wishart matrix with spiked covariance matrix for different dimensions and values of the coefficients σ_1 and σ_2. Figure 1 shows the p.d.f. of the four eigenvalues for the case of $n = M = 4$ with $\sigma_2 = 1$, and for three different values $\sigma_1 = 2, \sigma_1 = 5$, and $\sigma_1 = 10$. The comparison between the three figures shows that the variation of σ_1 has a negligible impact on the smallest eigenvalues λ_3 and λ_2 but has a considerable impact on the distribution of λ_1; in particular, the distribution of λ_1 tends to widen as the value of σ_1 increases. The same behaviour can be observed for the cases $n = M = 5$ shown in Fig. 2. As expected, the comparison between Fig. 1 and Fig. 2 reveals that the effect of widening of the distribution is emphasized by the size of the matrix. Finally, we report in Fig. 3 the p.d.f. of the eigenvalues for $n = 10$ and $M = 5$, for the same values of σ_1 and σ_2 as in the previous figures. By comparison with Fig. 2, we can see that here the distribution of λ_1 has a larger mean value. These distributions can be used for designing tests on the parameters of the Wishart matrix.



(a) $\sigma_1 = 10$, $\sigma_2 = 1$

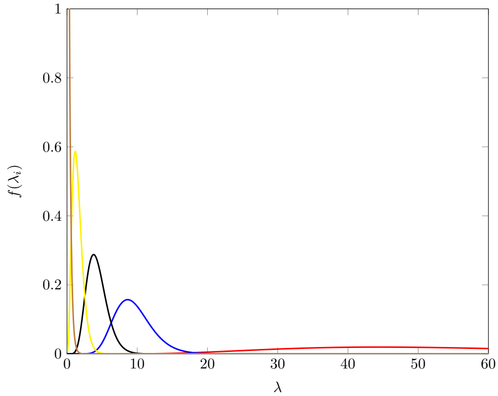


(b) $\sigma_1 = 5$, $\sigma_2 = 1$

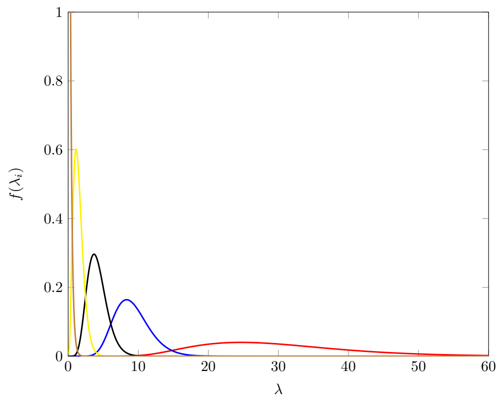


(c) $\sigma_1 = 2$, $\sigma_2 = 1$

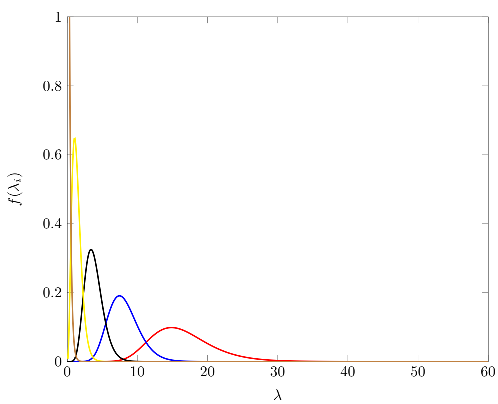
Fig. 1. p.d.f. of the eigenvalues λ_i, $i = 1, \ldots, 4$ for $n = M = 4$.



(a) $\sigma_1 = 10$, $\sigma_2 = 1$

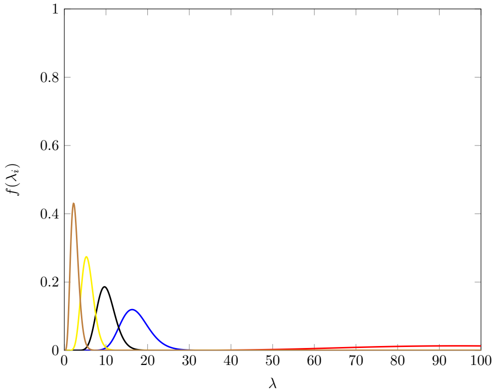


(b) $\sigma_1 = 5$, $\sigma_2 = 1$

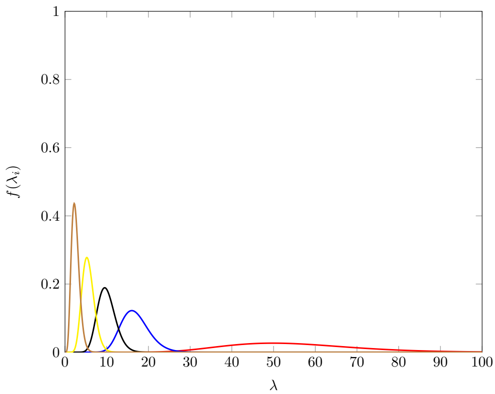


(c) $\sigma_1 = 2$, $\sigma_2 = 1$

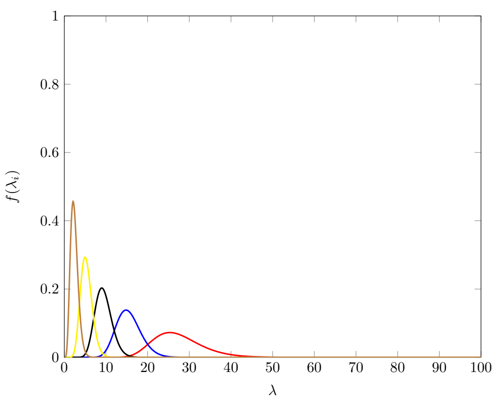
Fig. 2. p.d.f. of the eigenvalues λ_i, $i = 1, \ldots, 5$ for $n = M = 5$.



(a) $\sigma_1 = 10$, $\sigma_2 = 1$



(b) $\sigma_1 = 5$, $\sigma_2 = 1$



(c) $\sigma_1 = 2$, $\sigma_2 = 1$

Fig. 3. p.d.f. of the eigenvalues λ_i, $i = 1, \ldots, 5$ for $n = 10, M = 5$.

7. Conclusions

In this paper, we have considered the problem of the evaluation of the p.d.f. of the eigenvalues of complex central Wishart matrices with spiked covariance matrix. We have provided a formula for the p.d.f. of the $\ell^{\underline{\text{th}}}$ eigenvalue which is expressed as a sum of terms of the form $x^{\beta}\mathrm{e}^{-x\delta}$. This form is particularly useful to calculate a large variety of statistical averages without the need for numerical integration. To this aim, expressions for some expectations of functions of eigenvalues, which find applications in several fields, have been also derived.

Appendix: Proof of Theorem 1

We start the proof from the expression for the joint p.d.f. of the ordered eigenvalues for the spiked covariance case given by (4) and recall that the unordered distribution can be simply written as

$$f_{\boldsymbol{\lambda}}(x_1,\dots,x_M) = \frac{1}{M!} f_{\boldsymbol{\lambda}}^{(\text{ordered})}(x_1,\dots,x_M)\,. \tag{36}$$

Furthermore, we can obtain the joint p.d.f. of a subset of s unordered eigenvalues by integrating out the remaining $M-s$ eigenvalues. For instance, if we consider the subset $\lambda_1,\dots,\lambda_s$ (since they are unordered, the index of the eigenvalues does not have any impact), we get

$$f(x_1,x_2,\dots,x_s) = \frac{K}{M!}\int_0^{\infty}\!\cdots\!\int_0^{\infty} |\boldsymbol{\Upsilon}(\boldsymbol{x})|\cdot|\boldsymbol{E}(\boldsymbol{x},\boldsymbol{\phi})|\prod_{k=1}^{M} x_k^{n-M}\mathrm{d}x_{s+1}\dots\mathrm{d}x_M\,. \tag{37}$$

Using the Leibniz formula for the determinants, we obtain

$$\begin{aligned}
f(x_1,x_2,\dots,x_s) &= \frac{K}{M!}\sum_{\boldsymbol{\alpha}}\sum_{\boldsymbol{\mu}}\mathrm{sgn}(\boldsymbol{\alpha})\mathrm{sgn}(\boldsymbol{\mu})\int_0^{\infty}\cdots\int_0^{\infty}\prod_{k=1}^{M}\upsilon_{\alpha_k}(x_k)e_{\mu_k,k} \\
&\quad\times\prod_{k=1}^{M}x_k^{n-M}\mathrm{d}x_{s+1}\dots\mathrm{d}x_M \\
&= \frac{K}{M!}\sum_{\boldsymbol{\alpha}}\sum_{\boldsymbol{\mu}}\mathrm{sgn}(\boldsymbol{\alpha})\mathrm{sgn}(\boldsymbol{\mu})\int_0^{\infty}\cdots\int_0^{\infty}\prod_{k=1}^{M}\upsilon_{\alpha_k}(x_k)\bar{e}_{\mu_k}(x_k) \\
&\quad\times\prod_{k=1}^{M}x_k^{n-M}\mathrm{d}x_{s+1}\dots\mathrm{d}x_M \\
&= \frac{K}{M!}\sum_{\boldsymbol{\alpha},\boldsymbol{\mu}}\mathcal{C}(\boldsymbol{\alpha},\boldsymbol{\mu})\prod_{k=1}^{s}\upsilon_{\alpha_k}(x_k)\bar{e}_{\mu_k}(x_k)x_k^{n-M}\,,
\end{aligned} \tag{38}$$

where we have exploited the fact that

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \prod_{k=s+1}^{M} \upsilon_{\alpha_k}(x_k)\bar{e}_{\mu_k}(x_k)x_k^{n-M}\mathrm{d}x_{s-1}\cdots\mathrm{d}x_M \\
&= \prod_{k=s+1}^{M} \int_0^\infty \upsilon_{\alpha_k}(x)\bar{e}_{\mu_k}(x)x^{n-M}\mathrm{d}x \\
&= \prod_{k=s+1}^{M} (-1)^{\alpha_k-1} \begin{cases} \int_0^\infty x^{\alpha_k-1+n-M}\mathrm{e}^{-x/\sigma_1}\mathrm{d}x\,, & \mu_k = 1\,, \\ \int_0^\infty x^{\alpha_k-1+n-\mu_k}\mathrm{e}^{-x/\sigma_2}\mathrm{d}x\,, & \mu_k > 1\,, \end{cases} \\
&= \prod_{k=s+1}^{M} (-1)^{\alpha_k-1} \begin{cases} \sigma_1^{\alpha_k+n-M}(\alpha_k+n-M-1)! & \mu_k = 1\,, \\ \sigma_2^{\alpha_k+n-\mu_k}(\alpha_k+n-\mu_k-1)! & \mu_k > 1\,, \end{cases}
\end{aligned} \tag{39}$$

where

$$\bar{e}_{i,j} = \bar{e}_i(x_j) \triangleq \begin{cases} \mathrm{e}^{-x_j/\sigma_1}\,, & i = 1\,, \\ x_j^{M-i}\mathrm{e}^{-x_j/\sigma_2}\,, & i > 1\,, \end{cases} \tag{40}$$

and $\sum_{\boldsymbol{\alpha},\boldsymbol{\mu}} \mathcal{C}(\boldsymbol{\alpha},\boldsymbol{\mu})$ is defined in (7).

Once $f(x_1, x_2, \ldots, x_s)$ has been derived, to obtain $f_{\min:s:M}(x)$, we start with (38) by recalling that

$$\begin{aligned}
f_{\min:s:M}(x_s) &= \int_{x_s}^\infty \cdots \int_{x_2}^\infty f^{(\text{ordered})}(x_1, x_2, \ldots, x_s)\mathrm{d}x_1 \ldots \mathrm{d}x_{s-1} \\
&= \frac{1}{(s-1)!} \int_{x_s}^\infty \cdots \int_{x_s}^\infty f^{(\text{ordered})}(x_1, x_2, \ldots, x_s)\mathrm{d}x_1 \ldots \mathrm{d}x_{s-1} \\
&= s \int_{x_s}^\infty \cdots \int_{x_s}^\infty f(x_1, x_2, \ldots, x_s)\mathrm{d}x_1 \ldots \mathrm{d}x_{s-1}
\end{aligned} \tag{41}$$

and, consequently,

$$f_{\min:s:M}(x) = \frac{sK}{M!} \sum_{\boldsymbol{\alpha},\boldsymbol{\mu}} \mathcal{C}(\boldsymbol{\alpha},\boldsymbol{\mu}) \upsilon_{\alpha_s}(x) \bar{e}_{\mu_s}(x) x^{n-M}$$

$$\times \int_x^\infty \cdots \int_x^\infty \prod_{k=1}^{s-1} \upsilon_{\alpha_k}(x_k) \bar{e}_{\mu_k}(x_k) x_k^{n-M} \mathrm{d}x_1 \ldots \mathrm{d}x_{s-1}$$

$$= \frac{sK}{M!} \sum_{\boldsymbol{\alpha},\boldsymbol{\mu}} \mathcal{C}(\boldsymbol{\alpha},\boldsymbol{\mu}) \upsilon_{\alpha_s}(x) \bar{e}_{\mu_s}(x) x^{n-M} \prod_{k=1}^{s-1} \int_x^\infty \upsilon_{\alpha_k}(y)\, \bar{e}_{\mu_k}(y) y^{n-M} \mathrm{d}y\,. \quad (42)$$

By using (40) in (42), the integral in (42) can be written as

$$\int_x^\infty \upsilon_{\alpha_k}(y)\, \bar{e}_{\mu_k}(y) \mathrm{d}y = \hat{\sigma}_k^{\alpha_k+n-a_k} \Gamma\left(\alpha_k + n - a_k, \frac{x}{\hat{\sigma}_k}\right), \quad (43)$$

where $\Gamma(\cdot,\cdot)$ is the upper incomplete Gamma function, $\hat{\sigma}_k$ and a_k are defined in (8) and (9), respectively.

By substituting (43) in (42), we get

$$f_{\min:s:M}(x) = \frac{sK}{M!} \sum_{\boldsymbol{\alpha},\boldsymbol{\mu}} \mathcal{C}(\boldsymbol{\alpha},\boldsymbol{\mu}) (-1)^{\alpha_s-1} x^{\alpha_s-1+n-a_s} \mathrm{e}^{-x/\hat{\sigma}_s}$$

$$\times \left[\prod_{k=1}^{s-1} \hat{\sigma}_k^{\alpha_k+n-a_k} \Gamma\left(\alpha_k+n-a_k, \frac{x}{\hat{\sigma}_k}\right)\right]. \quad (44)$$

The product in (44) can be further manipulated by recalling that

$$\Gamma(s,x) = (s-1)!\,\mathrm{e}^{-x} \sum_{k=0}^{s-1} \frac{x^k}{k!},$$

so that the product becomes

$$\left(\prod_{k=1}^{s-1} \hat{\sigma}_k^{\alpha_k+n-a_k} (\alpha_k+n-a_k-1)!\, \mathrm{e}^{-x/\hat{\sigma}_k}\right) \left[\prod_{k=1}^{s-1} \sum_{t_k=0}^{\alpha_k+n-a_k-1} \frac{(x/\hat{\sigma}_k)^{t_k}}{t_k!}\right] =$$

$$= \left(\prod_{k=1}^{s-1} \hat{\sigma}_k^{\alpha_k+n-a_k} (\alpha_k+n-a_k-1)!\, \mathrm{e}^{-x/\hat{\sigma}_k}\right) \sum_{t_1=0}^{\alpha_1+n-a_1-1} \cdots \sum_{t_{s-1}=0}^{\alpha_{s-1}+n-a_{s-1}-1} \prod_{k=1}^{s-1} \frac{(x/\hat{\sigma}_k)^{t_k}}{t_k!}$$

$$= \mathrm{e}^{-x\sum_{k=1}^{s-1}\hat{\sigma}_k^{-1}} \sum_{t_1=0}^{\alpha_1+n-a_1-1} \cdots \sum_{t_{s-1}=0}^{\alpha_{s-1}+n-a_{s-1}-1} \left(x^{\sum_{k=1}^{s-1} t_k} \prod_{k=1}^{s-1} \frac{\hat{\sigma}_k^{\alpha_k+n-a_k-t_k} (\alpha_k+n-a_k-1)!}{t_k!}\right)$$

$$(45)$$

and we finally obtain (6).

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