# BALLISTIC LÉVY WALK WITH RESTS: ESCAPE FROM A BOUNDED DOMAIN

A. Kamińska, T. Srokowski

Institute of Nuclear Physics Polish Academy of Sciences, 31-342 Kraków, Poland

*(Received January 13, 2020; accepted May 11, 2020)*

The Lévy walk model that takes into account a waiting of a walker between consecutive displacements is analysed. The motion is restricted to a finite region, bounded by two absorbing barriers, and quantities describing the escape from this region are determined. Simple expression for a mean first passage time is derived for a ballistic version of the Lévy walk. Two limits emerge from the model: of short waiting time, that corresponds to Lévy walks without rests, and long waiting time which exhibits properties of a Lévy flights model. The analytical results are compared with Monte Carlo trajectory simulations.

DOI:10.5506/APhysPolB.51.1773

## 1. Introduction

The Lévy walk model [1–5] lets a walker move with a finite velocity, in contrast to a Lévy flights model when displacements are instantaneous and the moments of a position density distribution are infinite due to long tails of a jump-size density. More precisely, the latter process consists of a sequence of jumps with infinite velocity and the walker rests between consecutive jumps for a random time. The parameter $\alpha$ in a time-of-flight distribution of the Lévy walk, $\tau^{-1-\alpha}$, singles out two qualitatively different processes: when $0 < \alpha < 1$ and $1 < \alpha < 2$. In the first case, the mean time of flight diverges resulting in a ballistic diffusion: the mean-squared displacement $\langle x^2(t)\rangle$ rises with time as $t^2$. The diffusion is weaker for $1 < \alpha < 2$ [4], $\langle x^2(t)\rangle \sim t^{3-\alpha}$, while $\langle x^2(t)\rangle \sim t\ln t$ for $\alpha = 2$; for $\alpha > 2$, diffusion becomes normal, $\langle x^2(t)\rangle \sim t$. The case of $\alpha = 1$ is a transition point between enhanced and ballistic diffusion for which $\langle x^2(t)\rangle \sim t^2/\ln t$ holds. In the Lévy walk model, the subsequent time intervals the walker spends in flight are mutually independent but taking into account that the time of flight is finite introduces memory; as a consequence, the process is semi-Markovian [6]. Processes characterised by $\alpha$ from the lower interval are discussed in context of such phenomena as some properties of nanocristals [7] and blinking quantum dots [8]. The Lévy walk model usually assumes that a new jump takes place immediately after finishing the previous one. However, it is natural to expect that the walker may rest between consecutive jumps and then a finite waiting time has to be included in the model [9–12]. Though this version of the Lévy walk model is highly realistic, it is rarely discussed. If walker moves in a nonhomogeneous environment, the distribution of the waiting time may be position-dependent [13, 14].

The aim of this paper is to study one-dimensional ballistic Lévy walks ($\alpha < 1$) restricted to a finite interval by two absorbing barriers. The quantities that characterise the escape from a bounded domain are often discussed and applied in many physical problems [15]. One asks about a time required to reach the barrier for the first time (a first passage time) and its mean $T$ (MFPT) which, if exists, provides a simple estimation of the escape rate.

The first passage time problem is important in the modelling of animal movements, like search strategies of predators. In this case, the search time is the time taken for a predator to find a prey and the first passage time corresponds to arriving at a prey location. Then the equation for mean first passage time can be interpreted as the average time taken by animals beginning at the same start location to reach the fixed location [17]. The process of a mutual interaction between predator and pray, which includes a local pray density, a consumption rate of predators and a searching time, was formalised by the functional response method in terms of a Holling disc equation [18]. The simple random-walk models are unable to take into account that animal paths involve large spatial or temporal scales that turn out to be a combination of walk clusters with long travels between them, whose pattern corresponds to the Lévy walks; they allow a higher efficiency in random search scenarios possessing such fundamental properties as superdiffusivity and scale invariance [19].

The properties of the escape process change after substituting instantaneous jumps by walks with a finite velocity which effect is especially pronounced if $\alpha < 1$: the numerical analysis [16], performed for the Lévy walks without rests, demonstrates, in particular, that MFPT scales with the barrier position as $L$, while for the Lévy flights, $T \propto L^\alpha$ holds [16, 20]. In this paper, we derive expressions for the first passage time characteristics taking into account a finite and random waiting time between consecutive displacements. In Section 2, we define the Lévy walk process with rests in the presence of the absorbing barriers. The density distribution describing that process is derived and the first passage time statistics is evaluated in Section 3.

## 2. Definition of the process

The Lévy walk trajectory consists of a sequence of displacements where the walker moves with a constant velocity $v$. Before the next jump, a new direction is chosen: walker may depart to the left or to the right with the same probability. The time of a single flight, $\tau$, is a random variable determined by a density distribution $\psi(\tau)$ which is one-sided and has the asymptotics $\tau^{-1-\alpha}$, where $0<\alpha<1$. That power-law tail corresponds to the Laplace transform

$$\psi(s) = 1 - c_1 s^{\alpha}\,, \tag{1}$$

where $c_1 = \text{const}$. More precisely, we assume the following form of $\psi(\tau)$:

$$\psi(\tau) = \begin{cases} \alpha\epsilon^{\alpha}\tau^{-1-\alpha} & \text{for } \tau > \epsilon \\ 0 & \text{for } \tau \le \epsilon \end{cases}\,, \tag{2}$$

where $\epsilon = \text{const.}$ means the smallest duration of a single jump. Taking the Laplace transform from Eq. (2) and comparing the result with Eq. (1) yields $c_1$

$$c_1 = \lim_{s\to 0}[s^{-\alpha} - \alpha\epsilon^{\alpha}\Gamma(-\alpha,\epsilon s)] = \epsilon^{\alpha}\Gamma(1-\alpha)\,, \tag{3}$$

where we applied the expansion of an incomplete Gamma function, $\Gamma(a,b) = \Gamma(a) - b^a/a + b^{a+1}/(a+1) + \ldots$ [21]. Since the travelled distance $\xi$ is determined by $\tau$, both quantities are coupled in the jump density distribution

$$\bar{\psi}(\xi,\tau) = \tfrac{1}{2}\delta(|\xi| - v\tau)\psi(\tau)\,. \tag{4}$$

After walker ends its jump, and before the next direction and new time $\tau$ are sampled, it remains at rest. The resting time is a random quantity and follows from the exponential distribution with a rate $\nu$, then the mean waiting time is $1/\nu$. Both phases of the motion, namely of particles in flight and in rest, are quantified by two density distributions: $p_{\mathrm{v}}(x,t)$ and $p_{\mathrm{r}}(x,t)$, respectively. The total density, $p(x,t) = p_{\mathrm{r}}(x,t) + p_{\mathrm{v}}(x,t)$, is normalised to unity but the contribution of individual phases to the total probability may change with time: for $\alpha < 1$, $p_{\mathrm{r}}(x,t)$ decays and the flying phase prevails at long time.

The master equation can be constructed from an infinitesimal transition probability [13]. Let us assume that the particle rests in $x'$ at time $t$. Within a small time interval $\Delta t$, it may either continue its resting at $x = x'$ or moves on performing a flight for $t'$ which is determined by $\psi(t')$. Then the transition probability corresponds to a transition from $x' \to x$ and it is infinitesimal in respect of waiting time $\Delta t$. The transition probability reads

$$\begin{aligned} p_{\mathrm{tr}}\left(x, t+\Delta t | x', t\right) &= [1-\nu\Delta t]\delta\left(x-x'\right)\delta\left(|x-x'| - vt'\right) \\ &\quad + \nu\Delta t \tfrac{1}{2}\psi\left(t'\right)\delta\left(|x-x'| - vt'\right)\,, \end{aligned} \tag{5}$$

and the density distribution resulting from Eq. (5), $p_{\rm r}(x,t)$, corresponds to the walks terminating at $x$. The multiplication of Eq. (5) by a probability of the condition and integration over all possible $x'$ and $t'$ yields

$$p_{\rm r}(x,t+\Delta t)=\int_0^t\int p_{\rm tr}\left(x,t-t'+\Delta t|x',t-t'\right)p_{\rm r}\left(x',t-t'\right){\rm d}t'{\rm d}x'\,. \tag{6}$$

Passing to the limit of small $\Delta t$,

$$\frac{\partial}{\partial t}p_{\rm r}(x,t)=\lim_{\Delta t\to 0}\left[p_{\rm r}(x,t+\Delta t)-p_{\rm r}(x,t)\right]/\Delta t \tag{7}$$

yields the master equation

$$\frac{\partial}{\partial t}p_{\rm r}(x,t)=-\nu p_{\rm r}(x,t)+\nu\int_0^t\int p_{\rm r}\left(x',t-t'\right)\frac{1}{2}\psi\left(t'\right)\delta\left(|x-x'|-vt'\right){\rm d}t'{\rm d}x'\,. \tag{8}$$

Moreover, we have to take into account the particles that at $x$ are still in flight. First, let us evaluate a probability density that the particle remains in flight at $t$ and at a position $x$ under a condition that the latter jump started at $x'$ and at a time in the interval $(0,t)$. We divide this interval into $n$ small subintervals of length $\Delta t$, and assume that particle started at $t_i$. The probability density that this particle arrives at $x$ still being in flight is $\Psi(t-t_i)\delta(|x-x'|-v(t-t_i))$, where $\Psi(t)=\int_t^\infty\psi(t'){\rm d}t'$. The summation over all the time subintervals, multiplication by a probability that particle remains at any $x'$ at time $t_i$ and taking the limit $\Delta t\to 0$, yields the density of particles in flight

$$p_{\rm v}(x,t)=\nu\int\int_0^t\Psi\left(t'\right)\delta\left(|x-x'|-vt'\right)p_{\rm r}\left(x',t-t'\right){\rm d}x'{\rm d}t'\,. \tag{9}$$

We assume that the motion is restricted to the interval $(-L,L)$ by introducing absorbing barriers at $\pm L$ which means boundary conditions

$$p(\pm L,t)=0\,. \tag{10}$$

The first passage time density distribution is defined as a probability that the time needed to reach the barrier for the first time lies within the interval $(t,t+{\rm d}t)$ [15]. The survival probability, namely the probability that the particle never reached those barriers up to time $t$, is given by

$$S(t)=\int_{-L}^{L}p(x,t){\rm d}x\,. \tag{11}$$

The first passage time density distribution reflects the change of the survival probability with time

$$\wp(t) = -\mathrm{d}S(t)/\mathrm{d}t\,, \tag{12}$$

and MFPT is given by the integral

$$T = \int_0^\infty t\wp(t)\mathrm{d}t = \int_0^\infty S(t)\mathrm{d}t\,. \tag{13}$$

## 3. Fractional equations and mean first passage time

The analysis of the first passage time characteristics requires a differential equation, instead of the integral equation (8), for which the boundary conditions (10) can be applied. The derivation of such an equation is only possible in a limit of small arguments of the Fourier and Laplace transform, $k$ and $s$. However, the limit $\{k,s\} \to \{0,0\}$ is not unique and the order of taking the limit over $k$ and $s$ may influence the final density distribution and fluctuations. In particular, the proper asymptotics of the density distributions in the Lévy walk model (without rests) is achieved when the limits $s \to 0$ and $k \to 0$ are taken simultaneously [22] but this procedure does not apply for our case characterised by the divergent mean time of flight. Alternatively, passing $k \to 0$ for a given (small) $s$, one can reproduce the behaviour of the density close to the origin [4]; this procedure does not lead to a diffusion equation and a mean square displacement cannot be determined. They can be determined if one first assumes a given (small) value of $k$ and next takes the limit $s \to 0$. Though the density distributions obtained by taking a specific limit for $s$ and $k$ may not coincide with exact solutions of the master equation, the agreement for MFPT may still be achieved since it is only sensitive on the Laplace transform from $\wp(t)$ near $s = 0$: $T = -\frac{\mathrm{d}}{\mathrm{d}s}\wp(s=0)$. In the following, we apply the latter procedure for the limit $\{k,s\} \to \{0,0\}$ and demonstrate that then the resulting MFPT is consistent with numerical simulations.

Accordingly, after taking the Fourier and Laplace transform from (8) and keeping the lowest terms in the expansion in powers of $k$ and $s$, the equation for $p_{\mathrm{r}}(x,t)$ reads [14]

$$sp_{\mathrm{r}}(k,s) - P_0(k) = -c_1\nu\left[s^\alpha + Bv^2k^2s^{\alpha-2}\right]p_{\mathrm{r}}(k,s)\,, \tag{14}$$

where $B = \alpha(1-\alpha)/2$ and $P_0(x)$ stands for an initial condition. The expression determining the density of particles in flight follows from Eq. (9); the application of the Laplace transform yields

$$p_{\mathrm{v}}(k,s) = c_1\nu\left[s^{\alpha-1} - s^{\alpha-3}\tfrac{1}{2}(1-\alpha)(2-\alpha)v^2k^2\right]p_{\mathrm{r}}(k,s)\,. \tag{15}$$

The inversion of Eq. (15) reads

$$p_{\mathrm{v}}(x,t) = c_1\nu\left[{}_0D_t^{\alpha-1} + \frac{v^2}{2}(1-\alpha)(2-\alpha){}_0D_t^{\alpha-3}\frac{\partial^2}{\partial x^2}\right]p_r(x,t)\,, \tag{16}$$

which is a fractional equation [23] and involves a fractional Riemann–Liouville integral defined as [24]

$${}_0D_t^{-\beta}f(t) = \frac{1}{\Gamma(\beta)}\int_0^t \mathrm{d}t'\frac{f\left(t'\right)}{\left(t-t'\right)^{1-\beta}}\,, \tag{17}$$

where $\beta > 0$. Note that the superscript in the above operator is negative which differentiates this definition from a fractional differential operator. Since $\delta'(x)$ is an odd function, the time derivative from $p_{\mathrm{v}}(x,t)$ can be evaluated from Eq. (9)

$$\begin{aligned}\frac{\partial p_{\mathrm{v}}(x,t)}{\partial t} &= \nu\int\int_0^t \Psi\left(t'\right)\delta'\left(|x-x'|-vt'\right)p_{\mathrm{r}}\left(x',t-t'\right)\mathrm{d}x'\mathrm{d}t'\\ &= -\nu\int\int_0^t \Psi\left(t'\right)\delta\left(|x-x'|-vt'\right)\partial p_{\mathrm{r}}\left(x',t-t'\right)/\partial t'\mathrm{d}x'\mathrm{d}t'\,. \end{aligned}\tag{18}$$

Passing to the limit of small $s$ yields a Poisson equation

$$\frac{\partial p_{\mathrm{v}}(x,t)}{\partial t} = -c_1\nu\left[\frac{\partial^2}{\partial t^2} + \frac{v^2}{2}(1-\alpha)(2-\alpha)\frac{\partial^2}{\partial x^2}\right]{}_0D_t^{\alpha-2}p_{\mathrm{r}}(x,t)\,, \tag{19}$$

for an unknown function ${}_0D_t^{\alpha-2}p_{\mathrm{r}}(x,t)$ where l.h.s. is regarded as a source. Then Eq. (19) is a partial differential equation, which will be solved with given initial and boundary conditions, and the time evolution of the total density $p(x,t)$ can be determined from the expression

$$\frac{\partial p(x,t)}{\partial t} = c_1v^2\nu(1-\alpha)\frac{\partial^2}{\partial x^2}{}_0D_t^{\alpha-2}p_r(x,t) \tag{20}$$

that results from the combining (14) with (19).

Equation (19) will be solved by a variable separation and evaluating eigenfunctions corresponding to both variables. The expansion of the densities reads

$$p_{\mathrm{r}}(x,t) = \sum_{n=0}^{\infty}X_n(x)T_n(t) \tag{21}$$

and

$$p_{\mathrm{v}}(x,t) = \sum_{n=0}^{\infty} X_n^{(v)}(x) T_n^{(v)}(t)\,. \tag{22}$$

Inserting (21) and (22) into (19) yields for each $n$

$$-\frac{\mathrm{d}T_n^{(v)}(t)}{\mathrm{d}t} X_n^{(v)}(x) = \nu c_1 \left[\frac{\partial^2}{\partial t^2} + \frac{1}{2}(2-\alpha)(1-\alpha)v^2 \frac{\partial^2}{\partial x^2}\right] {}_0D_t^{\alpha-2}[X_n(x)T_n(t)]\,, \tag{23}$$

and the separation of variables produces an equation that determines the eigenfunctions $X_n(x)$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X_n(x) + \lambda_n X_n(x) = 0\,. \tag{24}$$

Equation (19) can only be solved if the eigenfunctions $X_n(x)$ are of the same form as those corresponding to the term of nonhomogeneity that contains the functions $X_n^{(v)}(x)$. More precisely, there are two possibilities: either (a) $X_n^{(v)}(x) = -X_n(x)$ or (b) $X_n^{(v)}(x) = X_n(x)$ and, for version (a), Eq. (19) yields

$$\frac{\mathrm{d}T_n^{(v)}(t)}{\mathrm{d}t} = \nu c_1 \left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{1}{2}(2-\alpha)(1-\alpha)v^2\lambda_n\right] {}_0D_t^{\alpha-2} T_n(t)\,. \tag{25}$$

Next, we evaluate the intensities of both phases of the motion, $\phi_r(t) = \int p_{\mathrm{r}}(x,t)\mathrm{d}x$ and $\phi_v(t) = \int p_{\mathrm{v}}(x,t)\mathrm{d}x$, from Eq. (8) and Eq. (18) by the integration over $x$ of the convolutions in those equations. Taking the Laplace transform and dropping higher terms in the expansion in the fractional powers of $s$ yields expressions which, after inverting the transforms, read

$$\phi_r'(t) = \phi_v'(t) = -\nu c_1 {}_0D_t^{\alpha} \phi_r(t)\,. \tag{26}$$

The inserting into Eq. (25) produces the equation

$$\sum_{n=0}^{\infty} \phi_n \frac{\mathrm{d}T_n^{(v)}(t)}{\mathrm{d}t} = -\nu c_1 {}_0D_t^{\alpha} \sum_{n=0}^{\infty} \phi_n T_n(t)\,, \tag{27}$$

where $\phi_n = -\int X_n(x)\mathrm{d}x$. Finally, we insert Eq. (25) into the above equation and, since it has to be satisfied for any choice of the basis functions $X_n(x)$, we obtain for any $n$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} {}_0D_t^{\alpha-2} T_n(t) - C^2 \lambda_n {}_0D_t^{\alpha-2} T_n(t) = 0\,, \tag{28}$$

where $C = v\sqrt{(1-\alpha)(2-\alpha)}/2$.

Version (b) does not apply since it leads to unphysical results. Indeed, the counterpart of Eq. (25) reads

$$-\frac{\partial T_n^{(v)}(t)}{\partial t} = \nu c_1 \left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{1}{2}(2-\alpha)(1-\alpha)v^2\lambda_n\right] {}_0D_t^{\alpha-2}T_n(t)\,, \tag{29}$$

and then $\lambda_n {}_0D_t^{\alpha-2}T_n(t) = 0$ which, according to Eq. (20), would mean a stationary state. Therefore, we continue with version (a).

We solve Eq. (24) with the boundary conditions (10). They imply $X_n(-L) = X_n(L) = 0$ yielding the solution in the form of

$$X_n(x) = \cos\left(\sqrt{\lambda_n}x\right)\,, \tag{30}$$

where the eigenvalues $\lambda_n = \pi^2 n^2/4L^2$ ($n = 1, 3, 5, \ldots$). Inserting these eigenvalues into Eq. (28) and solving the equation yields

$${}_0D_t^{\alpha-2}T_n(t) = a_n \mathrm{e}^{-C\sqrt{\lambda_n}t} + b_n \mathrm{e}^{C\sqrt{\lambda_n}t}\,, \tag{31}$$

and the solution of Eq. (23) is

$$\begin{aligned} {}_0D_t^{\alpha-2}[X_n(x)T_n(t)] \;=\; & \left[a_{2n+1}\exp\left(-\frac{C\pi(2n+1)t}{2L}\right)\right. \\ & \left. + b_{2n+1}\exp\left(\frac{C\pi(2n+1)t}{2L}\right)\right]\cos\frac{\pi(2n+1)x}{2L}\,. \end{aligned} \tag{32}$$

To evaluate the total density $p(x,t)$, we sum the above result over $n$ and insert into Eq. (20)

$$\begin{aligned} p(x,t) \;=\; & -\frac{c_1\nu}{2}\sqrt{\frac{1-\alpha}{2-\alpha}}\sum_{n=0}^{\infty}\left[a'_{2n+1}\exp\left(-\frac{C\pi(2n+1)t}{2L}\right)\right. \\ & \left. + b'_{2n+1}\exp\left(\frac{C\pi(2n+1)t}{2L}\right)\right]\cos\frac{\pi(2n+1)x}{2L}\,, \end{aligned} \tag{33}$$

where the new coefficients $a'_{2n+1}$ and $b'_{2n+1}$ can be determined from the following conditions: $p(x,\infty) = 0$, which implies $b'_{2n+1} = 0$, and the initial condition $p(x,0) = \delta(x)$, which, after taking into account the orthonormality of the cosine function, implies $a'_{2n+1} = -\frac{4}{Lc_1\nu}\sqrt{\frac{2-\alpha}{1-\alpha}}$. The final expression for the total density reads

$$p(x,t) = \frac{1}{L}\sum_{n=0}^{\infty}\exp\left(-\frac{C\pi(2n+1)t}{2L}\right)\cos\frac{\pi(2n+1)x}{2L}\,. \tag{34}$$

The survival probability follows from a direct integration of $p(x,t)$ over $x$

$$
\begin{aligned}
S(t) &= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\exp\left(-\frac{C\pi(2n+1)t}{2L}\right) \\
&= 1-\frac{4}{\pi}\arctan\left[\tanh\left(\frac{C\pi t}{4L}\right)\right], \qquad (35)
\end{aligned}
$$

and we conclude from the series in Eq. (35) that the decay pattern at large time is exponential. This form of relaxation may be attributed to the semi-Markovian property, which is characterised by a fast memory loss [6], of the Lévy walk process. The integration of $S(t)$ yields MFPT

$$
T = \frac{8L}{C\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = \frac{16L}{v\pi^2}\frac{\mathcal{G}}{\sqrt{(1-\alpha)(2-\alpha)}}\,, \qquad (36)
$$

where $\mathcal{G} = 0.916\ldots$ is a Catalan constant [21].

On the other hand, the density distributions can be obtained from a numerical simulation of individual trajectories by sampling the waiting time from the exponential distribution with the rate $\nu$ and the time of flight from the power-law distribution, according to Eq. (2). The analytically evaluated MFPT, Eq. (36), is compared with those calculations in Fig. 1 where the dependence $T \propto L$, well-known for the Lévy walk process without rests [16], is illustrated. Both results perfectly agree if $\alpha$ is small but for $\alpha = 0.8$, the curves are mutually shifted. The discrepancies at small $L$ are natural since then the condition $s \to 0$ is not satisfied. Moreover, they can be attributed to a relatively short time the walker spends in flight whose effect depends on $\alpha$: for large $\alpha$, the steps are short and the number of rests becomes large. Therefore, Eq. (36) better corresponds to numerical simulations if one reduces the waiting time compared to the time of flight by increasing the parameter $\nu$; Fig. 1 shows that a reasonable agreement has been achieved for $\nu = 100$ ($\alpha = 0.8$). On the other hand, Fig. 1 shows that one observes scaling $T \propto L^{\alpha}$ the case of which corresponds to the Lévy flight process [20, 25]. Then the resting phase, which is characterised by different scaling pattern than the flying phase, becomes essential for the process properties. The limit $\nu \to 0$ means a long waiting time compared to the time of flight. The relation $T \propto L$ still holds for small $\nu$ but this requires much larger values of $L$ than those in the figure.

MFPT as a function of $\alpha$ is presented in Fig. 2. For $\nu = 1$, Eq. (36) agrees with simulations at small $\alpha$, while in the region close to $\alpha = 1$ substantial discrepancies emerge. They diminish after taking the limit $\nu \to \infty$, similarly to the dependence presented in Fig. 1. Note that this limit corresponds to the Lévy walk process without rests.

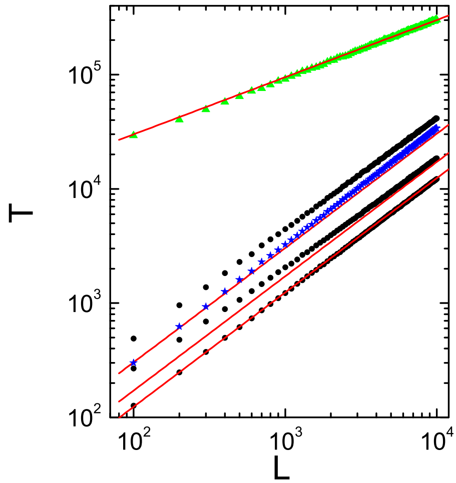


Fig. 1. (Colour on-line) MFPT as a function of barrier position for $\alpha = 0.2$, 0.5 and 0.8, calculated from trajectory simulations with $\nu = 1$ (black points from bottom to top, respectively). Blue stars correspond to the case of $\alpha = 0.8$ and $\nu = 100$. The red solid lines mark the result calculated from Eq. (36). Curve presented as green triangles was obtained from simulations with $\nu = 10^{-3}$; it has a shape $c_L L^\alpha$, marked by the upper red solid line, where a constant $c_L$ was adjusted to fit simulation data.

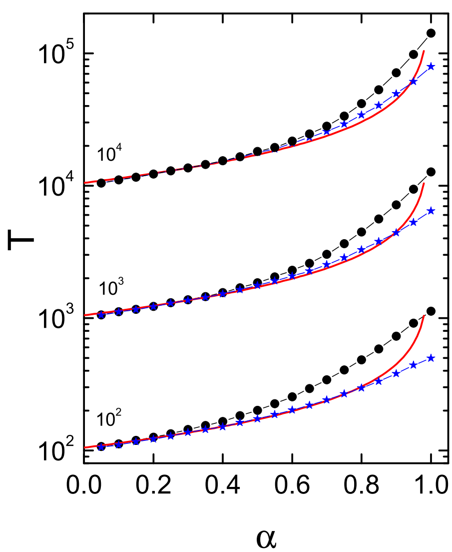


Fig. 2. (Colour on-line) MFPT as a function of $\alpha$ for $\nu = 1$ (black points) and $\nu = 100$ (blue stars). Solid red lines mark the dependence (36). The barrier position $L$ is indicated for each bunch of the curves.

## 4. Summary and conclusions

We have discussed the Lévy walk model that includes finite and random waiting times between consecutive velocity renewals. The motion is restricted to a finite interval by two absorbing barriers; we have derived time characteristics of the escape process from that interval. The combined density distribution for flights and rests, satisfying boundary conditions at barrier positions $\pm L$, has been evaluated by using an auxiliary equation in the form of the Poisson equation. The simple expression for MFPT has been obtained and dependences on $L$ and $\alpha$ established. The mean waiting time, $1/\nu$, that enters the model as a parameter, establishes the relative duration of resting and moving. Therefore, we may observe, as limiting cases, both the Lévy walks without rests (large $\nu$) and the Lévy flights ($\nu \to 0$) when the time of flight needed to reach the barrier becomes negligible compared to the resting time. In contrast to Eq. (36) that predicts the proportionality of MFPT to $L$, the latter case is characterised by the dependence MFPT $\propto L^\alpha$ which is a well-known feature of Lévy flights and also emerges in our numerical simulations; if $\nu$ is small, Eq. (36) can only be valid for very large $L$. In the limit of large $\nu$, Eq. (36) agrees with the simulations.

## REFERENCES

[1] T. Geisel, J. Nierwetberg, A. Zacherl, «Accelerated Diffusion in Josephson Junctions and Related Chaotic Systems», *Phys. Rev. Lett.* **54**, 616 (1985).

[2] G. Zumofen, J. Klafter, «Scale-invariant motion in intermittent chaotic systems», *Phys. Rev. E* **47**, 851 (1993).

[3] J. Klafter, A. Blumen, M.F. Shlesinger, «Stochastic pathway to anomalous diffusion», *Phys. Rev. A* **35**, 3081 (1987).

[4] V. Zaburdaev, S. Denisov, J. Klafter, «Lévy walks», *Rev. Mod. Phys.* **87**, 483 (2015).

[5] D. Froemberg, M. Schmiedeberg, E. Barkai, V. Zaburdaev, «Asymptotic densities of ballistic Lévy walks», *Phys. Rev. E* **91**, 022131 (2015).

[6] J. Janssen, N. Limnios (Eds.) «Semi-Markov Models and Applications», *Kluwer Academic Publishers*, The Netherlands 1999.

[7] X. Brokmann *et al.*, «Statistical Aging and Nonergodicity in the Fluorescence of Single Nanocrystals», *Phys. Rev. Lett.* **90**, 120601 (2003).

[8] G. Margolin, E. Barkai, «Nonergodicity of Blinking Nanocrystals and Other Lévy-Walk Processes», *Phys. Rev. Lett.* **94**, 080601 (2005).

[9] J. Klafter, G. Zumofen, «Lévy statistics in a Hamiltonian system», *Phys. Rev. E* **49**, 4873 (1994).

[10] V.Yu. Zaburdaev, K. Chukbar, «Enhanced superdiffusion and finite velocity of Lévy flights», *J. Exp. Theor. Phys.* **94**, 252 (2002).

[11] J.P. Taylor-King, E. van Loon, G. Rosser, S.J. Chapman, «From Birds to Bacteria: Generalised Velocity Jump Processes with Resting States», *Bull. Math. Biol.* **77**, 1213 (2015).

[12] G. Cristadoro, T. Gilbert, M. Lenci, D.P. Sanders, «Lévy walks on lattices as multi-state processes», *J. Stat. Mech.* **2015**, P05012 (2015).

[13] A. Kamińska, T. Srokowski, «Lévy walks in nonhomogeneous environments», *Phys. Rev. E* **96**, 032105 (2017).

[14] A. Kamińska, T. Srokowski, «Lévy walks with variable waiting time: A ballistic case», *Phys. Rev. E* **97**, 062120 (2018).

[15] S. Redner, «A Guide to First-passage Processes», *Cambridge University Press*, 2001.

[16] B. Dybiec, E. Gudowska-Nowak, E. Barkai, A.A. Dubkov, «Lévy flights *versus* Lévy walks in bounded domains», *Phys. Rev. E* **95**, 052102 (2017).

[17] H.W. McKenzie, M.A. Lewis, E.H. Merrill, «First Passage Time Analysis of Animal Movement and Insights into the Functional Response», *Bull. Math. Biol.* **71**, 107 (2009).

[18] C. Holling, «The Components of Predation as Revealed by a Study of Small-Mammal Predation of the European Pine Sawfly», *Can. Entomol.* **91**, 293 (1959).

[19] F. Bartumeus, M.G.E. da Luz, G.M. Viswanathan, J. Catalan, «Animal search strategies: A quantitative random-walk analysis», *Ecology* **86**, 3078 (2005).

[20] A. Zoia, A. Rosso, M. Kardar, «Fractional Laplacian in bounded domains», *Phys. Rev. E* **76**, 021116 (2007).

[21] I.S. Gradshteyn, I.M. Ryzhik, «Table of Integrals, Series, and Products», *Elsevier Inc.*, London, UK 2007.

[22] M. Schmiedeberg, V.Y. Zaburdaev, H. Stark, «On moments and scaling regimes in anomalous random walks», *J. Stat. Mech.* **2009**, P12020 (2009).

[23] I. Podlubny, «Fractional Differential Equations», *Elsevier Academic Press*, 1999.

[24] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, «Theory and Applications of Fractional Differential Equations», *Elsevier*, Amsterdam 2006.

[25] R.K. Getoor, «First passage times for symmetric stable processes in space», *Trans. Amer. Math. Soc.* **101**, 75 (1961).