CHAOTIC AND PERIODIC BEHAVIOR IN A FRACTIONAL-ORDER BIOLOGICAL SYSTEM

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(Received January 1, 2020; accepted June 9, 2020)

We investigated the effects of variation in the non-integer order of a fractional differential equation modeling activated enzyme molecules in brain wave. The dynamical changes in the system trajectories in both the chaotic and the periodic regimes of an existing second order differential equation model are numerically examined when the orders of the biological system are assigned non-integer values. The simulation showed that the dynamics of the system can be altered through the order of the derivatives. In particular, the integer-order system can be driven from chaotic oscillation into periodic state by adopting an appropriate non-integer orders when the system is associated with innate memory.

DOI:10.5506/APhysPolB.51.1885

1. Introduction

Differential equations containing arbitrary (non-integer) order derivatives are generally referred to as fractional differential equations (FDEs) (also known as extraordinary differential equation). A system is said to be a fractional order system when the dynamics of the system can be modeled by a fractional differential equation, in contradistinction to an ordinary integer-order differential equation which contains derivatives in whole numbers [1]. Fractional differential equations are viewed as alternative models to nonlinear differential equations. FDEs have been reported to exhibit chaotic attractors and other fascinating features of nonlinear differential equations [2].

Generally, fractional calculus has been receiving increased interest by researchers in many fields of science and it has been employed as a crucial tool in modeling many physical phenomena in applied sciences. For instance, studies in fractional-order differential equations have found applications in areas such as electromagnetism, viscoelasticity, signal processing, population

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dynamics, electrochemistry, acoustics and material sciences. These applications are not limited to mathematics but cut across other disciplines such as biology, chemistry, physics, engineering, computational fluid mechanics, systems identification, signal and image processing, finance, economics, food supplement, climate and several other physical processes [3–8]. Interestingly, the subject (fractional differentiation and integration) is as old as calculus itself and the recent increase in its application is attributable to the fact that fractional calculus is not a local or point property [9], rather it considers the system's history and non-local distributed effect. This essential property of FDEs is known as the memory and it makes the fractional calculus more realistic than its integer-order counterpart. In fact, the memory associated with natural systems is better modeled with non-local operators contained in delayed differential equations (DDEs) and FDEs compared to the ordinary integer-order derivatives [10]. Consequently, this makes FDE more applicable to solving real-life problems in social and physical sciences.

The popularity of fractional-order calculus is fast growing, evidently due to the benefits stemming from using its concepts in various social and scientific fields, ranging from system modelings to automatic controls. The accessibility of researchers to more efficient and powerful computational tools that are based on computer algebraic systems in commercially available softwares such as MatLab, Mathematica and Python has seriously increased interest in fractional differentiation, and as such, it has unplowed new possibilities for appraising the theoretical aspects of fractional calculus in its various applications. For instance, in biology, FDEs have put an undying demand on both theoretical and experimental research in studying life and living organism which makes the field an open area for ever expanding research and new discoveries. Since research in theoretical biology uses mathematical methods and quantitative models, it is constantly evolving due to new insights in mathematics and their application [11].

Numerical and analytic solutions of FDEs are typically difficult to compute. This is a major challenge mitigating the full implementation and application of FDEs. Currently, determining the approximate, numerical and exact solutions of FDEs is facile. The fact that different computational tools are now in use in analyzing dynamics of natural systems and the uniqueness in the existence of solutions of associated fractional differential equations have attracted the attention of many researchers, making it more roomy for its wide-spread applications in various fields [12–17]. Therefore, the need for analyzing the role of the order of differentiation on the dynamics of fractional differential equations can never be overemphasized.

The FDEs have been adapted to model and study biological systems [11] due to the need to incorporate long-range temporal memory effects (present in many population systems) into existing biological models. In ecology,

memory simply describes the ability of the past state or history of the community to effectively influence the present or future responses of the community. It also describes the degree to which the current state of an ecological process is being influenced by its past modifications of a landscape [18]. By incorporating the characteristic unlimited memory associated with FDEs into models of physical systems, scientists are able to give more concise and accurate information about the dynamical behavior of these systems. Biological systems have metastable states with a very high electric polarization, hence they exhibit rich variety of dynamical behaviors that require detailed description. In particular, selective transport processes in enzymatic substrate reaction with ferroelectric behavior in brain waves tend to increase the level of substrate influx, thus initiating specific chemical reactions that are dependent on the history of activated enzymes concentration, growth of unexcited enzymes and the number of the substrate molecules. Therefore, incorporating the memory capabilities of FDE into an existing integer-order model of brain waves can better describe the dynamics of enzyme–substrate interactions.

In addition, fractional models have been shown by many researchers to adequately describe the operation of a variety of biological processes, which includes models of cancer treatment [7], immune-tumor [19], viscoelasticity [20], and neurons [21]. Many intriguing results have been obtained from studies relating to the dynamics of biological systems modeled as FDEs, for instance, the stability conditions, bifurcations and chaos of fractional-order prey–predator system have been well highlighted [18]. Many of the interesting dynamical behaviors of integer-order ODE models of physical systems are reproducible using FDEs and some hidden behaviors are also being revealed when the systems are modeled as FDEs [22–25]. For example, the Mackey–Glass equation with fractional-order derivative was shown to exhibit more complex and richer dynamics with the cooperation of a delay term and fractional-order derivative [26]. Jahanshahi *et al.* [27] presented the dynamics of a novel four-dimensional fractional order system and showed that it exhibits rich dynamical behavior including self-excitation and hidden chaotic attractors. In addition, several control algorithms and synchronization techniques including active controller, adaptive controller, linear feed-back technique, backstepping controller, sliding mode controller (SMC), hybrid controllers and fuzzy logic controller (see [9, 23, 28–30] and references therein) that are widely applicable to integer-order equations of megastable, chaotic and hyperchaotic systems can be adopted to stabilize, synchronize or/and control these oscillators [27, 31, 32]. These techniques have been applied to synchronize and control FDE models of Bloch system [33], financial system [34, 35], hyper-chaotic economic system [31].

In this direction, this paper investigates the changes in the dynamics of an existing integer-order biological model, when the orders of the model are changed to arbitrary (non-integer) orders, assuming that all values of an arbitrary real fractional derivative are admissible. The rest of this paper is organized as follows: the dimensionless model of the unidirectional biological system is presented in Section 2. The relevant theories on how to solve fractional differential equations are briefly introduced in Section 3, while specifics on numerical simulations are detailed in Section 4. Results and relevant discussion are presented in Section 5.

2. Description of the model

The biological model used in this paper is a second order non-autonomous differential equation modeling activated enzyme molecules in the brain wave. The model is based on long range coherence interactions which are capable of initiating specific chemical reactions and transport processes in enzymes as highlighted by Fröhlich *et al.* [36]. This model reduces to a multi-limit cycles Van der Pol oscillator when the system is unforced. It exhibits both the periodic and the chaotic dynamics [37].

Kadji *et al.* [37] consider the biological model as a system of S substrate molecules with N and Z representing the population of excited enzymes and unexcited enzymes, respectively. The enzymes and substrate are characterized by selective long-range interactions that increase their level by influx. They [37] assumed that the rate of increase of the activated enzymes is proportional to their concentration N, the concentration Z of the remaining unexcited enzymes and the number of substrates S, so that the system of nonlinear differential equations in variables N, S and Z are, respectively,

$$\frac{\mathrm{d}N}{\mathrm{d}\tau} = \nu NRS - \xi N\,, \tag{1a}$$

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = \gamma S - \nu NRS\,, \tag{1b}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}\tau} = \xi N - \nu NZS - \lambda(Z - C)\,, \tag{1c}$$

where the nonlinear enzyme–substrate reaction strength is denoted as ν, the rate at which excited enzymes decay to the ground (or weakly polar) state is ξ and γ is the range attraction of the substrate particles initiated by spontaneous catalytic reactions. $\lambda(Z - C)$ is obtained from the long-range interaction where C denotes the equilibrium concentration of unexcited enzymes molecules where both the excited enzyme and the substrate are absent, that is, $N = S = 0$.

The system described by Eq. (1) is simplified by using the adiabatic elimination of fast variable, that is, by assuming the equilibrium of the unexcited enzyme concentration can be attained fast. Thus, the system described by Eqs. (1a) and (1b) becomes the well-known Lotka–Voltera equation [38]. Equations (1a) and (1b) can be respectively redefined in terms of excess concentrations of activated enzymes ε and substrate η molecules beyond their equilibrium values by considering the perturbed activated enzymes and substrate molecules around the non-trivial steady state.

Frohlich [36, 39] considered that the available chemical energy from substrate–enzyme reactions is capable of initiating two possible forms of oscillations around the equilibrium state in large regions of the proteins, substrates, ions and structured water. The first of these oscillations is a very low-frequency chemical oscillation in the substrate and activated enzyme molecules, while the second oscillation is in a form of electric vibrations induced by partially screened high dipole moment of the excited enzyme, which leads the system towards ferroelectric instability. By accounting for the system's innate resistance towards ferroelectric tendency and possible external chemical fluctuations or contributions to the electric field F from thermal fluctuations, and an externally applied field on the excited enzyme, the combined dielectric and chemical contribution for small values of excess concentrations of activated enzymes ε and substrate η leads to a modified form of equations (1a) and (1b), given by

$$\begin{aligned} \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} &= \gamma\eta + \left(\kappa^2 \mathrm{e}^{-\psi^2\varepsilon^2} - \sigma^2\right)\varepsilon + \nu C\eta\varepsilon + F(\tau)\,, \\ \frac{\mathrm{d}\eta}{\mathrm{d}\tau} &= -\xi\varepsilon - \nu C\eta\varepsilon\,. \end{aligned} \tag{2}$$

σ is the relaxation term from the contribution of electric resistance to ferroelectricity with polarization P assumed to be proportional to the excited enzyme molecules $\varepsilon(\tau)$.

By re-scaling the variables in the equation obtained from the third order expansion of the function $\mathrm{e}^{-\psi^2\varepsilon^2}$ in Eq. (2) which accounts for some nonlinearities in the excess concentration of the activated enzymes as

$$\begin{gathered} t = \omega_0\tau\,, \quad \omega_0^2 = \xi\gamma\,, \quad x = \Xi\varepsilon\,, \quad \Xi = \sqrt{\frac{3}{\kappa^2-\sigma^2}}\kappa\psi\,, \quad \mu = \frac{\kappa^2-\sigma^2}{\omega_0}\,, \\ E(t) = \frac{\Xi}{\omega_0^2}\frac{\mathrm{d}}{\mathrm{d}t}F\left(\frac{t}{\omega_0}\right)\,, \quad \alpha = \frac{5}{18\kappa^2}\left(\kappa^2-\sigma^2\right)\,, \quad \beta = \frac{7}{162\kappa}\left(\kappa^2-\sigma^2\right)^2\,, \end{gathered} \tag{3}$$

we obtain the dimensionless biological system as

$$\ddot{x} - \mu\left(1 - x^2 + \alpha x^4 - \beta x^6\right)\dot{x} + x = E\cos\Omega t\,, \tag{4}$$

where the dots denote differentiation w.r.t. time. α and β are positive coefficients of higher-order nonlinearities in damping parameter, μ is the amplitude of damping coefficients, while E and Ω denote the amplitude and the frequency of the externally applied input signal, respectively.

The biological system governed by Eq. (4) is well-studied and it exhibits a rich catalogue of dynamical behavior including periodic, quasi-periodic, and chaotic oscillations [37, 40, 41]. In this research, we focus on the dynamical changes induced by a modulation of the order of the differential equation governing the system.

3. Solution of fractional-order differential equations

Several definitions exist for fractional-order derivative, among which researchers take advantage of the Riemann–Liouville, the Caputo, or the Grünwald–Letnikov definition. For a wide class of functions, these three definitions are equivalent under certain conditions (zero initial condition and lower terminal $a = 0$) [2]. The Riemann–Liouville fractional derivative is defined as

$$\frac{\mathrm{d}^{\lambda} f}{\mathrm{d}t^{\lambda}} = {}_{a}\mathcal{D}_{t}^{\lambda} = \frac{\mathrm{d}^{m}}{\mathrm{d}t^{m}} \left[\frac{1}{\Gamma(m-\lambda)} \int_{a}^{t} \frac{f(\tau)}{(t-\tau)^{\lambda-m+1}} \mathrm{d}\tau \right], \quad (m-1 < \lambda < m), \tag{5}$$

where $m \in N$ and $\Gamma(\cdot)$ is a gamma function. λ is the non-integer order of the derivative which is usually in the range of $0 < \lambda < 2$. In spite of the fact that the definition guarantees some pleasant and useful mathematical properties, its practical implementation is often rigorous. For example, it is usually difficult to assign some physical meaning to the initial conditions in many situations. This is because the fractional derivative of a constant is not identically zero. To navigate the aforementioned challenge, an alternative definition introduced by Caputo is given by

$$\frac{\mathrm{d}^{\lambda} f}{\mathrm{d}t^{\lambda}} = {}_{a}\mathcal{D}_{t}^{\lambda} = \frac{\mathrm{d}^{\lambda} f(t)}{\mathrm{d}t^{\lambda}} = \frac{1}{\Gamma(m-\lambda)} \int_{a}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\lambda-m+1}} \mathrm{d}\tau, \quad (m-1 < \lambda < m). \tag{6}$$

A major advantage of Caputo's definition of fractional derivative is that initial conditions can take on the same standard form, $x(0) = x_0$, as that of associated integer-order variant, which is sufficient for practical problems [2]. From the fact that there exist an integral and a gamma function in the definitions, it also appears difficult to obtain the numerical solution of a fractional-order differential equations using the above definitions in Eq. (6). Therefore, a more simpler definition like the Grünwald–Letnikov definition,

which is well-known for its lack of sophistication in the discretization of fractional-order operators can be employed. This definition is given by

$$\frac{\mathrm{d}^{\lambda} f}{\mathrm{d}t^{\lambda}} = {}_{a}\mathcal{D}_{t}^{\lambda} = \lim_{h\to 0} \frac{1}{h^{\lambda}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^{j} \binom{\lambda}{j} f(t-jh)\,, \tag{7}$$

where the pair of square brackets [.] in the upper limit of the summation sign implies the integer part, while a is the length of the memory. The binomial coefficients are given by $\binom{\lambda}{0} = 1$, and $\binom{\lambda}{j} = \frac{\lambda(\lambda-1)\ldots(\lambda-j+1)}{j!}$, for $j \geq 1$.

To obtain the solution of a fractional-order differential system, one often refers to approximation methods through discretization of corresponding fractional-order differentiator, which requires the history (information about the previous state) of the system, a condition known as memory effect [2, 25, 42]. There are two popular approximation methods for numerical computation of fractional-order differential equations, which are: the frequency domain methods and the time domain methods. The Adams–Bashforth–Moulton algorithm which is an improved predictor–correctors scheme is a time domain method. Solving fractional-order differential equations in time domain has been shown to be more reliable in detecting chaos, however, simulation time may be longer with complicated algorithm compared to simulations in frequency domain methods due to long memory characteristics [43].

4. Numerical simulation

Here, we apply the Grünwald–Letnikov derivative (Eq. (7)) to numerically solve Eq. (4) by applying the short memory principle to the Grünwald–Letnikov definition so that the length of a system memory is substantially reduced and the numerical algorithm produces reliable results [43]. The principle is convenient for the consideration of dynamical properties of systems. The technique is dependent on the approximation that at large time t Grünwald–Letnikov's definition coefficients which correspond to the value of the function around the initial condition (say $t = 0$) have little significant contribution to the solution. The approximation of the numerical solution is based on the recent history $[t - L, t]$ of the system (where L is "memory length") with the derivative computed over a moving low limit. In general, the approach is that if $f(a) = 0$, the three definitions of fractional derivative (Riemann–Liouville, Caputo and Grünwald–Letnikov) are equivalent for a wide class of functions [2, 25, 42]. Thus, the explicit numerical approximation of Eq. (7) at the points $t_k = kh, (k = 1, 2, \ldots)$ in accordance with the short memory principle is of the form of

$$ {}_L\mathcal{D}^{\lambda}_{t_k} \approx \lim_{h\to 0} \frac{1}{h^{\lambda}} \sum_{j=0}^{[N(t)]} (-1)^j \binom{\lambda}{j} f(t_{k-j}) = \lim_{h\to 0} \frac{1}{h^{\lambda}} \sum_{j=0}^{[N(t)]} c_j^{(\lambda)} f(t_{k-j}) \,, \tag{8} $$

where h is the integration time step for the numerical computation and $N(t) = \min(\frac{t_k - L}{h}, \frac{L}{h})$. This relation helps to eliminate our dependence on initial conditions before $t = 0$ as normally required for systems with memory. The $c_j^{(\lambda)}$ are coefficients which can be computed by using the following recursive relation:

$$ c_0^{(\lambda)} = 1 \,, \qquad c_j^{(\lambda)} = \left(1 - \frac{1+\lambda}{j}\right) c_{j-1}^{(\lambda)} \,. \tag{9} $$

By factoring in the memory capabilities of contributing system properties into the biological system described by Eq. (4), the model can be written as a fractional differential equation of the form of

$$ \frac{\mathrm{d}^{\lambda_2} x}{\mathrm{d}t^{\lambda_2}} - \mu \left(1 - x^2 + \alpha x^4 - \beta x^6\right) \frac{\mathrm{d}^{\lambda_1} x}{\mathrm{d}t^{\lambda_1}} + x = E \cos \Omega t \,, \tag{10} $$

where λ_1 and λ_2 are the arbitrary non-integer orders. By setting $y = \frac{\mathrm{d}^{\lambda_1} x}{\mathrm{d}t^{\lambda_1}}$, Eq. (10) can be expressed in a form of a coupled lower-order differential equation of the form of

$$ \begin{aligned} \frac{\mathrm{d}^{\lambda_1} x}{\mathrm{d}t^{\lambda_1}} &= y \,, \\ \frac{\mathrm{d}^{\lambda_2} y}{\mathrm{d}t^{\lambda_2}} &= \mu \left(1 - x^2 + \alpha x^4 - \beta x^6\right) y - x + E \cos \Omega t \,. \end{aligned} \tag{11} $$

The discretization of the system is achieved by using the relation for the explicit approximation of fractional derivatives (Eq. (8)) in Eq. (10), such that Eq. (11) is written in the discretized form as

$$ \begin{aligned} x(t_k) &= (y(t_{k-1}))h^{\lambda_1} - \sum_{j=1}^{N-1} c_j^{(\lambda_1)} x(t_{k-j}) \,, \\ y(t_k) &= \Big(- x(t_k) + \mu \left(1 - x(t_k)^2 + \alpha x(t_k)^4 - \beta x(t_k)^6\right) y(t_{k-1}) \\ &\quad + E \cos(\Omega h t_k) \Big) h^{\lambda_2} - \sum_{j=1}^{N-1} c_j^{(\lambda_2)} y(t_{k-j}) \,. \end{aligned} \tag{12} $$

To generate the numerical results, Eq. (12) was solved with zero initial conditions $(x(0) = 0, y(0) = 0)$. Since the aforementioned definitions of fractional derivatives (given by Riemann–Liouville, Caputo and Grünwald–

Letnikov) take the same form at zero initial conditions, this choice is usually sufficient for numerical integration of FDEs. All simulations were implemented on MatLab with integration step length of $h = 0.005$ and a total simulation time $T_{\text{sim}} = 500$ (see [43, 45] and references therein for short tutorials on how to implement the algorithm on MatLab). The first 1000 iterates (initial iterates) were discarded to account for transient effects. The phase portraits and the time series plots were produced by varying the fractional integer coefficients.

5. Results and discussion

5.1. Modulation in chaotic dynamics

For clarity, we introduce the basic dynamics of the system through one-parameter bifurcation diagrams with respect to system parameters μ and α as shown in Fig. 1 (a) and (b), respectively, for the preliminary case ($\lambda_1 = 1$, $\lambda_2 = 1$) described by integer-order differential equation (Eq. (4)). Other system parameters are set as: $\beta = 1.70$, $E = 8.27$, and $\Omega = 3.465$. We

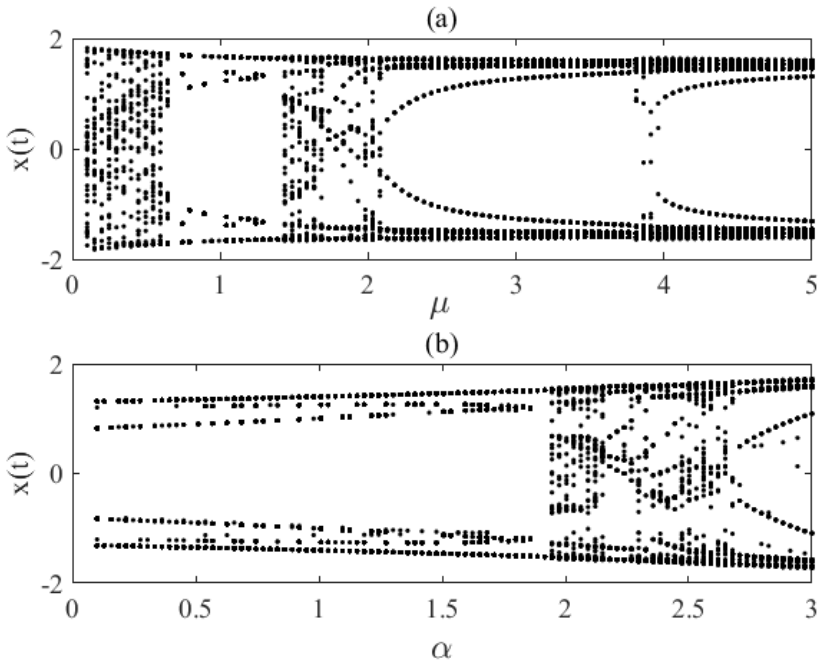

Fig. 1. The bifurcation diagram for the variation of oscillation with a system parameter for the preliminary biological system (Eq. (4)), $\lambda_1 = 1, \lambda_2 = 1$, presented as (a) *x versus* μ when $\alpha = 2.55$, $\mu \in [0.01, 5]$, (b) *x versus* α when $\mu = 2.00$, $\alpha \in [0.01, 3]$. Other system parameters are set as: $\beta = 1.70$, $E = 8.27$ and $\Omega = 3.465$.

remark that for $\lambda_1 = 1$, $\lambda_2 = 1$, the coefficients in Eq. (9) vanishes, that is, $c_j^{(\lambda)} = 0 \quad (j = 1, \ldots, k)$, and Eq. (12) is reduced to the standard biological model (Eq. (4)) without the memory effect. From Fig. 1 (a), the choice of μ within this parameter regime determines the state of the system. At the onset, the system is chaotic, afterwards it settles into a periodic window before transiting into another chaos. The onset of chaos exists till $\mu \simeq 1.59$ where the system transitions into periodic orbits within $\mu \in (0.59, 1.44)$. The oscillation exits this periodic window into chaos within $\mu \in (1.44,\ 2.08)$, followed by prevalent periodic transitions. The dependence of the system oscillations on α is presented in Fig. 1 (b) with obvious transitions from periodic motion to chaos reminiscent of transitions observed in the system dynamics with variation in μ (Fig. 1 (a)). As seen in Fig. 1, the choice of $\mu = 2.0$ and $\alpha = 2.55$ with other parameter values unchanged ensures the preliminary system is chaotic within the set parameter regime. The corresponding phase plots and time series are presented in Fig. 2 (a) and Fig. 3 (a), respectively. We shall analyze the changes in the system dynamics within this chaotic regime when the integer orders of the system derivatives are assigned arbitrary non-integer values, assuming such variations in system orders is physically plausible.

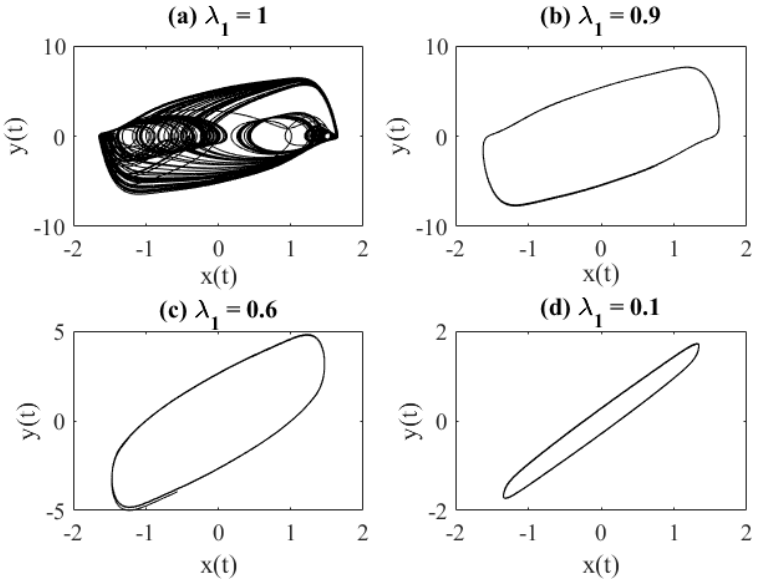

Fig. 2. $\lambda_1 (= 1, 0.9, 0.6, 0.1)$ for $\lambda_2 = 1$ on the system trajectories in phase space (a)–(d), respectively. Other parameters are set as: $\mu = 2.0$, $\alpha = 2.55$, $\beta = 1.70$, $E = 8.27$ and $\Omega = 3.465$.

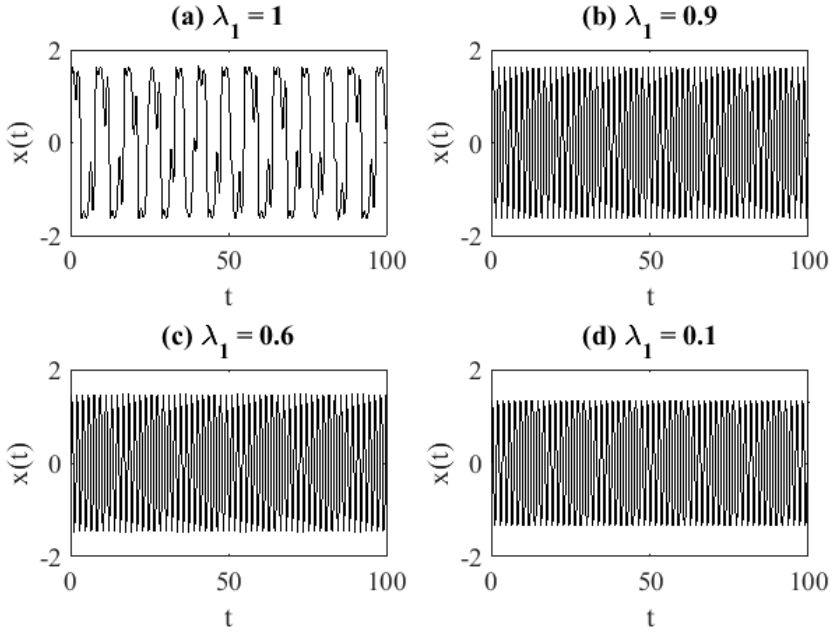

Fig. 3. The time evolution of the system (Eq. (10)) corresponding to the phase plots presented in Fig. 2 (a)–(d).

Figure 2 shows the system's phase portraits for four values of arbitrary real order $\lambda_1 (= 1, 0.9, 0.6, 0.1)$ when $\lambda_2 = 1$. The values of other parameter are set as: $\mu = 2.0$, $\alpha = 2.55$, $\beta = 1.70$, $E = 8.27$, and $\Omega = 3.465$. Figure 2 (a) is the phase portrait of the preliminary biological system with $\lambda_1 = \lambda_2 = 1$. The system's trajectories correspond to the chaotic attractor reported by Kadji *et al.* [37] for the integer-order ODE (Eq. (4)). In Fig. 2 (b)–(d), the values of λ_1 were chosen as $\lambda_1 = 0.9$, $\lambda_1 = 0.6$ and $\lambda_1 = 0.1$, respectively. Clearly, the system's chaotic dynamics (presented in Fig. 2 (a)) is altered by reducing the value of the arbitrary real order λ_1 as shown in Fig. 2 (b). Consequently, the system is driven into periodic oscillation as λ_1 and further reduced as presented in Fig. 2 (c) and Fig. 2 (d). The same behavior is verified in the time series evolution of the system as shown in Fig. 3 (a)–(d) for the same parameter values used in Fig. 2.

To further investigate the dynamics of the system with respect to variations in the non-integer orders, the arbitrary real order which was previously varied is kept constant (at $\lambda_1 = 1$), while the other λ_2 is varied. A noticeable change was observed in dynamics of the system, as depicted in Fig. 4 (a)–(d). Starting from the preliminary case in Fig. 4 (a) which corresponds to the plot presented in Fig. 2 (a) for $\lambda_1 = \lambda_2 = 1$, the trajectory of the system changes significantly presenting chaotic attractors which appear to have

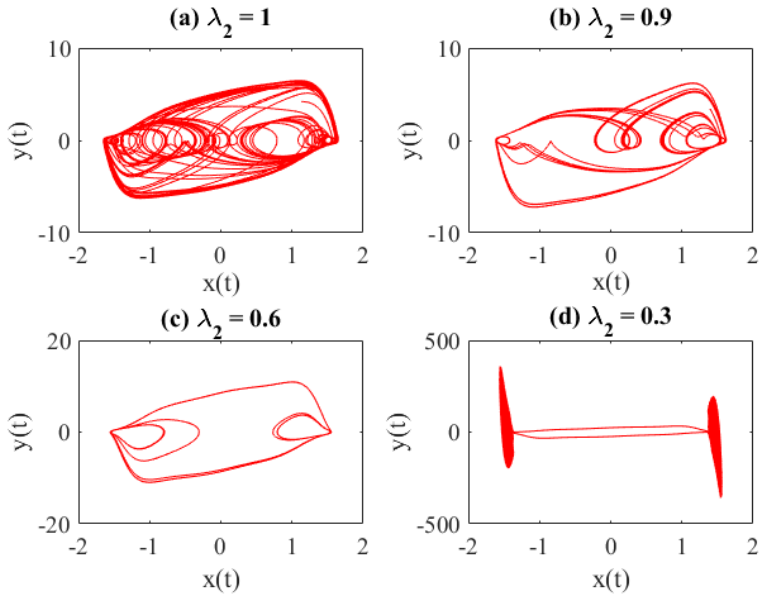

Fig. 4. The phase portrait of the biological system (Eq. (10)) computed from Eq. (12), depicting the effects of four values of arbitrary real order $\lambda_2(= 1, 0.9, 0.6, 0.3)$ at $\lambda_1 = 1$ on the system trajectories in phase space (a)–(d), respectively. Other parameters are set as; $\mu = 2.0$, $\alpha = 2.55$, $\beta = 1.70$, $E = 8.27$ and $\Omega = 3.465$.

different trajectories (as shown in Fig. 4 (b)–(d)). At $\lambda_2 = 0.6$ (shown in Fig. 4 (c)), the trajectory spirals into a double-scroll attractor with each on half sides of a period. This is well-captured in Fig. 4 (d) ($\lambda_2 = 0.3$) which shows the dynamics of the system with double-scroll attractors that are in equilibrium. Though, modulation of the system dynamics with λ_2 does not drive the system into periodic oscillation, it significantly changes the nature of its chaotic behavior. The chaotic dynamics of the system appears to be more controllable from the values of λ_2 as presented in Fig. 4 (a)–(d) and Fig. 5 (a)–(d).

It has earlier been reported by [37] that the dissipation term can be used to control the chaotic behavior of the system. However, from the above, it can be said that unlike the contributions of the dissipation term in controlling the dynamics of the system, the system's inherent memory property which is described by the non-integer order can be directly used to control the chaotic behavior of the system. Further evidence that validates this statement is presented in Fig. 6 (a)–(b) showing bifurcation diagrams of the system (Eq. (10)) with respect to the arbitrary (non-integer) order for (a) x *vs.* λ_2 when $\lambda_1 = 1$ in the upper panel, and (b) x *vs.* λ_1 when $\lambda_2 = 1$ in the lower panel. Other parameters are set as follows: $\mu = 2.0$, $\alpha = 2.55$, $\beta = 1.70$, $E = 8.27$,

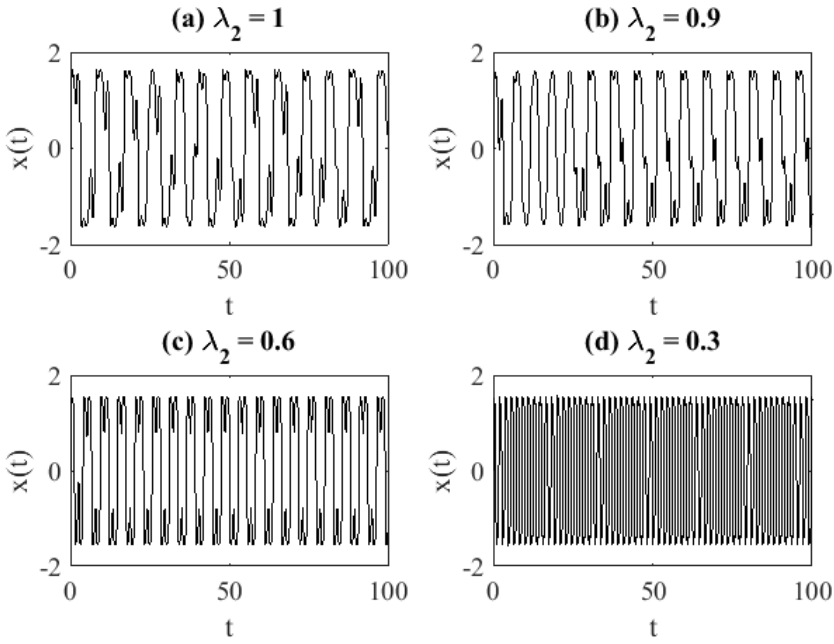

Fig. 5. The time evolution of the system (Eq. (10)) corresponding to the phase plots presented in Fig. 4 (a)–(d).

and $\Omega = 3.465$. These bifurcation diagrams are in agreement with the phase portraits and time evolutions presented for the variations of λ_1 and λ_2. Expectedly, the system is chaotic for a choice of $\lambda_1 = 1,\ \lambda_2 = 1$ in both panels. The periodic and chaotic windows correspond to the same dynamics presented through the phase portraits for the three non-integer values of $\lambda_1 (= 0.9,\ 0.6,\ 0.1)$ considered in Fig. 2 (b)–(d), and $\lambda_2 (= 0.9,\ 0.6,\ 0.3)$ considered in Fig. (4) (b)–(d). For instance, the periodicity shown in Fig. 2 (b) at $(\lambda_1, \lambda_2) = (0.9,\ 1)$ is confirmed on the upper panel of Fig. 6. Besides, the chaotic attractor revealed in Fig. 4 (b) for $(\lambda_1, \lambda_2) = (1,\ 0.9)$ falls within the chaotic window of Fig. 6 (b). This validates our submission that the dynamical behavior of the biological system can be determined in part by the arbitrary real orders. Hence, the arbitrary real orders can play complementary roles to the dissipative parameters in controlling the system behavior.

5.2. Modulation in periodic oscillation

Next, we extend the results by analyzing possible dynamical changes in the system's behavior by variation of arbitrary nonlinear orders when the preliminary system is periodic. For $\lambda_1 = \lambda_2 = 1$, and other parameters set as $\mu = 0.1,\ \alpha = 0.1,\ \beta = 0.2,\ E = 1$ and $\Omega = 0.7$, the system is periodic.

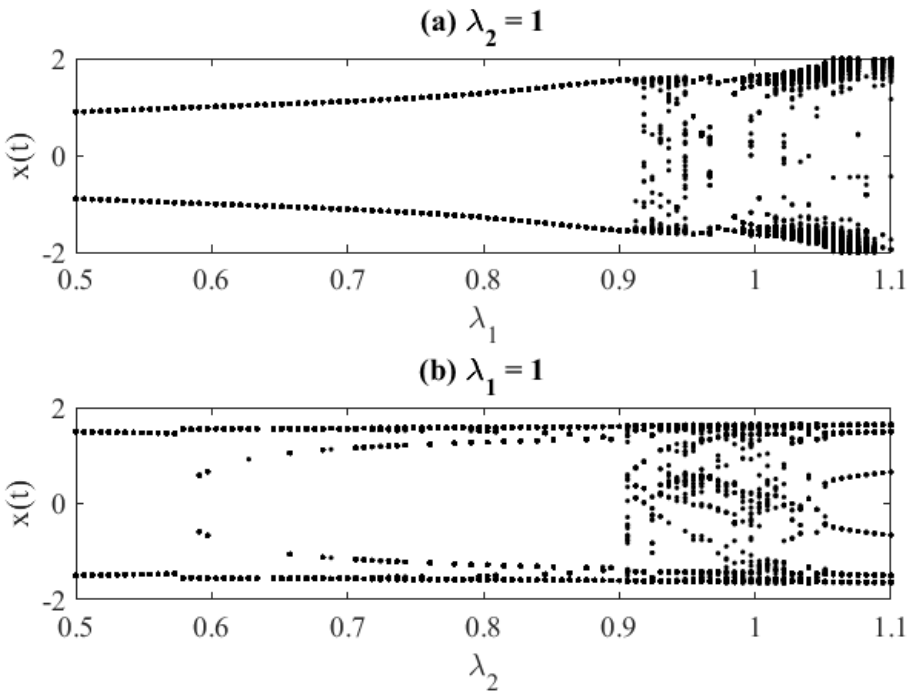

Fig. 6. The bifurcation diagram for the fractional-order biological system (Eq. (10)) computed from Eq. (12) with respect to variations in the arbitrary real fractional orders, presented as x-coordinate *versus* arbitrary real order of the derivative within $\lambda \in [0.5, 1.1]$ for (a) x *vs.* λ_2 when $\lambda_1 = 1$, (b) x *vs.* λ_1 when $\lambda_2 = 1$. Other parameters are set as: $\mu = 2.0$, $\alpha = 2.55$, $\beta = 1.7$, $E = 8.27$, and $\Omega = 3.465$.

In this regime, the variation of the non-integer orders constrained the system to a period-1 attractors as shown in Figs. 7 and 8. Figure 7 (a)–(d) depicts the effect of varying λ_1 for a fixed value of λ_2. The periodic oscillation of the preliminary system at $\lambda_1 = \lambda_2 = 1$ remained at other non-integer values of λ_1, that is, no transition to chaos was induced. It can be observed from Fig. 7 (a)–(d) and Fig. 8 (a)–(d) that reducing the value of the system's real order (λ_1) reduces the width of the associated periodic attractor. This effect is more pronounced in Fig. 7 (c) and Fig. 8 (d), respectively. In addition, at constant $\lambda_1 (= 1)$, λ_2 was varied to study it effects on the dynamical behavior of the system. Again, no transition to chaos occurred as the system remains periodic (see Fig. 9), but the corresponding time series in Fig. 10 show reduction in amplitude of oscillation. The bifurcation diagrams in Fig. 11 also confirms that within the chosen parameter regime, the system remains periodic.

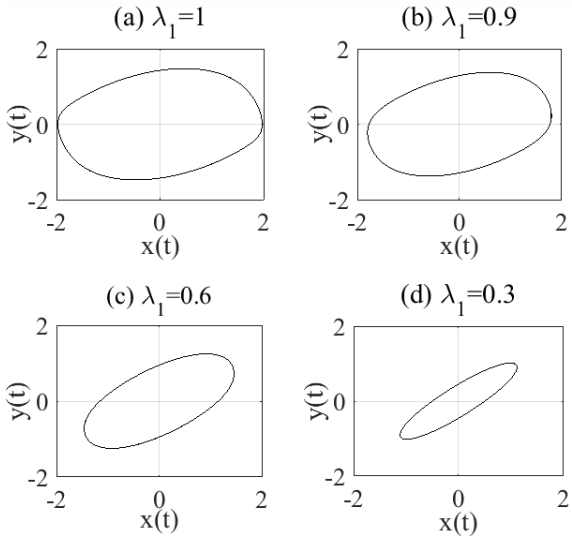

Fig. 7. The phase portrait of the system (Eq. (10)) computed from Eq. (12), depicting the effects of four values of arbitrary real order $\lambda_2(= 1, 0.9, 0.6, 0.3)$ at $\lambda_1 = 1$ on the system trajectories in phase space (a)–(d), respectively. Other parameters are set as: $\mu = 0.1,\ \alpha = 0.1,\ \beta = 0.2,\ E = 1$, and $\Omega = 0.7$.

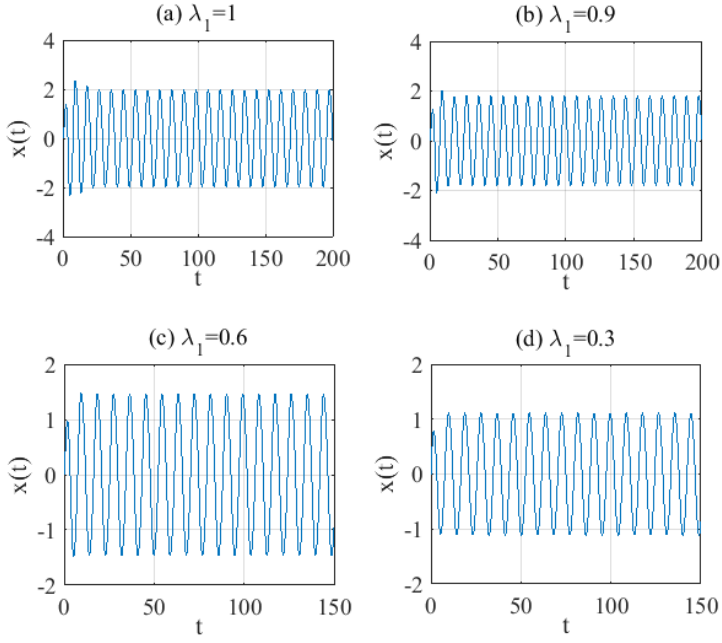

Fig. 8. The time evolution of the system with same parameters as in Fig. 7.

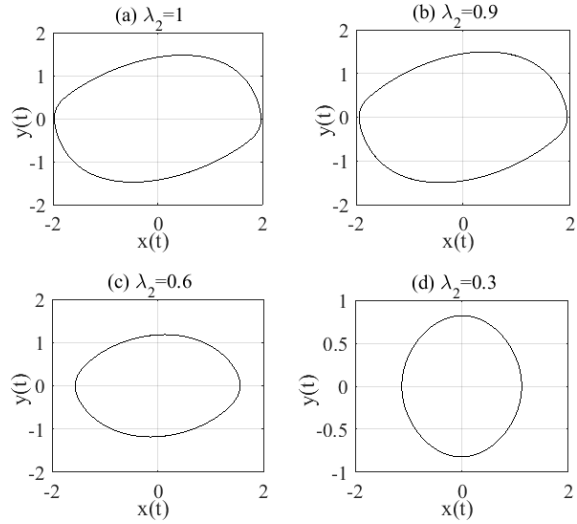

Fig. 9. The phase portrait of the system (Eq. (10)) computed from Eq. (12), depicting the effects of four values of arbitrary real order $\lambda_2(= 1, 0.9, 0.6, 0.3)$ at $\lambda_1 = 1$ on the system trajectories in phase space (a)–(d), respectively. Other parameters are set as: $\mu = 0.1,\ \alpha = 0.1,\ \beta = 0.2,\ E = 1,\ \Omega = 0.7$ and $\lambda_1 = 1$.

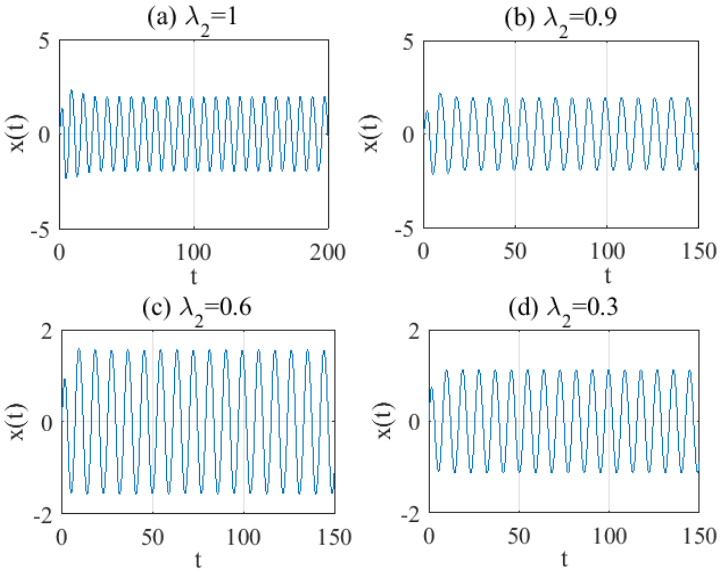

Fig. 10. The time evolution of the system (Eq. (10)) with same parameters as in Fig. 9.

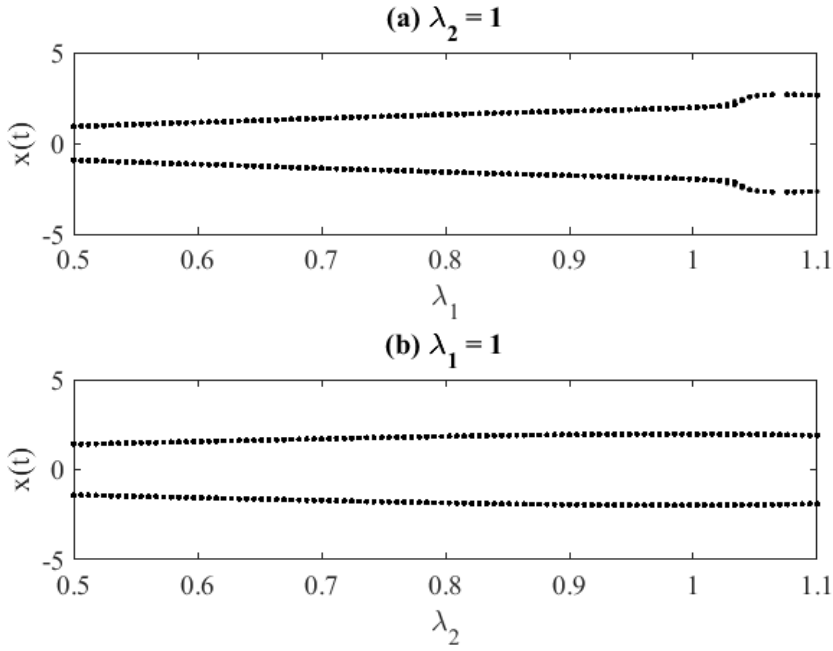

Fig. 11. The bifurcation diagram for the fractional-order biological system (Eq. (10)) computed from Eq. (12) with respect to variations in the arbitrary (non-integer) fractional order showing x-coordinate *versus* arbitrary real order of the derivative within $\lambda \in [0.5, 1.1]$ presented as (a) x *vs.* λ_2 when $\lambda_1 = 1$, (b) x *vs.* λ_1 when $\lambda_2 = 1$. Other system parameters are set as: $\mu = 0.1$, $\alpha = 0.1$, $\beta = 0.2$, $E = 1$, and $\Omega = 0.7$.

6. Conclusion

We leverage on the memory capabilities inherently found in most biological systems to imagine a fractional-order differential model for a type of biological system in place of the integer-order ODE as considered by Kadji *et al.* [37]. The effects of variations in the non-integer-order values of the system derivatives were examined in both the chaotic and the periodic regimes of a preliminary integer-order variant. Results are presented in phase plots and times series, and bifurcation diagrams depicting dynamical changes in system trajectories. The investigation revealed that the dynamics of the system is significantly dependent on the orders of the system derivatives, which implies that not only the bifurcation parameters of the system but also the memory parameters or a cooperation of both can be used to alter or control the system. In particular, we have shown that the chaotic oscillation of the integer-order system can be driven into periodic state by adopting an appropriate non-integer order when the system is modeled as a fractional

differential equation. In this direction, the results of our investigation can be of importance to researchers, especially in the field of nonlinear sciences and complex systems where emphases are placed on bifurcation parameters of physical systems in order to control their complex fascinating phenomena.

However, the fractional calculus model of the biological system (Eq. (4)) was not derived *ab initio*, instead a modification to an existing ODE equivalent was used. Thus, our modified model may potentially be an approximation to the applicable FDE model of the biological system, and the contributions of some system properties might have been omitted. Importantly, the stability conditions of the system was not analyzed. Moreover, the control effects of the arbitrary non-integer orders together with development of an appropriate controller were not implemented. We believe the analysis of the stability of the controlled system and synchronization of coupled systems are still open and challenging. We shall follow up this research in this direction.

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