A HIDDEN SYMMETRY OF CONFORMALLY INVARIANT LAGRANGIANS

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In this paper, a hidden extra symmetry of conformally invariant Lagrangians occuring in physics is pointed out. This symmetry is most apparent in a metric-independent, *i.e.* in a Palatini-like presentation of the variational problem. In such a presentation, the usual conformal weight of fields can be encoded as local dilatation group gauge charges. The conventional conformal invariance of Lagrangians is then equivalent to dilatation gauge invariance. The claim of the paper is that the most commonly occurring conformally invariant Lagrangians turning up in physics are not only invariant to local dilatation gauge transformations, but they are also invariant to any change of the dilatation gauge connection, meaning an additional algebraic symmetry property. In terms of dimensional analysis and differential geometry, this additional symmetry means complete insensitivity of the Lagrangian to the choice of the parallel transport rule of local measurement units throughout spacetime.

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1. Introduction

Conformally invariant field theories form a very important class of models when it comes to the general relativistic (GR) model building related to particle physics. For instance, the kinetic terms of the general relativistically formulated Standard Model (SM) Lagrangian are all conformally invariant, or can be made conformally invariant with rather plausible generalization.

Given a general relativistic spacetime model (M, g), M being a four-dimensional real smooth manifold and g being a smooth Lorentz signature metric tensor field over it, a conformal transformation is a pair (ϕ, Ω), where ϕ is an $M \to M$ diffeomorphism and Ω is an $M \to \mathbb{R}^+$ positive valued smooth scalar field. A conformal transformation (ϕ, Ω) maps a metric tensor field g to another one by the group action $\Omega^2 \phi^* g$, where ϕ^* denotes the pullback operation of the diffeomorphism ϕ[1]. An important subgroup of these transformations are the conformal rescalings, also called *Weyl rescalings*. For Weyl rescalings, the diffeomorphism ϕ is the identity of M and Ω is kept to be an arbitrary positive-valued smooth scalar field. In a field theory, a group action of the Weyl rescalings on the fundamental fields may be specified, called to be the conformal weights, and then the conformal group can act on all the fields in the theory simultaneously. Whenever the field equations or the action functional of the model are invariant to such a group action, the theory is called conformally invariant [1–3].

In the above conventional definition of conformal invariance, one needs first a field equation or Lagrangian, and the spacetime metric must be one of the fundamental fields. Then, a group action of the Weyl rescalings of the metric on all the fields must be specified. Only if all these group actions are declared, the invariance of the field equations or the action functional can be stated.

In this paper, a less metric-dependent formulation of conformal invariance is used. In that approach, all the fundamental fields are regarded as fields with D(1) gauge charge, analogous to the conformal weight[2]. The spacetime metric tensor field may as well be a composite field, *i.e.* some function of more fundamental fields, such as its spinorial decomposition. In such D(1) gauge theory reformulation, the usual conformal invariance in terms of Weyl rescalings is equivalent to a requirement on the action functional to be D(1) gauge invariant. The claim of the paper is that the most common conformally invariant Lagrangians in physics have a slightly larger symmetry than mere conformal invariance: their action functional has an algebraic property of being not only D(1) gauge invariant, but to be completely invariant to the choice of the D(1) gauge connection.

The mentioned phenomenon can most easily be seen on a general relativistic Dirac Lagrangian $\boldsymbol{v}_\gamma \,\mathrm{Re}\left(\bar{\Psi}\gamma^a i\nabla_a\Psi\right)$, where γ_a is the Clifford map associated with a Lorentz metric g_{ab}, $\boldsymbol{v}_\gamma$ is the volume form subordinate to the Clifford map (or equivalently, to the metric), Ψ is a Dirac field, and ∇_b is the combined metric and U(1) gauge covariant derivation [4]. (Penrose abstract indices are used for the tangent indices.) This Lagrangian is very well known to be conformally invariant, *i.e.* invariant to the transformation $\left(\Psi,\, \gamma_a,\, \nabla_b\right) \mapsto \left(\Omega^{-\frac{3}{2}}\Psi,\, \Omega\gamma_a,\, \nabla_b - \frac{1}{2}(i\Sigma_b{}^c - \delta_b{}^c I)(\Omega^{-1}d_c\Omega)\right)$, where $\Sigma_{ab} := \frac{i}{2}\left(\gamma_a\gamma_b - \gamma_b\gamma_a\right)$ stands for the *spin tensor*. The transformation rule of ∇_b is very well known to be uniquely determined by the requirement that the transformation preserves the metricity, torsion-freeness and the compatibil-

[1] The group of conformal transformations is not to be confused with the conformal diffeomorphism group, which is a subgroup of the group of conformal transformations, satisfying the condition $\Omega^2\phi^*g = g$, also called conformal isometries.

[2] The *dilatation group*, D(1), is $\mathbb{R}^+$ with the real multiplication.

ity to the Clifford map. This ordinary Dirac Lagrangian may be generalized in a quite straightforward way: the covariant derivation ∇_b may be generalized by incorporating a D(1) gauge potential as well. In that case, the pertinent Dirac Lagrangian is invariant as well to a slightly different transformation $\big(\varPsi,\,\gamma_a,\,\nabla_b\big) \mapsto \big(\varOmega^{-\frac{3}{2}}\varPsi,\,\varOmega\gamma_a,\,\nabla_b+\varOmega^{-\frac{3}{2}}d_b\varOmega^{\frac{3}{2}}\big)$, which can be considered as a local D(1) gauge transformation. With this generalization, the gradients of $\varOmega$ are absorbed by the D(1) gauge potential, understood within ∇_b. In such variables, it is straightforward to verify that the Dirac Lagrangian is invariant to a further transformation: $\big(\varPsi,\,\gamma_a,\,\nabla_b\big) \mapsto \big(\varPsi,\,\gamma_a,\,\nabla_b + C_b\big)$, where C_b is any D(1) gauge potential, *i.e.* is an arbitrary real valued smooth covector field.

In this paper, we show that the pertinent shift symmetry

$$\nabla_b \;\mapsto\; \nabla_b + C_b \qquad (C_b \text{ being a D(1) gauge potential}) \tag{1}$$

is also the symmetry of any conformally invariant Lagrangian turning up in physics. This algebraic symmetry property is in addition to their conformal invariance, and it means that the pertinent type of Lagrangians are completely insensitive to the parallel transport rule of local measurement units throughout spacetime. Although conformal invariance has been studied extensively (see a comprehensive review in [5, 6]), to our knowledge such an algebraic symmetry property has not yet been explicitly pointed out in the literature.

The structure of the paper is as follows. First, we recall the Lagrangian formulation of classical field theories in metric-independent way, *i.e.* in a Palatini-like approach. Then, we invoke a simple formalism for endowing the fields with D(1) charges without referring to a metric. Following that, we shall present important examples of conformally invariant Lagrangians and show that they have the additional algebraic symmetry property of being invariant to the choice of the D(1) connection, which is the emphasis of the paper. Then, we show a (non-physical) quite trivial counterexample of a conformally invariant Lagrangian, which does not have the pertinent additional symmetry property. Finally, we conclude.

2. Non-metric formulation of classical field theories

In this section, the precise mathematical definition of classical field theories is recalled in terms of the Lagrangian and variational principles: for a comprehensive overview, see *e.g.* [7–9]. The used definition deliberately does not refer to an *a priori* known spacetime metric tensor field, and thus resembles basically to a Palatini-type formulation [1]. In the following, we shall denote the tangent bundle of a manifold M by $T(M)$, and by $T^*(M)$ the corresponding cotangent bundle. The vector bundle of maximal forms is denoted by $\overset{n}{\wedge}T^{*}(M)$, where $n := \dim(M)$. In the following, every differential geometrical object is assumed to be smooth for simplicity of presentation: strict differentiability counting is performed in [7]. The vector space of smooth sections of some vector bundle $V(M)$ over M is denoted by $\Gamma(V(M))$. The affine space of covariant derivations over $V(M)$ shall be denoted by $D(V(M))$. The corresponding dual vector bundle of $V(M)$ is denoted by $V^{*}(M)$. These notations are the usual ones in differential geometry literature. In addition, we shall use Penrose abstract indices [1, 2] for denoting tensor traces and expressions concerning the tensor powers of $T(M)$ and $T^{*}(M)$. The abstract indices of $T(M)$ shall be denoted by superscripted lower case Latin letters ($^{abcd...}$), whereas for $T^{*}(M)$ subscripted lower case Latin letters ($_{abcd...}$) shall be used. The index symmetrization operation shall be denoted by round brackets, *e.g.* $t_{(abc)}$, whereas the antisymmetrization operation shall be denoted by square brackets, *e.g.* $t_{[abc]}$, furthermore, their normalization convention shall be set as in *e.g.* [1].

Let us recall that the space of smooth sections $\Gamma(V(M))$ of some vector bundle $V(M)$ admits a natural $\mathcal{E}$ test function topology [7]: without any further assumption, it is meaningful to define convergence of a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $\Gamma(V(M))$ to a limit φ in $\Gamma(V(M))$ with requiring that the field $(\varphi-\varphi_k)_{k\in\mathbb{N}}$ and all of its gradients uniformly converge to zero on any compact region of M[3]. Whenever the manifold M is compact, or a fixed compact region $K \subset M$ is considered, the $\mathcal{E}$ topology naturally gives rise to a norm equivalence class on the fields over the pertinent region [7, 10]. Because of that, ordinary (Fréchet) derivatives of functionals of such local fields can be naturally defined without further mathematical assumptions.

As usual in the differential geometry literature [1], a covariant derivation on a vector bundle $V(M)$ may be uniquely extended to all the tensor powers of $V(M)$ and its dual bundle $V^{*}(M)$ by requiring Leibniz rule over tensor product, commutativity with tensor contraction, and correspondence to the exterior derivation over the scalar line bundle $M \times \mathbb{R}$. Similarly, given two different vector bundles over M along with covariant derivation on each, then they naturally give rise to a joint covariant derivation, which uniquely extends to all tensor powers of the pertinent vector bundles and their duals, by requiring analogous properties.

Remark 1. *If ∇ is a covariant derivation over $T(M)$, then there is a unique covariant derivation $\tilde{\nabla}$ over $T(M)$ associated to it, having vanishing torsion tensor and having the same geodesics as ∇. The covariant derivation $\tilde{\nabla}$*

[3] A pointwise change of the norms and covariant derivation operators acting on $\Gamma(V(M))$, used for the definition of such a convergence notion, form a norm equivalence class in each point of M, as *e.g.* summarized in [10] Appendix A Lemma 3. Due to that pointwise norm equivalence, the $\mathcal{E}$ convergence notion does not depend on the particular choice of these auxiliary mathematical objects.

is called the torsion-free part *of* ∇. *In explicit formulae: whenever* v^b *is a smooth section of* $T(M)$, *then one has* $\tilde{\nabla}_a v^b = \nabla_a v^b + \frac{1}{2}T(\nabla)^b_{ac}v^c$, *where* $T(\nabla)^b_{ac}$ *denotes the torsion tensor of* ∇.

Remark 2. *Let* $J^a_{[c_1\dots c_n]}$ *be a smooth section of* $T(M)\otimes\overset{n}{\wedge}T^*(M)$, *i.e. a maximal form valued tangent vector field. Then, given any covariant derivation* ∇ *on* $T(M)$, *one has that the expression* $\tilde{\nabla}_a J^a_{[c_1\dots c_n]}$ *is independent of the choice of the covariant derivation, where* $\tilde{\nabla}$ *denotes the torsion-free part of* ∇. *That is, the divergence of a maximal form valued vector field is naturally defined without further assumptions. Similarly, for a smooth section* $K^{[ab]}_{[c_1\dots c_n]}$ *of* $T(M)\wedge T(M)\otimes\overset{n}{\wedge}T^*(M)$, *one has that* $\tilde{\nabla}_a K^{[ab]}_{[c_1\dots c_n]}$ *is independent of the choice of the covariant derivation and thus the divergence of such a field is naturally defined without further assumptions.*

Given the above notions and observations, a classical field theory may be defined as a quartet

$$(M, V(M), \boldsymbol{L}, S)\ , \tag{2}$$

where M is some finite dimensional differentiable manifold possibly with boundary (this is called the *base manifold* — it models the spacetime or a compactified spacetime with or without a boundary), $V(M)$ is some finite dimensional smooth vector bundle over it, called the *vector bundle of matter fields*. The *Lagrange form* $\boldsymbol{L}$ is then a smooth pointwise fiber bundle morphism

$$V(M)\ \times\ T^*(M)\otimes V(M)\ \times\ T^*(M)\wedge T^*(M)\otimes V(M)\otimes V^*(M) \\ \to \overset{n}{\wedge}T^*(M)\,, \tag{3}$$

taking the triplet of matter fields, matter field gradients, and field strength tensors into a maximal form field. In particular, it acts on the sections as

$$\boldsymbol{L}: \\ \Gamma\left(V(M)\ \times\ T^*(M)\otimes V(M)\ \times\ T^*(M)\wedge T^*(M)\otimes V(M)\otimes V^*(M)\right) \\ \to \Gamma\left(\overset{n}{\wedge}T^*(M)\right), \\ (v, Dv, F)\mapsto \boldsymbol{L}(v, Dv, F)\,. \tag{4}$$

A pair $(v,\nabla)\in\Gamma(V(M))\times D(V(M))$ is called a *field configuration*, which forms an affine space over the vector space *field variations* $\Gamma(V(M))\times\Gamma(T^*(M)\otimes V(M)\otimes V^*(M))$. Given a field configuration (v,∇), the map $(v,\nabla)\mapsto\boldsymbol{L}(v,\nabla v,F(\nabla))$ is called the *Lagrangian expression*, where ∇v is the covariant derivative of v by ∇, and where $F(\nabla)$ denotes the curvature tensor of ∇. Then, the *action functional* $S(K)$ is defined on a compact region $K \subset M$ as the integral of the Lagrangian expression over K

$$\begin{aligned} S(K) : \\ \Gamma(V(M)) \times D(V(M)) \to \mathbb{R}\,, \\ (v,\nabla) \mapsto S_{v,\nabla}(K) := \int\limits_K \boldsymbol{L}(v, \nabla v, F(\nabla))\,. \end{aligned} \tag{5}$$

(As such, the action functional can be regarded as a Radon measure valued map $S : (v,\nabla) \mapsto S_{(v,\nabla)}(\cdot)$ from the field configurations.) As usually, the solutions of the field equation of the field theory shall be the stationary points of the action functional with the fields having fixed boundary value on ∂K. More concretely, the field $(v,\nabla) \in \Gamma(V(M)) \times D(V(M))$ is said to be a solution of the field theory whenever for all compact regions $K \subset M$ one has

$$D^\circ S_{v,\nabla}(K) = 0\,, \tag{6}$$

where $D^\circ S(K)$ denotes the Fréchet derivative $DS(K)$ of $S(K)$ restricted in its linear variable to the space of vanishing field variations on the boundary set ∂K. In the end, as quite expected [7], this is equivalent to the Euler–Lagrange equations

$$\begin{aligned} D_1\boldsymbol{L}(v,\nabla v, F(\nabla)) - \tilde{\nabla}_a D_2^a \boldsymbol{L}(v,\nabla v,F(\nabla)) &= 0\,, \\ D_2\boldsymbol{L}(v,\nabla v,F(\nabla))(\cdot)v - \tilde{\nabla}_a 2D_3^{[ab]}\boldsymbol{L}(v,\nabla v,F(\nabla))(\cdot) &= 0 \end{aligned} \tag{7}$$

for the fields (v,∇) throughout the interior of any compact region $K \subset M$ and thus throughout M. Here, $D_1\boldsymbol{L}$, $D_2\boldsymbol{L}$, $D_3\boldsymbol{L}$ mean the spacetime pointwise partial derivative of $\boldsymbol{L}$ with respect to its first, second and third argument, respectively, *i.e.* the derivative of the Lagrange form along the matter fields, the matter field gradients, and the curvature tensor. One should note that because of Remark 2, the covariant derivation may be chosen arbitrarily over $T(M)$ in the divergence expressions of Eq. (7).

Remark 3. *Note that whenever a model is considered in which M is compact (possibly with boundary), then the field equations can be written in a simpler form*

$$DS_{v,\nabla}(M) = 0\,. \tag{8}$$

This is quite similar to as in Eq. (6), but variation on the boundary does not need to be excluded. If the variation on the boundary is not suppressed, *then along with the Euler–Lagrange equations (Eq. (7)), one gets additional boundary field equations*

$$\begin{aligned} D_2^a \boldsymbol{L}(v, \nabla v, F(\nabla)) \; &= \; 0\,, \\ 2D_3^{[ab]} \boldsymbol{L}(v, \nabla v, F(\nabla))(\cdot) \; &= \; 0 \end{aligned} \tag{9}$$

over ∂M, which can eventually be used to impose boundary constraints on the fields.

For clarity, we note that in the standard GR terminology, the above approach resembles the Palatini action principle: the covariant derivation is varied independently from the field quantities, in particular, independently from the metric tensor field.

3. Non-metric formulation of conformal invariance

Given a metric-independent formulation of a field theory $(M, V(M), \boldsymbol{L}, S)$ as in the previous section, we introduce a metric-independent notion of conformal weights. For this, we assume that the vector bundle of fields $V(M)$ is composed of sectors having D(1) *gauge charges*, *i.e.*

$$V(M) = \underset{q\in\mathcal{Q}}{\oplus} V_q(M)\,, \tag{10}$$

where $\mathcal{Q}$ is a finite set of rational numbers and a D(1) gauge transformation is represented by a non-vanishing smooth field $\Omega : M \to \mathbb{R}^+$ acting as

$$\begin{aligned} v_q \; &\mapsto \; \Omega^q\, v_q & &(v_q \in V_q(M),\; q \in \mathcal{Q})\,, \\ \nabla \; &\mapsto \; \Omega^q \nabla \Omega^{-q} = \nabla - q\, \mathrm{d}\,(\ln \Omega) & &(\text{over } V_q(M),\; q \in \mathcal{Q}) \end{aligned} \tag{11}$$

on the fields and covariant derivations, where 'd' denotes exterior derivation. The numbers $q \in \mathcal{Q}$ are called D(1) *gauge charges*, and such transformations are called D(1) *gauge transformations*. Whenever a spacetime metric tensor field is present in the theory with non-zero D(1) gauge charge, then quite evidently, the field rescalings induced by the D(1) group can always be redefined such that the spacetime metric tensor field has D(1) gauge charge 2 by convention, *i.e.* belonging to $V_2(M)$, which is just an equivalent reformulation of the Weyl rescaling, *i.e.* that the metric transforms as $g_{ab} \mapsto \Omega^2 g_{ab}$ by convention. With these notions, a field theory $(M, V(M), \boldsymbol{L}, S)$ is said to be D(1) *gauge invariant* whenever its action functional is invariant to the D(1) gauge transformations as in Eq. (11).

It shall be shown in the following sections that the conformally invariant Lagrangians turning up in physics have a slightly larger symmetry than simple D(1) gauge invariance: their action functional does not depend on the

D(1) gauge connection at all. In terms of formulas, this means an algebraic property of the invariance of the Lagrangian expression $\boldsymbol{L}(v, \nabla v, F(\nabla))$ to the transformation

$$\begin{aligned} v &\mapsto v\,, \\ \nabla &\mapsto \nabla + q\,C \qquad \text{(over each sector } V_q(M),\, q \in \mathcal{Q}) \end{aligned} \tag{12}$$

for any real smooth covector field $C \in \Gamma(T^*(M))$. This property shall be called D(1) *connection invariance*, and is seen to be a further symmetry on top of invariance to simple Weyl rescalings. It means that the theory is invariant to the choice of the parallel transport of local measurement units throughout spacetime.

3.1. More geometric reformulation using measure line bundles

The notion of D(1) gauge charge can be reformulated in a geometrically even more elegant setting. The key idea is motivated by a work of Matolcsi [11] and of Janyška, Modugno, Vitolo [12], in which they proposed a simple mathematical framework for formal mathematical handling of physical units. In their concept, the mathematical model of special relativistic spacetime is considered to be a triplet (M, L, η), where M is a four-dimensional real affine space (modeling the flat spacetime), L is a one-dimensional vector space (modeling the one-dimensional vector space of length values), and $\eta : \overset{2}{\vee} T \to \overset{2}{\otimes} L$ is the flat Lorentz signature metric (constant throughout the spacetime), where T is the underlying vector space of M (tangent space). The important idea in that construction is: the field quantities, such as the metric tensor, are not simply real valued, but they take their values in the rational tensor powers of the *measure line* L [4]. Such a setting formalizes the physical expectation that quantities actually have physical dimensions (the metric carries length-square dimension in this case), and that quantities with different physical dimensions cannot be added since they reside in different vector spaces. It is seen that the technique of measure lines is nothing but the precise mathematical formulation of dimensional analysis.

Such a mathematically precise formulation of dimensional analysis, although may seem to be a relatively innocent idea at a first glance, becomes quite powerful tool when carried over to a general relativistic framework. Namely, let our base manifold M be some four dimensional real manifold (with or without boundary), and let $L(M)$ be a real vector bundle over M, with one-dimensional fiber. The fiber of $L(M)$ over each point of M shall

[4] The term *measure line* was introduced by [11], whereas the same concept is called *scale space* by [12]. Apparently, these two groups of authors discovered the pertinent rather useful notion independently.

model the vector space of length values, and the pertinent line bundle shall be called the *measure line bundle*, or *line bundle of lengths*. Just like proposed in [11, 12], the field quantities shall carry certain tensor powers of $L(M)$. For simplicity, the notation $L^n(M) := \overset{n}{\otimes} L(M)$ and $L^{-n}(M) := \overset{n}{\otimes} L^*(M)$ shall be used, for all non-negative integers n, conforming to the conventions of [11, 12], and also to our physical intuition of dimensional analysis[5]. The proposed idea can be physically formulated as: the field quantities are tagged with physical dimensions, but the units of the physical dimensions in different spacetime points are not necessarily comparable *a priori*, but a connection over $L(M)$ needs to be explicitly specified for that. Using these conventions and the analogy of Eq. (10), we assume that the structure of the vector bundle of matter fields takes the form of

$$\begin{aligned} V(M) &= \underset{q\in\mathcal{Q}}{\oplus} V_q(M)\,, \\ \text{with}\quad V_q(M) &= L^q(M)\otimes\mathcal{V}_q(M) \qquad (\text{for all } q\in\mathcal{Q})\,. \end{aligned} \tag{13}$$

This form of Eq. (10) helps to book-keep the physical dimensions of quantities in a quite transparent way: the rational numbers $q \in \mathcal{Q}$ are called *physical dimensions*, the factors $L^q(M)$ are seen to count the physical dimensions, and $\mathcal{V}_q(M)$ would represent the dimension-free form of field quantities, but the true physical fields reside in $V_q(M) = L^q(M) \otimes \mathcal{V}_q(M)$, carrying appropriate dimensions. The pointwise $L(M) \to L(M)$ vector bundle automorphisms are equivalent to D(1) gauge transformations as discussed previously, and the power q in the $L^q(M)$ factor shall then automatically correspond to the D(1) gauge charge. Thus, using the formalism of measure line bundle makes the D(1) gauge charge, *i.e.* the physical dimension of the fields explicit. From the dimensional analysis point of view, all this can simply be understood as: the fields are tagged by physical dimensions, but the unit of measurement might be spacetime-point-dependent. Since only pure real valued maximal form fields may be integrated throughout the manifold, $\boldsymbol{L}$ must be pure $\overset{n}{\wedge} T^*(M)$ valued, without physical dimension in terms of powers of $L(M)$. This consistency requirement already poses algebraic constraints on the possible Lagrangians and shows the advantage of not neglecting the physical dimensions of fields in the formalism.

It is evidently seen that the property of D(1) connection invariance of an above-type model is just equivalent to the independence of the Lagrangian expression $\boldsymbol{L}(v, \nabla v, F(\nabla))$ from the $L(M)$ connection. We shall call such

[5] Note that due to the one dimensionality of the measure line L, rational tensor powers are also well-defined as seen in [11, 12]. Roughly speaking, if L is a one-dimensional real vector space, then $\sqrt[n]{L}$ is defined to be the one-dimensional vector space obeying $\overset{n}{\otimes}\left(\sqrt[n]{L}\right) \equiv L$ for a given non-negative integer n.

models *measure line connection invariant*. In the coming section, it is shown that the conformally invariant Lagrangians turning up in physics possess this property, which means that these type of Lagrangians are insensitive to the parallel transport rule of measurement units throughout spacetime.

4. Examples

In the following, the important conformally invariant Lagrangians are recalled. It is shown that these all have an additional symmetry of being invariant to the choice of the connection on the measure line bundle, or equivalently, to the choice of the D(1) gauge connection.

4.1. Conformal invariant version of vacuum general relativity

For illustrative purpose, we present the formulation of the conformally invariant generalization of vacuum GR. The model is specified via a slightly generalized form of the Einstein–Hilbert Lagrangian. Namely, let the base manifold M be 4 real dimensional and oriented, and the vector bundle of fields to be $V(M) := L^{-1}(M) \oplus L^2(M)\otimes \overset{2}{\vee} T^*(M)$, where $L(M)$ is the line bundle of lengths. Let the symbol $\boldsymbol{v}(g)$ denote the canonical volume form field generated by the dimensional metric tensor field $g\in\Gamma\Big(L^2(M)\otimes\overset{2}{\vee}T^*(M)\Big)$, taking its values in $\Gamma\Big(L^4(M)\otimes\overset{4}{\wedge}T^*(M)\Big)$, *i.e.* having dimension length to the four, as physically expected. We take then the Lagrange form to be

$$\begin{aligned}
&\boldsymbol{L}: \\
&\Gamma\left(V(M)\times T^*(M)\otimes V(M)\times T^*(M)\wedge T^*(M)\otimes V(M)\otimes V^*(M)\right) \\
&\qquad\to\Gamma\left(\overset{4}{\wedge}T^*(M)\right), \\
&\Big((\varphi,g_{ab}),\,(D\varphi_c,Dg_{def}),\,\big(r_{gh},R_{ghi}{}^{j}\big)\Big)\;\mapsto\;\boldsymbol{v}(g)\varphi^2 g^{km}\delta^l{}_n R_{klm}{}^n\,. \qquad (14)
\end{aligned}$$

As already mentioned in Section 2, in our variational scheme, the quantities are varied independently, *i.e.* no *a priori* relation is assumed between the metric and covariant derivation, furthermore, also the torsion of the covariant derivation is not restricted initially. It is seen that Eq. (14) simply corresponds to the standard Einstein–Hilbert Lagrangian with a slight generalization: the inverse Planck length (here denoted by φ) is not assumed to be constant, but can (must) have location dependence, *i.e.* it is rather a field than a constant in this model, as it is set to be a section of the vector bundle $L^{-1}(M)$. With this simple generalization, the theory becomes measure-line-connection-invariant, and hence D(1) connection-invariant in terms of our definition in Section 3. This is verified by directly observing that at any field configuration $((\varphi, g_{ab}), \nabla_c)$, the Lagrangian expression

$$\boldsymbol{v}(g)\varphi^2 g^{ab} R(\nabla)_{acb}{}^{c} \tag{15}$$

is invariant to the transformation, Eq. (12), *i.e.* does not depend on the covariant derivation over the line bundle of lengths, where $R(\nabla)_{acb}{}^{d}$ is the Riemann tensor of ∇.

The field equations are derived by direct substitution of $\boldsymbol{L}$ in Eq. (14) into Eq. (7), along with the subsequent usage of the identities $\frac{\partial \boldsymbol{v}(g)}{\partial g_{ab}} = \frac{1}{2} g^{ab}\boldsymbol{v}(g)$ and $\frac{\partial g^{cd}}{\partial g_{ab}} = -\frac{1}{2}\left(g^{ca}g^{bd} + g^{cb}g^{ad}\right)$. Straightforward calculations show (see also [7]) that the field equations read as

$$\begin{aligned} \tilde{\nabla}_a\left(\varphi^2 g_{bc}\right) &= 0\,, \\ \varphi^2 E\left(\nabla, \varphi^2 g\right)_{ab} &= 0 \end{aligned} \tag{16}$$

throughout M, where

$$\begin{aligned} E\left(\nabla, \varphi^2 g\right)_{ab} &:= \\ \tfrac{1}{2}R(\nabla)_{acb}{}^{c} + \tfrac{1}{2}R(\nabla)_{bca}{}^{c} - \tfrac{1}{2}\left(\varphi^2 g_{ab}\right)&\left(\varphi^{-2}g^{ef}\right)R(\nabla)_{ecf}{}^{c} \end{aligned} \tag{17}$$

is the Einstein tensor defined by ∇_a and $\varphi^2 g_{bc}$, whereas $\tilde{\nabla}$ denotes the torsion-free part of the covariant derivation ∇, furthermore $R(\nabla)_{abc}{}^{d}$ is its Riemann tensor. Equation (16) can be transformed to a more familiar form via introducing the notation

$$\begin{aligned} \mathcal{T}\left(\nabla, \varphi^2 g\right)_{ab} &:= \\ \tfrac{1}{4}\Bigg(\tilde{\nabla}_a T(\nabla)^{g}_{bg} + \tilde{\nabla}_b T(\nabla)^{g}_{ag} &+ T(\nabla)^{h}_{ga} T(\nabla)^{g}_{bh} \\ -\tfrac{1}{2}\left(\varphi^2 g_{ab}\right)\left(\varphi^{-2}g^{ef}\right)\left(2\tilde{\nabla}_e T(\nabla)^{g}_{fg} + T(\nabla)^{h}_{ge} T(\nabla)^{g}_{fh}\right)&\Bigg), \end{aligned} \tag{18}$$

where $T(\nabla)^{c}_{ab}$ is the torsion tensor of ∇. Using this, Eq. (16) is equivalent to

$$\begin{aligned} \tilde{\nabla}_a\left(\varphi^2 g_{bc}\right) &= 0\,, \\ \varphi^2 E\left(\tilde{\nabla}, \varphi^2 g\right)_{ab} &= \varphi^2 \mathcal{T}\left(\nabla, \varphi^2 g\right)_{ab}\,, \end{aligned} \tag{19}$$

which is obtained by the well-known identity between the Riemann tensor of a covariant derivation and the Riemann tensor of its torsion-free part.

The obtained field equation is nothing but an ordinary vacuum Einstein equation for the dimension-free metric $\varphi^2 g_{ab}$, *i.e.* for the metric tensor measured in units of square Planck length φ^{-2} in each spacetime point. Quite obviously, presence of matter fields will generate contribution to Eq. (19) in terms of energy-momentum tensor as a source on the right-hand side. It should be noted that whenever the torsion tensor $T(\nabla)^{c}_{ab}$ is not assumed to be zero *a priori*, it contributes to the energy-momentum tensor as seen from Eq. (19).

Remark 4. *The presented variational problem may be reformulated on the closed affine subspace of torsion-free covariant derivations, in which case the torsion tensor $T(\nabla)^{c}_{ab}$ automatically vanishes and thus the source term $\mathcal{T}(\nabla, \varphi^2 g)_{ab}$ vanishes on the right-hand side of Eq. (19) along with having automatically $\tilde{\nabla}_a = \nabla_a$. That would mean ordinary vacuum Einstein equations for the dimension-free metric $\varphi^2 g_{ab}$.*

The field equations (Eq. (19)) may be re-expressed also in terms of the original metric g_{ab} which is not rescaled to be dimensionless. More specifically, Eq. (19) is seen to be equivalent to

$$\begin{aligned}
\tilde{D}_a(g_{bc}) &= 0\,, \\
E(\tilde{D}, g)_{ab} &= \mathcal{T}\left(\nabla, \varphi^2 g\right)_{ab} \\
&\quad +\varphi^{-1}\tilde{D}_a\tilde{D}_b\varphi + \varphi^{-1}\tilde{D}_b\tilde{D}_a\varphi \\
&\quad -2g_{ab}\,g^{ef}\varphi^{-1}\tilde{D}_e\tilde{D}_f(\varphi) \\
&\quad -4\varphi^{-1}\tilde{D}_a(\varphi)\,\varphi^{-1}\tilde{D}_b(\varphi) \\
&\quad +g_{ab}\,g^{ef}\varphi^{-1}\tilde{D}_e(\varphi)\,\varphi^{-1}\tilde{D}_f(\varphi)\,, \\
g^{ab}\tilde{D}_a\tilde{D}_b\varphi - \tfrac{1}{6}\mathcal{R}\left(\tilde{D}, g\right)\varphi &= \tfrac{1}{6}g^{ab}\mathcal{T}\left(\nabla, \varphi^2 g\right)_{ab}\varphi\,,
\end{aligned} \tag{20}$$

where in this case $\tilde{D}_a$ is a torsion-free covariant derivation over $L^{-1}(M)\otimes T(M)$ such that it is metric-compatible ($\tilde{D}_a(g_{bc}) = 0$), furthermore $E(\tilde{D}, g)_{ab}$ is the Einstein tensor of $\tilde{D}_a$ and g_{bc}, whereas $\mathcal{R}(\tilde{D}, g)$ is the Ricci scalar of $\tilde{D}_a$ and g_{bc}. The obtained field equation is seen to be nothing but the coupled conformally invariant Einstein–Klein–Gordon equation for g_{ab} and φ, along with some source term coming from a possible torsion contribution (which may be zeroed out by means of Remark 4). Again, when further matter fields are present, they contribute to the right-hand side in terms of an energy-momentum tensor. The field equations Eq. (20) are known to be conformally invariant in the conventional sense of Weyl rescalings.

Remark 5. *Whenever the base manifold M has a boundary ∂M and the variation on the manifold boundary is allowed as in Remark 3, the boundary field equations read as*

$$\varphi^2 g_{ab} = 0 \qquad \text{(throughout } \partial M)\,. \tag{21}$$

The field equations (Eq. (19) and Eq. (21)) mean together that the dimension-free metric $\varphi^2 g_{ab}$ and its Levi-Civita covariant derivation $\tilde{\nabla}_a$ obey vacuum Einstein equations with a possible additional source term originating from the torsion of ∇_a. Furthermore, the dimension-free metric $\varphi^2 g_{ab}$ is pressed to zero as approaching the boundary with a conformal scaling factor (just like the asymptotical behavior in the case of Friedman–Robertson–Walker cosmological solutions).

Remark 6. *It is worth to note that whenever the torsion is not zeroed out a priori, a dynamical torsion theory arises. However, due to our non-metric (Palatini-like) variational principle, the field equations will be slightly different than that of the Einstein–Cartan–Sciama–Kibble theory [13]. The essential difference is: not the original covariant derivation ∇_a is compatible with the dimension-free metric $\varphi^2 g_{ab}$ as in ECSK theory, but the torsion-free part of it. That is, if the torsion is not required to be zero a priori, the field equations are a simple Einstein theory for $\tilde{\nabla}_a$ and $\varphi^2 g_{ab}$, but the torsion $T(\nabla)^c_{ab}$ also contributes to the energy-momentum tensor. In addition, one obtains the constraint equation*

$$\varphi^{-2} g^{ab} \tilde{\nabla}_a \mathcal{T}\left(\nabla, \varphi^2 g\right)_{bc} = 0 \tag{22}$$

for the torsion tensor due to the automatic vanishing of the divergence of the Einstein tensor because of the Bianchi identities. It is seen that Eq. (19) along with Eq. (22) is slightly different than that of ECSK field equations [13].

4.2. Spinorial formulation of conformally invariant version of vacuum general relativity

The proposed metric-independent definition of conformal invariance becomes particularly useful when dealing with non-metric theories, *i.e.* with models in which the spacetime metric tensor is a derived quantity, not a fundamental one.

Remark 7. *A simple example for a model in which the metric tensor is not a fundamental quantity can be readily given with spinorial formulation [1, 2] of conformally invariant version of general relativity. In that approach, one has a spinor bundle $S(M)$ with two complex dimensional fibers over the real four manifold M. The Lagrange form is the spinorial representation of the conformally invariant Einstein–Hilbert Lagrangian (see also Eq. (14))*

$$\begin{aligned} &\boldsymbol{L} : \\ &\Gamma\left(V(M) \times T^*(M) \otimes V(M) \times T^*(M) \wedge T^*(M) \otimes V(M) \otimes V^*(M)\right) \\ &\qquad \to \Gamma\left(\overset{4}{\wedge} T^*(M)\right), \end{aligned}$$

$$\begin{aligned}&\Big(\Big(\varphi,\epsilon_{AB},\sigma_a^{AA'},\chi^B\Big),\Big(D\varphi_b,D\epsilon_{ABb},D\sigma_a^{AA'}{}_b,D\chi^B{}_b\Big),\\&\qquad\Big(r_{ab},\rho_{abAB}{}^{CD},\Pi_{abc}{}^{AA'd}{}_{BB'},P_{abA}{}^B\Big)\Big)\\&\mapsto\ \boldsymbol{v}(g(\sigma,\epsilon))\varphi^2g(\sigma,\epsilon)^{ac}\Big(\sigma_c^{AA'}\bar{P}_{abA'}{}^{B'}\sigma^b_{AB'}+\sigma_c^{AA'}P_{abA}{}^B\sigma^b_{BA'}\Big),\quad(23)\end{aligned}$$

where $V(M):=L^{-1}(M)\oplus L(M)\otimes\overset{2}{\wedge}S^*(M)\oplus T^*(M)\otimes\bar{S}(M)\otimes S(M)\oplus L^{-1}\otimes S(M)$*. Here,* $g(\sigma,\epsilon)_{ab}:=\sigma_a^{AA'}\sigma_b^{BB'}\bar{\epsilon}_{A'B'}\epsilon_{AB}$ *denotes the canonical Lorentz metric tensor generated by an* $\epsilon_{AB}\in\Gamma\Big(L(M)\otimes\overset{2}{\wedge}S^*(M)\Big)$ *and* $\sigma_a^{AA'}\in\Gamma\left(T^*(M)\otimes\bar{S}(M)\otimes S(M)\right)$*, furthermore,* $\boldsymbol{v}(g(\sigma,\epsilon))$ *denotes the canonical volume form generated by* $g(\sigma,\epsilon)_{ab}\in\Gamma\Big(L^2(M)\otimes\overset{2}{\vee}T^*(M)\Big)$*. In the notation, Penrose abstract indices were used according to the conventions of [1, 2]. It is seen by direct substitution that the model defined by this Lagrange form is measure-line-connection-invariant, and hence is* D(1) *gauge-connection-invariant in the sense of Section 3. This is verified by directly observing that for any field configuration* $\Big((\varphi,\epsilon_{AB},\sigma_a^{AA'},\chi^B),\nabla_b\Big)$*, the Lagrangian expression*

$$\boldsymbol{v}(g(\sigma,\epsilon))\varphi^2g(\sigma,\epsilon)^{ac}\Big(\sigma_c^{AA'}\bar{P}(\nabla)_{abA'}{}^{B'}\sigma^b_{AB'}+\sigma_c^{AA'}P(\nabla)_{abA}{}^B\sigma^b_{BA'}\Big)\quad(24)$$

is invariant to the transformation Eq. (12), i.e. does not depend on the covariant derivation over the line bundle of lengths, where $P(\nabla)_{abB}{}^D$ *is the spinorial curvature tensor of* ∇*.*

4.3. Dirac kinetic term

The below example shows how the definition of conformal invariance in terms of connection works for the Dirac kinetic term, which is a classic example of non-trivial Lagrangians known to be conformally invariant in terms of Weyl rescalings.

Remark 8. *For the definition of Dirac Lagrangian, we refer again to [1, 2] for spinorial notations. One has a spinor bundle* $S(M)$ *with two complex dimensional fibers over the real four manifold* M*. The Lagrangian reads as*

$$\begin{aligned}&\boldsymbol{L}:\\&\Gamma\left(V(M)\times T^*(M)\otimes V(M)\times T^*(M)\wedge T^*(M)\otimes V(M)\otimes V^*(M)\right)\\&\qquad\to\Gamma\Big(\overset{4}{\wedge}T^*(M)\Big),\\&\Big(\Big(\varphi,\epsilon_{AB},\sigma_a^{AA'},\chi^B,\bar{\xi}_{C'}\Big),\Big(D\varphi_b,D\epsilon_{ABb},D\sigma_a^{AA'}{}_b,D\chi^B{}_b,D\bar{\xi}_{C'b}\Big),\end{aligned}$$

$$\begin{aligned}&\Big(r_{ab}, \rho_{abAB}{}^{CD}, \Pi_{abc}{}^{AA'd}{}_{BB'}, P_{abA}{}^{B}, \bar{Q}_{abA'}{}^{B'}\Big)\Big)\\ \mapsto\; &\boldsymbol{v}(g(\sigma,\epsilon))\, g(\sigma,\epsilon)^{ab}\,\sqrt{2}\,\sigma_a^{AA'}\mathrm{Re}\Big(\bar{\epsilon}_{A'B'}\epsilon_{AB}\bar{\chi}^{B'}iD\chi^{B}{}_{b}+\xi_A iD\bar{\xi}_{A'b}\Big)\;, \quad (25)\end{aligned}$$

where $V(M) := L^{-1}(M) \oplus L(M)\otimes \overset{2}{\wedge} S^*(M) \oplus T^*(M)\otimes\bar{S}(M)\otimes S(M) \oplus L^{-2}(M)\otimes S(M) \oplus L^{-1}(M)\otimes\bar{S}^*(M)$*. The definition of* $g(\sigma,\epsilon)_{ab}$ *and* $\boldsymbol{v}(g(\sigma,\epsilon))$ *is the same as in Section 4.2. It is seen by direct substitution that the model defined by this Lagrange form is measure-line-connection-invariant, and hence is* D(1) *gauge-connection-invariant in the sense of Section 3. This is verified by directly observing that for any field configuration*

$$\Big((\varphi, \epsilon_{AB}, \sigma_a^{AA'}, \chi^B, \bar{\xi}_{C'}), \nabla_b\Big)\;,$$

the Lagrangian expression

$$\boldsymbol{v}(g(\sigma,\epsilon))\, g(\sigma,\epsilon)^{ab}\,\sqrt{2}\,\sigma_a^{AA'}\,\mathrm{Re}\left(\bar{\epsilon}_{A'B'}\epsilon_{AB}\bar{\chi}^{B'}i\nabla_b\left(\chi^B\right) + \xi_A i\nabla_b\left(\bar{\xi}_{A'}\right)\right) \quad (26)$$

is invariant to the transformation Eq. (12), i.e. does not depend on the covariant derivation over the line bundle of lengths.

4.4. Yang–Mills kinetic term

Our last example shows how the definition of conformal invariance in terms of connection works for the Yang–Mills kinetic term, which is another classic example of non-trivial Lagrangians known to be conformally invariant in terms of Weyl rescalings.

Remark 9. *For the formulation of the Yang–Mills Lagrangian, we postulate that our gauge group is a compact real Lie group. This implies that any element of a finite dimensional real linear representation of its Lie algebra has vanishing real part of its trace. The Lagrangian reads as*

$$\begin{aligned}&\boldsymbol{L}:\\ &\Gamma\left(V(M)\times T^*(M)\otimes V(M)\times T^*(M)\wedge T^*(M)\otimes V(M)\otimes V^*(M)\right)\\ &\qquad\to \Gamma\left(\overset{4}{\wedge}\, T^*(M)\right)\;,\\ &\left((\Phi, g_{ab}),\, (D\Phi_c, Dg_{def}),\, \left(F_{gh}, R_{ghi}{}^{j}\right)\right)\\ \mapsto\;& A\,\boldsymbol{v}(g)\, g^{ac}\, g^{bd}\;\mathrm{Tr}\left(\left(F_{ab} - \frac{1}{\mathrm{Tr}\, I} I\,\mathrm{Tr}\, F_{ab}\right)\left(F_{cd} - \frac{1}{\mathrm{Tr}\, I} I\,\mathrm{Tr}\, F_{cd}\right)\right)\\ &+ B\,\boldsymbol{v}(g)\, g^{ac}\, g^{bd}\;\mathrm{Im}\left(\mathrm{Tr}\, F_{ab}\right)\mathrm{Im}\left(\mathrm{Tr}\, F_{cd}\right)\;, \quad (27)\end{aligned}$$

where $V(M) := Y(M) \oplus L^2(M)\otimes \overset{2}{\vee}\, T^*(M)$*. Here,* $Y(M)$ *is the vector bundle of matter fields in the Yang–Mills theory, possibly internally also* *tagged with physical dimensions in terms of tensor powers of $L(M)$. The symbol $\boldsymbol{v}(g)$ means the canonical volume form generated by the metric tensor g_{ab} as previously, and I is the identity over the sections of $Y(M)$, whereas* Tr *denotes trace in terms of $Y(M)\otimes Y^*(M)$. In the formula, A and B are real numbers, determining the weights (coupling factors) of the non-Abelian part and the* U(1) *part of the gauge group. It is seen by direct substitution that the model defined by this Lagrange form is measure-line-connection-invariant, and hence is* D(1) *gauge-connection-invariant in the sense of Section 3. This is verified by directly observing that for any field configuration $((\Phi, g_{ab}), \nabla_b)$, the Lagrangian expression*

$$A\,\boldsymbol{v}(g)\,g^{ac}\,g^{bd}\,\mathrm{Tr}\left(\left(F_{\nabla ab}-\frac{1}{\mathrm{Tr}\,I}I\,\mathrm{Tr}\,F_{\nabla ab}\right)\left(F_{\nabla cd}-\frac{1}{\mathrm{Tr}\,I}I\,\mathrm{Tr}\,F_{\nabla cd}\right)\right) + B\,\boldsymbol{v}(g)\,g^{ac}\,g^{bd}\,\mathrm{Im}\,(\mathrm{Tr}\,F_{\nabla ab})\,\mathrm{Im}\,(\mathrm{Tr}\,F_{\nabla cd}) \tag{28}$$

is invariant to the transformation, Eq. (12), i.e. does not depend on the covariant derivation over the line bundle of lengths, where $F_{\nabla ab}$ denotes the curvature tensor of ∇ over $Y(M)$. This invariance property simply follows from the fact that a change of the covariant derivation on $L(M)$ could only give contribution through $\mathrm{Re}(\mathrm{Tr}\,F_{\nabla ab})$*, which is excluded from the Lagrangian expression by construction.*

4.5. A counterexample

Based on the examples presented, one could ask the question whether there is an counterexample, when a theory is locally D(1) gauge-invariant, but it does not possess the extra symmetry of being independent from the choice of the D(1) gauge connection. The answer is affirmative: for example, a Yang–Mills Lagrangian based on the curvature tensor of the D(1) gauge connection would be conformally invariant, would be locally D(1) gauge-invariant, but it would be an explicit function of the D(1) gauge connection, *i.e.* of a connection on the measure line bundle $L(M)$ and, therefore, it would not possess the property of being independent of the choice of the D(1) gauge connection. That is, the pertinent symmetry is not an inherent property of all conformally-invariant Lagrangians, but it seems that the conformally-invariant Lagrangians appearing in realistic models, such as the Standard Model, happen to possess this extra symmetry of being not dependent on the choice of a D(1) connection. With the notations as in Section 4.4, the pertinent non-invariant Lagrangian reads as

$$\boldsymbol{L}:\ \Gamma\left(V(M)\times T^*(M)\otimes V(M)\times T^*(M)\wedge T^*(M)\otimes V(M)\otimes V^*(M)\right) \to \Gamma\left(\overset{4}{\wedge}T^*(M)\right),$$

$$\begin{aligned} &((\Phi, g_{ab}),\ (D\Phi_c, Dg_{de\,f}),\ (F_{gh}, R_{ghi}{}^{j})) \\ &\mapsto\ \boldsymbol{v}(g)\, g^{ac}\, g^{bd}\ \mathrm{Re}\,(\mathrm{Tr}\, F_{ab})\, \mathrm{Re}\,(\mathrm{Tr}\, F_{cd})\ . \end{aligned} \tag{29}$$

Given a field configuration $((\Phi, g_{ab}), \nabla_b)$, the Lagrangian expression reads as

$$\boldsymbol{v}(g)\, g^{ac}\, g^{bd}\ \mathrm{Re}\,(\mathrm{Tr}\, F_{\nabla ab})\, \mathrm{Re}\,(\mathrm{Tr}\, F_{\nabla cd})\ . \tag{30}$$

Clearly, a connection on $L(M)$, possibly contained within the connection on $Y(M)$ will contribute to $\mathrm{Re}\,(\mathrm{Tr}\, F_{\nabla ab})$, and thus the pertinent Lagrangian is conformally invariant, but it is not invariant then to the choice of the connection on $L(M)$.

5. Concluding remarks

In this paper, a metric independent reformulation for the property of conformal invariance for classical field theories was shown. This was done by attaching a D(1) gauge charge (in the analogy of conformal weight) to each fundamental field. It was argued that the D(1) gauge charge is nothing but a physical dimension, and the D(1) gauge connection is nothing but the rule for parallel transport of measurement units between points of spacetime. The D(1) gauge invariance was seen to be equivalent to ordinary conformal invariance in terms of Weyl rescalings of the metric. It was also shown in addition, that conformally invariant Lagrangians turning up in physics have a slightly larger symmetry than that: their action functional is not only D(1) gauge-invariant, but does not depend on the D(1) gauge connection at all.

All this was also presented by a somewhat more elegant geometrical formulation using measure line bundles. In that approach, the field quantities are not simply real valued, but each fundamental field takes its value on the tensor powers of line bundle of lengths. The connection on this line bundle corresponds to the pertinent D(1) gauge connection.

The presented metric independent variational formulation has the advantage of direct applicability to models in which the spacetime metric tensor is not a fundamental quantity, but is some function of other fundamental fields. (Such a situation happens if the spinorial decomposition of the metric is considered to be the fundamental variables, rather than the metric.) The used metric-independent formalism also reveals the discussed hidden symmetry of conformally invariant Lagrangians in a fairly transparent way.

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