A RELATIVISTIC MODEL FOR THE CHARMONIUM SPECTRUM WITH A REDUCED NUMBER OF FREE PARAMETERS*

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A previously introduced reduction of the Dirac equation is used to study the charmonium spectrum. A regularized vector potential that only depends on the coupling constant and the regularization radius is adopted, considering the interacting quark as an extended source of the chromo-electric field. A scalar interaction is also introduced with some constraints for its parameters. A good description of the structure of the charmonium spectrum is obtained with only three free parameters.

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1. Introduction

In a previous work [1], we studied a relativistic reduction of the Dirac equation for quark-composed systems. In that work, we analyzed the theoretical fundaments of that reduced equation and showed that it was able to reproduce the charmonium spectrum with high accuracy but using a relatively large number of free parameters. More precisely, we used eight or nine parameters to reproduce the spectrum, taking into account a possible energy dependence of the interaction.

On the contrary, in the present work, by using the same reduced relativistic equation we shall try to reproduce the main structure of the charmonium spectrum with a very small number of parameters, possibly with evident physical meaning. To this aim, we shall *determine* two parameters of the model in order to reduce the total number of free parameters.

As in [1], we use a vector–scalar interaction model to represent the dynamics of charmonium. In more detail, we shall use a specific form of the vector interaction (possibly related to QCD) that corresponds to a regular-

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ized color interaction of the quarks. To this aim, we assume a non-pointlike distribution of their chromo-electric charge. This model has been studied in detail in work [2].

To avoid repetition, the reader will be directed to the specific parts of works [1] and [2], when necessary. In those works, the reader can also find the references on which the whole study is based. Recalling that the aim of this work is to study the charmonium spectroscopy with a *relativistic* model with only *three free parameters*, we briefly mention below (with no attempt of completeness), some relatively recent studies on this subject.

We first quote the non-relativistic models. In Ref. [3], two models with five parameters are constructed to study the charmonium resonances and their electromagnetic transitions. A momentum-helicity model [4], also with five parameters, was proposed for the charmonium spectrum. A model for the spectrum and decay rates with a Coulomb-like potential, a linear confining potential and a potential derived from the instanton vacuum [5], was studied. This model used four parameters. The instanton effects are also studied in another model with six parameters [6]. The mass spectrum was calculated in the framework of non-relativistic QCD, with seven parameters [7]. The charmonium properties were studied by solving the Schrödinger equation with the discrete variable representation method [8]. This last model used five parameters.

A semirelativistic model with a Coulomb plus linear potential using five parameters [9] was studied.

As for the fully relativistic models, we recall the studies performed by means of the Covariant Spectator Theory, with vector, scalar and pseudoscalar interactions [10, 11]. The authors studied, by means of the same model, heavy and heavy–light mesons. They used three free parameters and a fixed cutoff parameter to regularize the momentum space integrals. Other parameters are the constituent quark masses, and the weight of scalar and pseudoscalar coupling for the confining interaction. The authors analyzed the dependence of the results on these last parameters by means of different calculations in which they are considered as fixed or as free parameters.

Another relativistic model was based on the use of a momentum space integral equation with positive energy Dirac spinors. A complete one-gluon exchange interaction with other phenomenological scalar terms was used [12]. In that work, two different potentials were considered, with seven and eight free parameters, respectively. The same model was applied to the study of the bottomonium spectrum [13].

A covariant four-dimensional approach based on the Schwinger–Dyson equations with a vector contact interaction was used to study the first radial excitations of heavy quarkonium [14]. For the charmonium case, the authors used five parameters and fixed the c-quark current mass at 1.09 GeV. Fur-

thermore, they obtained other results adjusting three parameters to get the experimental mass of the $\eta_c'(2S)$. The model was also generalized to study the masses of light and heavy mesons and baryons [15].

We conclude observing that the construction of a consistent model for the study of charmonium (and, in more general, quarkonium) is still a very active field of investigation.

Going back to the present paper, its remaining content is organized as follows. In Subsection 1.1, we briefly explain the symbols and the notation used in the work. In Section 2, we recall the main aspects of the reduced relativistic equation. In Section 3, the general form of the interaction is introduced. In Subsections 3.1 and 3.2, the details of the vector and scalar interactions are discussed, respectively. Finally, the results of the model are presented and analyzed in Section 4. Some conclusions are drawn in Section 5.

1.1. Symbols and notation

For the Dirac matrices (in the standard representation), for all the other operators and wave functions, we use the notation introduced in [1] and also $\hbar = c = 1$. We shall use the generic word *quark* to denote both the c-quark and the antiquark $\bar{c}$ of the charmonium system. The word antiquark will be used only when strictly necessary. For the argument of the color charge distribution, in Eqs. (12) and (15), we use $x = |\boldsymbol{x}|$.

2. The reduced Dirac equation

Following [1], we summarize here the main aspects of the reduction of the Dirac equation that is used in this work. The starting point is represented by the one-body reduction operator that, for the i^{th} particle, takes the form of

$$K_i = K(m_i, E_i; \boldsymbol{p}_i, \boldsymbol{\sigma}_i) = \begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma}_i \cdot \boldsymbol{p}_i}{m_i + E_i} \end{pmatrix} , \tag{1}$$

where m_i, E_i, $\boldsymbol{p}_i$, $\boldsymbol{\sigma}_i$, respectively, represent the mass, energy, momentum and Pauli matrix of the i^{th} constituent. The operator K_i, introduced in Eq. (21) of [1], is applied to a two-component spinor and gives rise (for one particle) to a four-component vinculated Dirac spinor. Note that K_i is a *local* operator, so that the complete equation is also local and can be solved in the coordinate space.

The two-body reduced equation, given in Eq. (40) of [1], can be formally written as

$$K_1^\dagger \cdot K_2^\dagger \left(D_1 + D_2 + W_{(2)} \right) K_1 \cdot K_2 |\Phi\rangle = 0 \, , \tag{2}$$

where D_i $(i = 1, 2)$ represents the standard operator of the free Dirac equation

$$D_i = D(m_i, E_i; \boldsymbol{p}_i, \boldsymbol{\alpha}_i, \beta_i) = \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_i + \beta_i m_i - E_i \,. \tag{3}$$

In Eq. (2), we have also introduced the two-body Dirac interaction operator $W_{(2)}$.

We consider the case of two equal mass particles $m_1 = m_2 = m$, in the center-of-mass (CM) reference frame, where the total momentum $\boldsymbol{P}$ is vanishing. In this frame, the following relation holds:

$$\boldsymbol{p}_1 = -\boldsymbol{p} \,, \qquad \boldsymbol{p}_2 = \boldsymbol{p} \,, \tag{4}$$

where $\boldsymbol{p}$ represents the relative momentum operator, canonically conjugated to the relative distance vector

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1 \,. \tag{5}$$

Furthermore, we assume that, in the CM, the two particles have the same energy

$$E_1 = E_2 = \frac{E_{\mathrm{T}}}{2} = \frac{M}{2} \,, \tag{6}$$

where $E_{\mathrm{T}} = M$ represents the mass off the resonant state.

In this way, we obtain the reduced equation in the form given by Eqs. (43) and (44) of [1], that is

$$\left[\left(1 + \frac{\boldsymbol{p}^2}{(E_{\mathrm{T}}/2 + m)^2} \right) \left(\frac{2\boldsymbol{p}^2}{E_{\mathrm{T}}/2 + m} + 2m - E_{\mathrm{T}} \right) + \hat{W}_{(2)} \right] |\Phi\rangle = 0 \,, \tag{7}$$

where we have also introduced the two-body reduced interaction operator

$$\hat{W}_{(2)} = K_1^\dagger \cdot K_2^\dagger \; W_{(2)} \; K_1 \cdot K_2 \,. \tag{8}$$

The expressions for the reduced scalar and vector two-body interactions have been given in Eqs. (C.3) and (C.5) of [1].

Equation (7) is a local, energy-dependent equation, free from continuum dissolution disease [1, 16] that can be advantageously used to study the spectroscopy of charmonium and of other mesons.

We also recall that Eq. (7), being an energy-dependent effective equation, can be solved by means of the technique explained in detail in Section 7 of [1].

3. The general structure of the interaction

As in Section 6 of [1], for the two-body interaction $W_{(2)}$ that appears in Eqs. (7) and (8), we consider a standard sum of a vector and a scalar contribution, in the form of

$$W_{(2)} = W^{\mathrm{v}}_{(2)} + W^{\mathrm{s}}_{(2)} \,. \tag{9}$$

For the vector interaction, we take the following standard expression:

$$W^{\mathrm{v}}_{(2)} = V^{\mathrm{v}}_{(2)}(r)\gamma^0_1\gamma^0_2 \cdot \gamma^\mu_1\gamma^\nu_2 g_{\mu\nu} \,. \tag{10}$$

The potential function $V^{\mathrm{v}}_{(2)}(r)$ will be discussed in Subsection 3.1.

In order to have a local interaction operator, as explained in [1], we have not included retardation contributions, consistently with Eq. (6): we make the hypothesis that the quark energies $E_i = E_{\mathrm{T}}/2$ are *fixed*; in other words, we assume that the quarks do not interchange energy with the effective gluonic vector field that mediates the interaction.

For the scalar interaction, we take the expression

$$W^{\mathrm{s}}_{(2)} = V^{\mathrm{s}}_{(2)}(r)\gamma^0_1\gamma^0_2 \,. \tag{11}$$

Many trials have been performed to determine the specific form of the potential functions $V^{\mathrm{v}}_{(2)}(r)$ and $V^{\mathrm{s}}_{(2)}(r)$ in order to reproduce the charmonium spectrum with a very small number of free parameters. In the two following subsections, the specific properties of the two potential functions will be discussed in detail.

3.1. The vector interaction

The vector interaction is constructed according to the model proposed in work [2]. In that model, the quarks are considered as extended sources of the chromo-electric field. In consequence, the color charge distribution of these sources determines the form of the potential and the value of the self-energy (*i.e.*, the zero-point potential energy) that is *not* introduced as an extra parameter.

Considering the objective of the present work, the contents of Section 2 of [2] can be synthetically rewritten in the following way.

Given a color charge distribution $\rho(x)$ (obviously rotationally invariant) for each quark, the attractive interaction energy between the quark and antiquark has the form of

$$V^{\mathrm{int}}(r) = -\frac{4}{3}\alpha_{\mathrm{v}} \int \mathrm{d}^3x \int \mathrm{d}^3x' \rho(x)\rho\left(x'\right) \frac{1}{|\boldsymbol{x} - \boldsymbol{x}' + \boldsymbol{r}|} \,. \tag{12}$$

(This interaction energy was denoted as $W^{\rm int}_{q\bar{q}}$ in [2].) Now, we recall that 4/3 is the color factor for the quark–antiquark interaction, $\alpha_{\rm v}$ is the color (vector) effective coupling constant (more frequently denoted as $\alpha_{\rm strong}$). Note that the last factor of Eq. (12) represents the Coulombic term. Due to the presence of that term, the interaction energy $V^{\rm int}(r)$ is also Coulombic at a large distance.

As shown in [2], the color charge distributions give rise to a positive zero-point self-energy that will be denoted as $\bar{V}_{\rm v}$ in the present work (while the same quantity was defined $W^{\rm self}$ in [2]). It is given by the following relation:

$$\bar{V}_{\rm v} = -V^{\rm int}(r=0)\,. \tag{13}$$

Furthermore, we note that the time component of the vector interaction, studied in [2], corresponds to $V^{\rm v}_{(2)}(r)$ of Eq. (10). In conclusion, for this quantity, we have

$$V^{\rm v}_{(2)}(r) = \bar{V}_{\rm v} + V^{\rm int}(r)\,. \tag{14}$$

The color charge distributions of the quarks *regularize* the interaction potential at $r=0$ and produce the self-energy $\bar{V}_{\rm v}$. As a result, we obtain for $V^{\rm v}_{(2)}(r)$ a potential that is *vanishing* at $r=0$ and approaches the maximum value $\bar{V}_{\rm v}$ (with a Coulombic behavior) as $r\to\infty$.

The best reproduction of the experimental data has been obtained with a Gaussian color charge distribution, of the form of

$$\rho(x) = \frac{1}{(2\pi d^2)^{3/2}} \exp\left(-\frac{\boldsymbol{x}^2}{2d^2}\right)\,. \tag{15}$$

With this distribution, $\bar{V}_{\rm v}$ and $V^{\rm int}(r)$ can be calculated analytically. The results are

$$\bar{V}_{\rm v} = \frac{4}{3}\frac{\alpha_{\rm v}}{d}\frac{1}{\sqrt{\pi}} \tag{16}$$

and

$$V^{\rm int}(r) = -\frac{4}{3}\frac{\alpha_{\rm v}}{r}{\rm erf}\left(\frac{r}{2d}\right)\,. \tag{17}$$

The same regularization function $F_{\rm v}(r) = {\rm erf}\left(\frac{r}{2d}\right)$ was also used in Eq. (61) of [1], with $2d = d_{\rm v}$. The relevant properties of this regularization function are also explained there.

Finally, note that $\bar{V}_{\rm v}$ is *not* a free parameter but is determined by $\alpha_{\rm v}$ and d that represent the only free parameters of the vector interaction.

3.2. The scalar interaction and an additional constraint

In order to reproduce with reasonable accuracy the experimental data of the charmonium spectrum, we have verified that it is strictly necessary to introduce a scalar interaction.

However, in the present context, it has not been possible to construct a more fundamental model to represent this interaction.

After trying different forms for $V_{\rm s}(r)$, we have found that a *negative* function, regular at $r = 0$, that goes to zero as $r \to \infty$, is needed to reproduce the spectrum. The simplest expression, with only two free parameters, is a Gaussian function

$$V_{\rm s}^{\rm G}(r) = -\bar{V}_{\rm s} \exp\left(-\frac{r^2}{r_{\rm s}^2}\right) . \tag{18}$$

We also report, in the results of the following section, a *very simple test* with a constant interaction

$$V_{\rm s}^{\rm C}(r) = -\bar{V}_{\rm s} . \tag{19}$$

In this case, the reproduction of the spectrum is obviously *worse* than that obtained with $V_{\rm s}^{\rm G}(r)$.

Furthermore, we have used a *two-region* potential, studying the possibility that the scalar interaction is related to the interchange of a scalar particle of mass m_b.

After trying different parametrizations, we found that, in any case, it is necessary to consider two spatial regions: an inner region, with a relatively soft dependence on r and an outer region in which the scalar interaction is represented by a standard Yukawa function related to the interchange of a mass m_b. The expression that has been used has the following form:

$$V_{\rm s}^{\rm T}(r) = \begin{cases} -\bar{V}_{\rm s}\left[1 - b\left(\frac{r}{r_{\rm s}}\right)^p\right] & \text{for } r \le r_{\rm s}, \\ -\frac{\beta}{r}\exp\left(-\frac{r}{r_b}\right) & \text{for } r > r_{\rm s} . \end{cases} \tag{20}$$

In the previous expression, we use the same symbol $r_{\rm s}$ introduced for the Gaussian potential. But now it has a different meaning: it fixes the limit between the inner and outer spatial regions; b and p are adimensional parameters whose numerical value will be given in the next section, β represents the adimensional coupling constant of the Yukawa interaction and, finally, $r_b = 1/m_b$.

We require that $V_{\rm s}^{\rm T}(r)$ and its first derivative are *continous* at $r = r_{\rm s}$. These two conditions respectively give

$$\beta = \bar{V}_{\rm s} r_{\rm s}(1 - b)\exp\left(\frac{r_{\rm s}}{r_b}\right) \tag{21}$$

and

$$r_b = r_{\rm s}\frac{1-b}{bp-1+b}\,. \tag{22}$$

Further discussions about the scalar interaction and its specific form $V_{\rm s}^{\rm T}(r)$ are postponed to the next section.

Finally, in order to reduce the number of free parameters of the scalar interaction, we introduce a phenomenological constraint on the parameter $\bar{V}_{\rm s}$. Recalling the discussion in the previous subsection about the self-energy of the vector interaction, analogously to Eq. (13), we assume here that, for the scalar potentials of Eqs. (18), (19) and (20), $V_{\rm s}(r=0) = -\bar{V}_{\rm s}$ represents the (negative) self-energy of the scalar interaction.

Considering this starting point, we shall use the following phenomenological *balance* equation:

$$\bar{V}_{\rm v} = 2m_q - \bar{V}_{\rm s}\,. \tag{23}$$

It means that the self-energy of the vector interaction equals the rest energy of the quarks plus the negative self-energy of the scalar interaction.

In this way, solving Eq. (23) with respect to $\bar{V}_{\rm s}$, we avoid to introduce this quantity as a free parameter; on the contrary, it is *determined* by the other parameters of the model.

Some more comments will be given in Section 4 when analyzing the results of the calculation.

4. The result for the charmonium spectrum

In this section, we apply the model to study the charmonium spectrum with the interaction introduced in Section 3.

The relativistic, energy-dependent equation (7) is solved with the same technique explained in Section 7 of [1], to which we refer the reader.

We use a variational basis of harmonic oscillator (HO) wave functions that, in the coordinate space, have the form given in Eq. (63) of [1]

$$\Phi_{n;L,S,J}(\boldsymbol{r}) = \langle \boldsymbol{r}|n;L,S,J\rangle = R_{n,L}(r;\bar{r})[Y_L(\hat{r})\otimes\chi_S]_J\,. \tag{24}$$

In the previous equation, the trial radial function is represented by $R_{n,L}(r;\bar{r})$, n being the principal HO quantum number and $\bar{r}$ the variational parameter with the dimension of longitude; $Y_{L,M_L}(\hat{r})$ is the corresponding spherical harmonic and χ_{S,M_S}, with $S=0,1$ is the $c\bar{c}$ coupled spin function. The orbital angular momentum and the spin are standardly coupled to the total angular momentum J, M_J. For brevity, we do not write M_J because it is irrelevant for the calculations of rotationally scalar operators.

Finally, for simplicity, we do not consider the possibility of mixing between states with different values of L, because these effects are usually considered negligible in these calculations.

The analytic form of the radial HO functions is given in Eq. (64) of [1].

As for the fit procedure, we have determined the free parameters of the model by minimizing the quantity

$$D^2 = \sum_i \left(E_i^{\rm th} - M_i^{\rm exp} \right)^2 , \tag{25}$$

where $E_i^{\rm th}$ and $M_i^{\rm exp}$, respectively, represent the calculated energy and the experimental mass (rest energy) of the $i^{\rm th}$ resonance.

Due to the higher number of parameters (more precisely, eight or nine) used in [1], in that work, a better reproduction of the spectrum was obtained. On the other hand, we try here to fit the whole spectrum with *three* parameters only.

To this aim, we also fix the mass of the c-quark at the value given by the PDG as the "running" mass in the $\overline{\rm MS}$ scheme. This value is presently $m_q = 1.27$ GeV [17].

All the results for the spectrum are given in Table I.

In the last column of this table, we give the experimental values of the resonances. In particular, we have considered all the eight experimentally observed resonances, whose energies are below the open charm threshold $D\,\bar{D}$; we have also taken eight not controversial resonances at higher energies. For a discussion about the phenomenological interpretation of the resonances in different models, the interested reader is referred to Ref. [12].

In the columns denoted by Gauss, Const. and Two Reg., we report the theoretical results given respectively by the Gaussian scalar potential of Eq. (18), by the constant scalar potential of Eq. (19) and by the *two-region* scalar potential of Eq. (20).

Considering Eq. (25), we report, for simplicity, in the last line of Table I the quantity $Q = D^2/100$ MeV^2 in order to give an indication about the quality of the fit for the three scalar potentials.

The values of the parameters of the model are shown in Table II.

In particular, we give the values obtained with the fit procedure for the independent parameters: the effective coupling constant $\alpha_{\rm v}$ and the regularization radius d for the effective vector interaction, and the radius $r_{\rm s}$ of the scalar interaction. We also give the values of dependent parameters $\bar{V}_{\rm v}$ of Eq. (16) and $\bar{V}_{\rm s}$, determined by means of Eq. (23).

The same notation (for the different scalar potentials) as in Table I is used in Table II.

TABLE I

Comparison between the experimental average values [17] of the charmonium spectrum (last column) and the theoretical results of the model. All the masses are in MeV. The quantum numbers n, L, S and J have been introduced in Eq. (24); they represent the principal quantum number, the orbital angular momentum, the spin and the total angular momentum, respectively. The results of the columns Gauss, Const. and Two Reg. refer to the different forms of the scalar interaction, as specified in the text. A line divides the resonances below and above the open Charm threshold. At the bottom, the quantity Q gives an indication of the quality of the fit, as explained in the text.

Name	$n^{2S+1}L_J$	Gauss	Const.	Two Reg.	Experiment
η_c	1^1S_0	2989	3007	2990	2983.9 ± 0.5
J/ψ	1^3S_1	3092	3100	3092	3096.9 ± 0.006
χ_{c0}	1^3P_0	3420	3386	3419	3414.71 ± 0.30
χ_{c1}	1^3P_1	3499	3461	3497	3510.67 ± 0.05
h_c	1^1P_1	3511	3464	3509	3525.38 ± 0.11
χ_{c2}	1^3P_2	3565	3556	3562	3556.17 ± 0.07
η_c'	2^1S_0	3649	3680	3643	3637.5 ± 1.1
ψ'	2^3S_1	3680	3708	3673	3686.097 ± 0.010
$\psi(3770)$	1^3D_1	3797	3756	3791	3778.1 ± 1.2
$\psi(3823)$	1^3D_2	3832	3828	3826	3822.2 ± 1.2
$\chi_{c1}(3872)$	2^3P_1	3893	3914	3890	3871.69 ± 0.17
$\chi_{c2}(3930)$	2^3P_2	3928	3949	3926	3927.2 ± 2.6
$\psi(4040)$	3^3S_1	4014	4027	4020	4039 ± 1
$\chi c1(4140)$	3^3P_1	4145	4135	4158	4146.8 ± 2.4
$\psi(4230)$	4^3S_1	4214	4195	4220	4218.7 ± 2.8
$\chi c1(4274)$	4^3P_1	4272	4258	4269	4274 ± 7
Q		22.6	138	17.8	

The main results of this work are those obtained with the Gaussian potential $V_{\rm s}^{\rm G}(r)$ of Eq. (18) for the scalar interaction. As shown in Table I, a good overall reproduction of the spectrum is obtained with only *three* free parameters.

As anticipated in Subsection 3.2, we have also tried to reproduce the spectrum (as a very simple test) with a constant scalar potential. In this case, we need only two free parameters: $\alpha_{\rm v}$ and d.

The corresponding value of Q in Table I shows that the quality of the fit is considerably worse than that given by the Gaussian scalar potential.

Finally, with the *two-region* potential, we have explored the possibility that the scalar interaction is given, at least in the outer region, by the standard exchange of a scalar particle, by using the potential $V_{\rm s}^{\rm T}(r)$ of Eq. (20), with the continuity conditions of Eqs. (21) and (22).

TABLE II

Numerical values of the free and dependent parameters of the model; m_q is fixed at the value of Ref. [17], as explained in the text. The reported numerical values represent the results of the fits of the free parameters $\alpha_{\rm v}$, d and $r_{\rm s}$; $\bar{V}_{\rm v}$ and $\bar{V}_{\rm s}$ are dependent parameters, as explained in the text.

				Units
m_q	1.27			GeV
	Gauss	Const.	Two Reg.	
$\alpha_{\rm v}$	1.864	3.991	1.865	
d	0.1526	0.2665	0.1531	fm
$r_{\rm s}$	1.879		1.991	fm
$\bar{V}_{\rm v}$	1.813	2.223	1.808	GeV
$\bar{V}_{\rm s}$	0.7268	0.3170	0.7321	GeV

For this case, we point out that, after some trials, we have fixed (for simplicity) the power p of the inner part of the potential at $p = 3/2$ and the parameter b at the value $b = 2^{-1/2}$. We have verified that no significant improvement is obtained varying these values.

We also give the values of other *dependent* parameters of this interaction:

(i) for the range of the Yukawa interaction r_b, fixed by Eq. (22), we have obtained $r_b = 0.7594$ fm, corresponding to a scalar boson mass of $m_b = 0.2598$ GeV;

(ii) for the Yukawa coupling costant β, fixed by Eq. (21), we have found $\beta = 29.75$.

Considering the results of Table I, we observe that no significant improvement for the charmonium spectrum is obtained with respect to the case of the Gaussian scalar potential. Furthermore, the radius of the inner region $r_{\rm s}$ is large with respect to the range r_b of the (hypothetical) Yukawa interaction.

In conclusion, the model does not show clear evidence for the exchange of a scalar particle.

Finally, for all the scalar interactions we have also tried to consider $\bar{V}_{\rm v}$ and $\bar{V}_{\rm s}$ as free parameters, ignoring Eqs. (16) and (23), but no significant improvement in the reproduction of the spectrum has been obtained.

As in [1], we have also used the reduced equation obtained by the relativistic Mandelzweig and Wallace equation [18, 19]; the obtained results are *very similar* to those discussed above (obtained by using the reduced Dirac-like equation (7)). For this reason, they have not been shown in the paper. Some more comments are given in Conclusions.

5. Conclusions

In this paper, we have shown that a relativistic, energy-dependent, local equation can be used to describe the charmonium spectrum with good accuracy using only three free parameters. A standard mixture of a vector and scalar interaction has been considered. As for the vector part of the interaction, the regularization radius fixes the quark self-energy that is determined in this way as a *dependent* parameter.

For the scalar interaction, a phenomenological Gaussian potential is taken; the possibility of the interchange of a scalar particle has been also explored. A balance equation is used to determine the value of the scalar potential at $r = 0$.

Further investigation is needed to understand in more detail the nature and the origin of the scalar interaction.

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