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CALCULATION OF THE ONE-LOOP BOX INTEGRAL AT FINITE TEMPERATURE AND DENSITY*

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Calculation of hadronization, decay or scattering processes at nonzero temperatures and densities within the Nambu–Jona-Lasinio-like models requires some techniques for computation of Feynmann diagrams. Decomposition of Feynmann diagrams at the one-loop level leads to the appearance of elementary integrals with one, two, three, and four fermion lines. For example, evaluation of the $\pi\pi$ scattering amplitude requires calculating of a box diagram with four fermion lines. In this work, the real and imaginary parts of the box integral at the one-loop level are provided in the form suitable for numerical evaluation. The obtained expressions are applicable to any value of temperature, particle mass, and chemical potential. We pay special attention to the conditions for the existence of the appearing improper integrals. As a result, we have obtained constraints on possible values of particle momenta. Among the expressions for the box integral, the general formulas for the integral with an arbitrary number of lines are derived for the case with zero or collinear fermion momenta.

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1. Introduction

Due to the complexity of the QCD Lagrangian and the nonperturbativity of QCD at energies relevant for the deconfinement/chiral phase transition, effective theories of strongly interacting matter at nonzero temperatures/densities are under intensive study, see, *e.g.* [1–4], and references therein.

An extension of QCD Lagrangian-inspired effective models such as the Nambu–Jona-Lasinio (NJL) one [5–10] to finite temperatures/densities is performed by the imaginary time Matsubara formalism [11]. The description of hadronization processes starts with Feynman diagrams which can contain two (a polarization loop), three or four (a box diagram) fermion lines, and the obtained amplitudes at the one-loop level are the integrals that contain $\prod_i (q_i^2 - m_i^2)^{-1}$. For calculations at zero temperature, it was shown [12] that all occurring transition amplitudes at the one-loop level can be decomposed into a number of elementary integrals, which in turn can be classified by the number of particle lines they contain. This decomposition technique can be generalized to nonzero temperatures using the Matsubara formalism [13]. The authors of the mentioned paper gave a solution of this task for diagrams with one, two, and three fermion lines, explicitly displaying their complex nature.

The study of meson scattering processes (for example, $\pi\pi$ scattering) requires the calculation of a box diagram with four fermion lines [2, 14]. In the previous works, the amplitude for $\pi\pi$ scattering at non-zero temperatures was calculated only in special kinematics with $s = 4M_\pi^2$ and $t = u = 0$ due to the complexity of calculations in the general case. The aim of this paper is to solve technical problems associated with such integrals following the corrected techniques developed in [13] and to give researchers, who are active in this field, the knowledge sufficient to calculate such functions. Expressions for both real and imaginary parts will be given in the form suitable for numerical evaluation.

We carefully investigate for what values of the parameters all integrals that appear, exist. This work brings us to the necessity of correcting the results [13] for the three-line elementary integral. We also provide some reasons to use the three-momentum cutoff regularization scheme with $|\boldsymbol{p}| < \Lambda$ though the integrals with three and more fermion lines are convergent.

The paper is organized as follows. Section 2 shows how the original integral can be represented as a sum of elementary integrals. In Section 3, we show how a careful study of the existence of elementary integrals leads to a difference from the result [13] for the three-line function. After that, the results for the four-line integral are given depending on configurations of particle momenta. The simplest cases when all three momenta are collinear or equal to zero are discussed in Section 4. More complicated *planar* configurations are considered in Section 5. The main original result of the paper is formulated in Section 6, where the expression for the most general case of independent momenta is given. We also shortly discuss how to calculate a one-loop integral with any number of fermion lines in Section 7. Finally, the results are summarized in Conclusions. Some technical details are presented in two Appendices.

2. Decomposition to elementary integrals

In the general case, after replacing the integration over the time component of the four-momentum by the sum over the Matsubara frequencies, the integral with $L+1$ fermion lines has the form of [13]

$$Z_0^{(L)}\left(\mathcal{X}_1^+,\ldots,\mathcal{X}_L^+;\mathcal{T}\right) = \frac{16\pi^2}{\beta}\lim_{\eta\to 0}\sum_n \mathrm{e}^{i\omega_n\eta}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{1}{(i\omega_n+\mu)^2-E^2} \times\prod_{l=1}^{L}\frac{1}{(i\omega_n+\mu-\lambda_l)^2-\widetilde{E}_l^2\left(\boldsymbol{k}_l\right)}\,, \tag{1}$$

where the arguments were written as a collection of ordered sets (vectors)

$$\mathcal{X}_l^{\pm} = (\pm\lambda_l,\boldsymbol{k}_l,m_l)\ , \qquad \mathcal{T} = (T,\mu,m)\ . \tag{2}$$

Here T is the temperature, $\beta = 1/T$, m and m_l denote the masses of fermions, μ and μ_l are the fermion chemical potentials, λ_l is defined as $\lambda_l = \mu-\mu_l+i\nu_{j_l}$, and $\boldsymbol{k}_l$ are the three momenta. All considered integrals are integrated for $p = |\boldsymbol{p}| \leq \Lambda$, *i.e.* Λ is the three-momentum cutoff, and

$$E = \sqrt{p^2+m^2}\,, \qquad \widetilde{E}_l(\boldsymbol{k}) = \sqrt{(\boldsymbol{p}-\boldsymbol{k})^2+m_l^2}\,. \tag{3}$$

The Matsubara summation is carried out over the bosonic $i\nu_{j_l} = 2\pi j_l T$ and fermionic $\omega_n = (2n+1)\,\pi\,T$ frequencies. After the Matsubara summation over n is carried out, the complex bosonic frequencies are analytically continued to their values on the real axis and become the zeroth components associated with the corresponding three momentum, $i\nu_{j_l}\to k_l^0$. The limit $\eta\to 0$ has to be taken after the Matsubara summation.

The Matsubara summation in Eq. (1) can be easily performed [13, 15]. There are $2(L+1)$ poles in Eq. (1)

$$i\omega_n = -\mu\pm E\,, \qquad i\omega_n = \lambda_l-\mu\pm\widetilde{E}_l(\boldsymbol{k}_l)\,, \tag{4}$$

and after the summation over n, which is performed according to the rule

$$T\sum_n\frac{1}{i\omega_n\pm z} = \frac{1}{\mathrm{e}^{\pm\beta z}+1} = f(\pm z) \tag{5}$$

with the Fermi–Dirac distribution function $f(\pm z)$, one obtains

$$
\begin{aligned}
Z_0^{(L)}\left(\mathcal{X}_1^+,\dots,\mathcal{X}_L^+;\mathcal{T}\right) \;=\; 16\pi^2 \Bigg[& \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{f(E-\mu)}{2E}\prod_{l=1}^{L}\frac{1}{(E-\lambda_l)^2-\widetilde{E}_l^2(\boldsymbol{k}_l)} \\
& -\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{f(-E-\mu)}{2E}\prod_{l=1}^{L}\frac{1}{(E+\lambda_l)^2-\widetilde{E}_l^2(\boldsymbol{k}_l)} \\
& +\sum_{s=1}^{L}\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{f(E_s-\mu_s)}{2E_s}\frac{1}{(E_s+\lambda_s)^2-\widetilde{E}^2(\boldsymbol{k}_s)} \\
& \times\prod_{l=1,l\neq s}^{L}\frac{1}{(E_s+\lambda_{sl})^2-\widetilde{E}_l^2(\boldsymbol{k}_s-\boldsymbol{k}_l)} \\
& -\sum_{s=1}^{L}\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{f(-E_s-\mu_s)}{2E_s}\frac{1}{(E_s-\lambda_s)^2-\widetilde{E}^2(\boldsymbol{k}_s)} \\
& \times\prod_{l=1,l\neq s}^{L}\frac{1}{(E_s-\lambda_{sl})^2-\widetilde{E}_l^2(\boldsymbol{k}_s-\boldsymbol{k}_l)}\Bigg]. \qquad (6)
\end{aligned}
$$

Above, the substitution $\boldsymbol{p}\to\boldsymbol{k}_s-\boldsymbol{p}$ was made in the last four lines, and[1]

$$
\begin{aligned}
\lambda_{ls} &= \lambda_l-\lambda_s=\mu_s-\mu_l+i\nu_{j_l}-i\nu_{j_s}\,, \qquad l\neq s\,, \qquad (7)\\
\widetilde{E}(\boldsymbol{k}) &= \sqrt{(\boldsymbol{p}-\boldsymbol{k})^2+m^2}\,, \qquad E_l=\sqrt{p^2+m_l^2}\,. \qquad (8)
\end{aligned}
$$

Thus, we see that the original integral can be written as a decomposition

$$
\begin{aligned}
& Z_0^{(L)}\left(\mathcal{X}_1^+,\dots,\mathcal{X}_L^+;\mathcal{T}\right)=\mathcal{Z}_L^+\left(\mathcal{X}_1^-,\dots,\mathcal{X}_L^-;\mathcal{T}\right)-\mathcal{Z}_L^-\left(\mathcal{X}_1^+,\dots,\mathcal{X}_L^+;\mathcal{T}\right)\\
& +\sum_{s=1}^{L}\left[\mathcal{Z}_L^+\left(\mathcal{X}_{s1}^+,\dots,\mathcal{X}_{sL}^+;\mathcal{T}_s\right)-\mathcal{Z}_L^-\left(\mathcal{X}_{s1}^-,\dots,\mathcal{X}_{sL}^-;\mathcal{T}_s\right)\right] \qquad (9)
\end{aligned}
$$

with

$$
\mathcal{T}_s=(T,\mu_s,m_s)\,, \qquad \mathcal{X}_{ss}^{\pm}=(\pm\lambda_s,\boldsymbol{k}_s,m)\,, \qquad \mathcal{X}_{sl}^{\pm}=(\pm\lambda_{sl},\boldsymbol{k}_s-\boldsymbol{k}_l,m_l)\,. \qquad (10)
$$

It is important to note here that to obtain the above decomposition, one has to make two crucial assumptions. The first one is that all poles (4) are different. Therefore, if parameters of at least two particles coincide, $m_l=m_s$, $\lambda_l=\lambda_s$, and $\boldsymbol{k}_l=\boldsymbol{k}_s$, $l\neq s$, or $m_l=m$, $\lambda_l=0$, and $\boldsymbol{k}_l=0$ for an index, we will have another decomposition instead of Eq. (6) or a single irreducible integral, see the discussion at the end of Section V in [13]. The

[1] Here and everywhere below, $l\neq s$ if we have two indices in the expression.

second implicit assumption was introduced when we performed the variable replacement in Eq. (6). This operation can be made only if each integral of the sum exists independently. Namely, the latter property is used in the paper.

In the spherical coordinates, the elementary integral is given by

$$\begin{aligned}\mathcal{Z}_L^{\pm}\left(\mathcal{X}_1^+,\ldots,\mathcal{X}_L^+;\mathcal{T}\right) &= \frac{1}{2^L\pi}\lim_{\epsilon\to 0}\int\limits_m^{\Lambda_E}\mathrm{d}E\,p\,f(\pm E-\mu)\\ &\quad\times\int\limits_{-1}^{1}\mathrm{d}x\int\limits_0^{2\pi}\mathrm{d}\phi\prod_{l=1}^{L}\frac{1}{\lambda_l(E-E_{0l}-i\epsilon)+\boldsymbol{p}\boldsymbol{k}_l}\\ &= \mathrm{Re}\mathcal{Z}_L^{\pm}+i\,\mathrm{Im}\mathcal{Z}_L^{\pm}\,,\end{aligned} \tag{11}$$

where $\boldsymbol{p} = p\left(\cos\phi\sqrt{1-x^2}, \sin\phi\sqrt{1-x^2}, x\right)$,

$$E_{0l} \equiv E_0(\varepsilon_l,\lambda_l) = \frac{\varepsilon_l}{\lambda_l}\,, \tag{12}$$

$$\varepsilon_l \equiv \varepsilon(\mathcal{X}_l^+,m) = \varepsilon(\mathcal{X}_l^-,m) = \frac{k_l^2+m_l^2-\lambda_l^2-m^2}{2}\,, \qquad k_l = |\boldsymbol{k}_l|\,, \tag{13}$$

and

$$\Lambda_E = \sqrt{\Lambda^2+m^2}\,.$$

Thus we have reduced the task to evaluating functions of the same kind. For further consideration, we need to write the components of $\boldsymbol{k}_l$

$$\boldsymbol{k}_l = k_l\left(\sin\delta_l\cos\phi_l, \sin\delta_l\sin\phi_l, \cos\delta_l\right)\,. \tag{14}$$

Let us also introduce

$$\cos\psi_{ls} = \frac{\boldsymbol{k}_l\boldsymbol{k}_s}{k_lk_s}\,, \qquad \sin\psi_{ls} = \left|\frac{\boldsymbol{k}_l\times\boldsymbol{k}_s}{k_lk_s}\right|\,, \tag{15}$$

where $\boldsymbol{a}\times\boldsymbol{b}$ and $\boldsymbol{a}\boldsymbol{b}$ denote the vector and scalar products of two three-vectors, respectively. As one can easily see,

$$\cos\psi_{ls} = \cos\delta_l\cos\delta_s + \sin\delta_l\sin\delta_s\cos(\phi_l-\phi_s)\,. \tag{16}$$

One should keep in mind that $\delta_l, \psi_{ls} \in [0,\pi]$ and so we always have $\sin\delta_l$, $\sin\psi_{ls} \geq 0$. Also, without loss of generality, one can choose

$$\boldsymbol{k}_1 = k_1(0,0,1)\,, \qquad \boldsymbol{k}_2 = k_2(\sin\delta_2, 0, \cos\delta_2)\,. \tag{17}$$

At the end of this section, let us discuss how Eq. (11) can be converted into integrals of real functions. The method is based on applying the famous formula

$$\lim_{\epsilon\to 0}\frac{1}{x-i\epsilon}=\mathcal{P}\frac{1}{x}+i\pi\delta(x)$$

which can be generalized as

$$\lim_{\epsilon\to 0}\frac{1}{x-i\lambda\epsilon}=\mathcal{P}\frac{1}{x}+i\pi\,\mathrm{sgn}(\lambda)\,\delta(x)\,,\qquad \lambda\neq 0\,. \tag{18}$$

Writing the product under the integral (11) as

$$\begin{aligned}&\prod_{l=1}^{L}\frac{1}{\lambda_l(E-E_{0l}-i\epsilon)+\boldsymbol{pk}_l}\\ &=\prod_{l=1}^{L}\left[\mathcal{P}\frac{1}{\lambda_l(E-E_{0l})+\boldsymbol{pk}_l}+i\pi\,\mathrm{sgn}\lambda_l\,\delta\left(\lambda_l(E-E_{0l})+\boldsymbol{pk}_l\right)\right]\end{aligned} \tag{19}$$

and expanding it, we obtain separate real and imaginary parts. Equation (19) assumes that one takes L different limits $\epsilon_l\to 0$. This approach coincides with the one used in [13] for calculation of the three-line integral for nonzero momenta. However, when $k_1=k_2=0$, the authors apply a different expansion, taking only one limit $\epsilon\to 0$. It is quite clear that the results obtained with both approaches must coincide since they cannot depend on how we take a separate limit. As is shown below, this is true only if one takes into account the conditions for the existence of the appearing improper multiple integrals. To underline this conclusion, we will use Eq. (19) excluding the simplest case when all momenta are equal to zero. As one can easily see, in such a case the product of two or more delta functions immediately vanishes if $E_{0l}\neq E_{0s}$.

3. Correction of the result [13] for the three-line integral

Before calculating $\mathcal{Z}_3^{\pm}$, we would like to make some remarks related to the results for $L=2$ obtained in [13], see Eq. (5.37). The authors of [13] derived the formulae

$$\begin{aligned}&\mathcal{P}\int_{-1}^{1}\mathrm{d}x\frac{\mathrm{sgn}(z+x\cos\psi)\,\Theta(\varDelta(x,z,\cos\psi))}{(x+y)\sqrt{\varDelta(x,z,\cos\psi)}}\\ &=\frac{\varXi(y,z,\cos\psi)\,\Theta(\varDelta_0(y,z,\cos\psi))+\mathscr{G}(y,z,\cos\psi)\,\Theta(-\varDelta_0(y,z,\cos\psi))}{\sqrt{|\varDelta_0(y,z,\cos\psi)|}}\,,\end{aligned} \tag{20}$$

where $\sin\psi > 0$,

$$\begin{aligned}\varDelta(a,b,c) &= a^2+b^2+c^2+2\,abc-1 = (c+ab)^2-\left(1-a^2\right)\left(1-b^2\right)\,, &(21)\\ \varDelta_0(a,b,c) &= \varDelta(-a,b,c)=\varDelta(a,-b,c)\,, &(22)\end{aligned}$$

and

$$\begin{aligned}\mathscr{G}(a,b,c) &\triangleq \operatorname{sgn}(b+c)\arccos\frac{(1+a)(a-bc)-\varDelta_0(a,b,c)}{(1+a)\sqrt{(1-b^2)(1-c^2)}}\\ &\quad -\operatorname{sgn}(b-c0\arccos\frac{-(1-a)(a-bc)-\varDelta_0(a,b,c)}{(1-a)\sqrt{(1-b^2)(1-c^2)}}\,,\\ \varDelta_0(a,b,c) &< 0\,, &(23)\end{aligned}$$

$$\begin{aligned}\varXi(a,b,c) &\triangleq \operatorname{sgn}(b+c)F(1;a,b,c)-\operatorname{sgn}(b-c)F(-1;a,b,c)\\ &\quad +\Theta\left(1-b^2\right)\left[\operatorname{sgn}(b-c)F(x^-;a,b,c)-\operatorname{sgn}(b+c)F(x^+;a,b,c)\right]\,,\\ \varDelta_0(a,b,c) &> 0 &(24)\end{aligned}$$

with[2]

$$F(x;a,b,c)=\pm\ln\left|\frac{(x+a)(a-bc)-\varDelta_0(a,b,c)\pm\sqrt{\varDelta_0(a,b,c)\varDelta(x,b,c)}}{x+a}\right|\,.$$

Above, $x^\pm$ are the roots of the quadratic x-trinomial $\varDelta(x,b,c)$,

$$x^\pm\equiv x^\pm(b,c)=-bc\pm\sqrt{d(b,c)}\,,\qquad d(b,c)\geq 0\,,$$

and

$$d(b,c)=\left(1-c^2\right)\left(1-b^2\right) \tag{25}$$

is the corresponding determinant.

Since to evaluate the box integral we need Eq. (20) or its generalization to a more complicated rational function, let us consider it in more detail. The first, cosmetic, disadvantage of Eqs. (23) and (24) is that they are not explicitly symmetric with respect to the transposal $y\leftrightarrow z$. The mentioned symmetry follows from Eq. (11), see [13] and below. The original expressions for both functions are also too complicated. Fortunately, we can simplify both functions by restoring explicit symmetry. First of all, it is more convenient to choose

$$F(x;a,b,c)=\frac{1}{2}\,\ln\left|\frac{\varDelta_0(a,b,c)-(x+a)(a-bc)-\sqrt{\varDelta_0(a,b,c)\varDelta(x,b,c)}}{\varDelta_0(a,b,c)-(x+a)(a-bc)+\sqrt{\varDelta_0(a,b,c)\varDelta(x,b,c)}}\right| \tag{26}$$

[2] The function $F(x;y,z,\psi)$ is defined up to an additive constant.

which has a nice property $F(x^{\pm};a,b,c) = 0$. As a result, we see that the function $\varXi(a,b,c)$ is independent of $x^{\pm}$ and is completely defined by the first line of Eq. (24). After some transformations, one can prove

$$\varXi(a,b,c) = \ln\left|\frac{c-ab-\sqrt{\varDelta_0(a,b,c)}}{c-ab+\sqrt{\varDelta_0(a,b,c)}}\right| . \tag{27}$$

Further, mentioning that

$$\mathrm{sgn}(b\pm c) = \mathrm{sgn}(b)\Theta\big(b^2-c^2\big) \pm \mathrm{sgn}(c)\Theta\big(c^2-b^2\big)$$

and using Eqs. (A.1), one can show

$$\begin{aligned} \mathscr{G}(a,b,c) &= \mathrm{sgn}(c-ab)\arccos\frac{|\varDelta_0(a,b,c)|-(c-ab)^2}{(1-a^2)(1-b^2)} \\ &= 2\arctan\frac{c-ab}{\sqrt{|\varDelta_0(a,b,c)|}} . \end{aligned} \tag{28}$$

The second, crucial, disadvantage of Eq. (20) is that it contains excess terms. This statement can be demonstrated in two ways. On the one hand, the l.h.s. of Eq. (20) is obtained after integration over ϕ with the help of [13]

$$\int\limits_0^{2\pi}\frac{\mathrm{d}\phi}{a+b\cos\phi} = \frac{2\pi}{\sqrt{a^2-b^2}}\,\mathrm{sgn}(a)\,\Theta\big(a^2-b^2\big) \tag{29}$$

which can be generalized as

$$\int\limits_0^{2\pi}\frac{\mathrm{d}\phi}{a+b\sin\phi+c\cos\phi} = \frac{2\pi}{\sqrt{a^2-b^2-c^2}}\,\mathrm{sgn}(a)\,\Theta\big(a^2-b^2-c^2\big) . \tag{30}$$

The proof is similar to that for Eq. (29) and can be made using [16]. As both last expressions are written, one assumes that the ϕ-integral is zero when $b^2+c^2>a^2$. However, let us remind that generally in calculating $\mathrm{Re}\,\mathcal{Z}_2^{\pm}$, when the spherical coordinates are chosen as (17), we deal with the triple improper integral of second kind

$$\int\limits_m^{\varLambda_E}\mathrm{d}E\int\limits_{-1}^{1}\mathrm{d}x\int\limits_0^{2\pi}\mathrm{d}\phi\,\frac{f(\pm E-\mu)}{p}\,\frac{1}{z_1+x}\,\frac{1}{z_2+x\cos\delta_2+\cos\phi\sin\delta_2\cos\phi_2\sqrt{1-x^2}} , \tag{31}$$

where
$$z_l = \frac{\lambda_l E - \varepsilon_l}{p\,k_l}\,. \tag{32}$$

The theorem on the unconditional convergence of an improper multiple integral of second kind states that the integral (31) unconditionally convergent is equivalent to the existence only if the same is true for the integral

$$\begin{aligned}&\mathcal{P}\int\limits_m^{\Lambda_E}\mathrm{d}E\int\limits_{-1}^{1}\mathrm{d}x\int\limits_0^{2\pi}\mathrm{d}\phi\left|\frac{f(\pm E-\mu)}{p}\,\frac{1}{z_1+x}\,\frac{1}{z_2+x\cos\delta_2+\cos\phi\sin\delta_2\cos\phi_2\sqrt{1-x^2}}\right|\\&=\mathcal{P}\int\limits_m^{\Lambda_E}\mathrm{d}E\,\frac{f(\pm E-\mu)}{p}\int\limits_{-1}^{1}\frac{\mathrm{d}x}{|z_1+x|}\int\limits_0^{2\pi}\frac{\mathrm{d}\phi}{\left|z_2+x\cos\delta_2+\cos\phi\sin\delta_2\cos\phi_2\sqrt{1-x^2}\right|}\,.\end{aligned} \tag{33}$$

It can be easily shown that $\int_0^{2\pi}\frac{\mathrm{d}\phi}{|a+b\sin\phi+c\cos\phi|}$ does not exist when $a^2 < b^2+c^2$. When $a^2 > b^2+c^2$,

$$\int\limits_0^{2\pi}\frac{\mathrm{d}\phi}{|a+b\sin\phi+c\cos\phi|} = \frac{2\pi}{\sqrt{a^2-b^2-c^2}}\,. \tag{34}$$

Continuing evaluation of Eq. (33), we need to calculate

$$\mathcal{P}\int\limits_{-1}^{1}\frac{\mathrm{d}x}{|z_1+x|\,\sqrt{\Delta_2(x)}}\,, \tag{35}$$

where we introduce $\Delta_l(x) = \Delta(x, z_l, \cos\delta_l)$ and $\Delta_2(x)$ plays the role of $a^2-b^2-c^2$. It can be proved that the integral (35) is finite only if $1-z_1^2 < 0$ and exists if $\Delta_2(x)$ is positive for any $x \in (-1,1)$ since otherwise, we have a contribution $\int\limits_{x^-}^{x^+}\mathrm{d}x\ldots$, where $\sqrt{\Delta_2(x)}$ is not a real function. As a result, we have to assume $1-z_2^2 < 0$.

Summarising, we conclude that the correct version of Eq. (20) is

$$\mathcal{P}\int\limits_{-1}^{1}\mathrm{d}x\frac{\operatorname{sgn}(z+x\cos\psi)}{(x+y)\sqrt{\Delta(x,z,\cos\psi)}} = \frac{\Xi(y,z,\cos\psi)}{\sqrt{\Delta_0(y,z,\cos\psi)}}\,, \tag{36}$$

where $1-z^2<0$ and $1-y^2<0$ are assumed. We see that the region of the existence of this integral is symmetric until the transposal $y \leftrightarrow z$.

On the other hand, if we believe that Eqs. (30) and (20) can be used without any changes, application of Eq. (20) to calculate $\mathrm{Re}\,\mathcal{Z}_2^{\pm}$ when the spherical coordinates are chosen as (17) and $k_{1,2} > 0$ immediately gives

$$\mathcal{P}\int_{-1}^{1} \mathrm{d}x \frac{\mathrm{sgn}(z_s + x\cos\delta_s)\,\Theta(\Delta_s(x))}{(x+z_l)\sqrt{\Delta_s(x)}} = \frac{\Xi_{ls}(E)\,\Theta(\Delta_{ls}) + \mathcal{G}_{ls}(E)\,\Theta(-\Delta_{ls})}{\sqrt{|\Delta_{ls}|}}\,,$$

where we introduce $\Delta_{ls} = \Delta_0(z_l, z_s, \cos\psi_{ls}) = \Delta_l(-z_s) = \Delta_s(-z_l)$, $\Xi_{ls}(E) = \Xi(z_l, z_s, \cos\psi_{ls})$, and $\mathcal{G}_{ls}(E) = \mathcal{G}(z_l, z_s, \cos\psi_{ls})$ for shortness.

However, we can use other spherical coordinates demanding that $\phi_1 = \phi_2$ while $\sin\delta_{1,2} > 0$. Then

$$\begin{aligned}
&\frac{1}{2\pi}\mathcal{P}\int_{-1}^{1}\mathrm{d}x\int_{0}^{2\pi}\mathrm{d}\phi\,\frac{1}{z_1 + x\cos\delta_1 + \sqrt{1-x^2}\sin\delta_1(\sin\phi\sin\phi_1 + \cos\phi\cos\phi_1)}\\
&\times\frac{1}{z_2 + x\cos\delta_2 + \sqrt{1-x^2}\sin\delta_2(\sin\phi\sin\phi_1 + \cos\phi\cos\phi_1)}\\
&= \sin\delta_1\int_{-1}^{1}\mathrm{d}x\frac{\mathrm{sgn}(z_1 + x\cos\delta_1)\,\Theta(\Delta_1(x))}{(\beta_{12} + \alpha_{12}x)\sqrt{\Delta_1(x)}}\\
&- \sin\delta_2\int_{-1}^{1}\mathrm{d}x\frac{\mathrm{sgn}(z_2 + x\cos\delta_2)\,\Theta(\Delta_2(x))}{(\beta_{12} + \alpha_{12}x)\sqrt{\Delta_2(x)}}\,.
\end{aligned}$$

Above we have used Eq. (30) and denoted

$$\alpha_{ls} = \cos\delta_s\sin\delta_l - \cos\delta_l\sin\delta_s\cos(\phi_l - \phi_s)\,, \tag{37}$$

$$\beta_{ls} = z_s\sin\delta_l - z_l\sin\delta_s\cos(\phi_l - \phi_s)\,, \tag{38}$$

for arbitrary spherical coordinates.

Applying Eq. (20) to every term separately, mentioning that $\alpha_{12}^2 = \sin^2\psi_{12}$,

$$\Delta_t\left(-\frac{\beta_{ls}}{\alpha_{ls}}\right) = \frac{\sin^2\delta_t}{\sin^2\psi_{ls}}\,\Delta_{ls}\,, \qquad t = l, s\,, \tag{39}$$

and using Eq. (A.3), we obtain

$$
\begin{aligned}
&\sin\delta_1 \int\limits_{-1}^{1} \mathrm{d}x \frac{\mathrm{sgn}(z_1 + x\cos\delta_1)\,\Theta(\Delta_1(x))}{(\beta_{12}+\alpha_{12}x)\sqrt{\Delta_1(x)}} \\
&-\sin\delta_2 \int\limits_{-1}^{1} \mathrm{d}x \frac{\mathrm{sgn}(z_2 + x\cos\delta_2)\,\Theta(\Delta_2(x))}{(\beta_{12}+\alpha_{12}x)\sqrt{\Delta_2(x)}} \\
&= \frac{\sin\psi_{12}}{\alpha_{12}} \left[\frac{\Xi(\beta_{12}/\alpha_{12}, z_1, \cos\delta_1)\,\Theta(\Delta_{12}) + \mathscr{G}(\beta_{12}/\alpha_{12}, z_1, \cos\delta_1)\,\Theta(-\Delta_{12})}{\sqrt{|\Delta_{12}|}} \right. \\
&\left. - \frac{\Xi(\beta_{12}/\alpha_{12}, z_2, \cos\delta_2)\,\Theta(\Delta_{12}) + \mathscr{G}(\beta_{12}/\alpha_{12}, z_2, \cos\delta_2)\,\Theta(-\Delta_{12})}{\sqrt{|\Delta_{12}|}} \right] \\
&= \frac{\Xi_{12}(E)\,\Theta(\Delta_{12}) + [\mathscr{G}_{12}(E) - \pi]\,\Theta(-\Delta_{12})}{\sqrt{|\Delta_{12}|}}
\end{aligned}
$$

which contains the additional π-term when $\Delta_{12} < 0$. Since the double integral cannot depend on the coordinate system, the mentioned inconsistency proves that the integral does not exist for $\Delta_{12} < 0$. Using the symmetry between z_1 and z_2, we conclude that the conditions for the existence of the integral are $1 - z_{1,2}^2 < 0$.

Besides Eq. (31), $\mathrm{Re}\mathcal{Z}_2^{\pm}$ contains a second term

$$
\begin{aligned}
&\int\limits_{m}^{\Lambda_E} \mathrm{d}E \int\limits_{-1}^{1} \mathrm{d}x \int\limits_{0}^{2\pi} \mathrm{d}\phi \frac{f(\pm E-\mu)}{p}\,\delta(z_1 + x) \\
&\times \delta\left(z_2 + x\cos\delta_2 + \cos\phi\sin\delta_2\cos\phi_2\sqrt{1-x^2}\right) \\
&= 2\int\limits_{m}^{\Lambda_E} \mathrm{d}E \frac{f(\pm E-\mu)}{p}\frac{\Theta(-\Delta_{12})}{\sqrt{|\Delta_{12}|}}
\end{aligned}
\tag{40}
$$

obtained with the help of the well-known result

$$
\int\limits_{-1}^{1} \mathrm{d}x\, f(x)\,\delta(a+x) = f(-a)\Theta\big(1-a^2\big) \tag{41}
$$

and

$$
\int\limits_{0}^{2\pi} \mathrm{d}\phi\, f(\phi)\,\delta\left(a + b\cos\phi + c\sin\phi\right) = \frac{f\left(\phi^+\right) + f\left(\phi^-\right)}{\sqrt{|a^2-b^2-c^2|}}\,\Theta\big(b^2+c^2-a^2\big)\,, \tag{42}
$$

where $\phi^\pm$ are defined by the relations

$$\cos\phi^\pm = \frac{-ab \pm c\sqrt{b^2+c^2-a^2}}{b^2+c^2}\,, \qquad \sin\phi^\pm = \frac{-ac \mp b\sqrt{b^2+c^2-a^2}}{b^2+c^2}\,.$$

The integral (40) evidently exists and is nonzero only when $\varDelta_{12} < 0$. We see that the integration of a δ-function does not influence the existence of a multiple integral and just leads to Θ-expressions. Saying more precisely, the conditions for the existence of an elementary integral can be investigated only after making all integrations of δ-functions. If the resulting expression is an ordinary integral, it definitely exists if it is finite.

Combining two considered contributions, we find that the second term vanishes

$$\mathrm{Re}\mathcal{Z}_2^\pm = \frac{1}{2}\int\limits_m^{\varLambda_E} \mathrm{d}E\, \frac{f(\pm E-\mu)\,\varXi_{12}(E)}{\sqrt{\varPhi_{12}(E)}}\,, \tag{43}$$

and the conditions $\varPhi_l(E) \le 0$, $l = 1,2$ must be satisfied. The quadratic trinomials $\varPhi_l = p^2 k_l^2(1-z_l^2)$ and $\varPhi_{ls} = p^2 k_l^2 k_s^2 \varDelta_{ls}$ are described in detail in Appendix B.

Generally, integrands of the integrals with three and more lines have good asymptotics $p^{-\alpha}$ with $\alpha \ge 2$, for $p \to \infty$ and the integrals do not need a regularization. Applying a condition for the existence of the elementary integral, one should guarantee that it is satisfied for the whole integration interval, $E \in [m, \varLambda_E]$, that in turn leads to constraints on possible values of parameters. For example, Eq. (B.8) results in the necessary conditions $k_l \le |\lambda_l|$ for the existence of the real part. Conversely, a simple cut off by the Θ-function does not restrict allowed values of k_l and λ_l. This is a key difference between our approach and [13]. A detailed investigation of the trinomial $\varPhi_l(E)$ demonstrates that conditions for the existence of the real part, $\varPhi_l(E) \le 0$, are fulfilled for two different cases. Allowed values of parameters in the first one do not depend on the momentum cutoff parameter, $\varLambda$, *i.e.* on the regularization scheme. This case is determined by the condition that the trinomial has no roots or two negative roots. The second subset of parameters depends on $\varLambda$ since both roots have to be greater than $\varLambda_E$. If $\varLambda_E \to \infty$, the second possibility disappears. However, if we apply the 3D cutoff regularization to integrals with one and two lines, from the consistency point of view, we can also use the finite cutoff for expressions with three and more lines.

It can be shown in a similar way that the factor $\Theta(\varDelta_{12}) = \Theta(\varPhi_{12})$ in the imaginary part $\mathrm{Im}\mathcal{Z}_2^\pm$ that can be extracted from Eq. (5.44) in [13], should be considered as a condition for the existence.

Let us again underline that the above proof is valid only if integrals $\mathcal{Z}_L^{\pm}\left(\mathcal{X}_1^+,\ldots,\mathcal{X}_L^+;\mathcal{T}\right)$ can be considered separately. Below, we suppose that this property is satisfied. If singularities of two contributions in Eq. (9) compensate one another, it is necessary to consider the existence of $\mathcal{Z}_0^{(L)}\left(\mathcal{X}_1^+,\ldots,\mathcal{X}_L^+;\mathcal{T}\right)$ as an entire object.

4. The Z_3^* for collinear momenta

Let us now start the detailed evaluation of $\mathcal{Z}_3^{\pm}$ for all possible orientations of momenta $\boldsymbol{k}_{1,2,3}$. As one can easily check, up to permutation of $\boldsymbol{k}_{1,2,3}$, we have to consider seven opportunities:

1. $k_1 = k_2 = k_3 = 0\,;$
2. $k_1 = k_2 = 0\,,\qquad k_3 > 0\,;$
3. $k_l > 0\,,\qquad \boldsymbol{k}_l \times \boldsymbol{k}_s = 0\,;$
4. $k_{1,2} > 0\,,\qquad k_3 = 0\,,\qquad \boldsymbol{k}_1 \times \boldsymbol{k}_2 = 0\,;$
5. $k_{1,2} > 0\,,\qquad k_3 = 0\,,\qquad \boldsymbol{k}_1 \times \boldsymbol{k}_2 \neq 0\,;$
6. $k_l > 0\,,\qquad \boldsymbol{k}_l \times \boldsymbol{k}_s \neq 0\,,\qquad \boldsymbol{k}_1(\boldsymbol{k}_2 \times \boldsymbol{k}_3) = 0\,;$
7. $k_l > 0\,,\qquad \boldsymbol{k}_l \times \boldsymbol{k}_s \neq 0\,,\qquad \boldsymbol{k}_1(\boldsymbol{k}_2 \times \boldsymbol{k}_3) \neq 0\,.$

The first four cases corresponding to "dot" and "line" configurations are very simple. For this reason, it is more convenient to derive a general expression for $\mathcal{Z}_L^{\pm}$ and then to apply it to $L = 3$. These cases will be considered below in this section. Planar configurations (items 5 and 6) when all three vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, and $\boldsymbol{k}_3$ lie in the same plane or, more precisely, when $\boldsymbol{k}_1(\boldsymbol{k}_2 \times \boldsymbol{k}_3) = 0$ are considered in the next section. The most complicated situation for three independent vectors is analysed separately in Section 6.

4.1. Calculation of $\mathcal{Z}_L^{\pm}$ when all $k_l = 0$

The simplest situation is when all k_l are equal to zero. Equation (11) is strongly simplified and takes the form of

$$\mathcal{Z}_L^{\pm}(\mathcal{X}_1^+,\ldots,\mathcal{X}_L^+;\mathcal{T}\,) = \frac{1}{2^{L-2}\rho_L}\lim_{\epsilon\to 0}\int\limits_m^{\Lambda_E}\mathrm{d}E\,p\,f(\pm E-\mu;T)\prod_{l=1}^{L}\frac{1}{E-E_{0l}-i\epsilon} \tag{44}$$

with

$$\rho_L = \prod_{l=1}^{L}\lambda_l\,. \tag{45}$$

To calculate the integral, we need to translate the product into a sum. Let us here remind the useful relation [16]

$$\frac{\varphi(x)}{f(x)} = \sum_{l=1}^{L} \frac{c_l'}{x - x_l}\,, \qquad c_l' = \frac{\varphi(x_l)}{f'(x_l)}\,, \tag{46}$$

where we assume that the function $f(x)$ has only simple roots x_l. A consequence of this wonderful expression is the relation

$$\prod_{l=1}^{L} \frac{1}{\alpha_l y - y_l} = \sum_{l=1}^{L} \frac{c_l^{(L)}}{\alpha_l y - y_l}\,, \tag{47}$$

where $y_s \neq y_l$ if $s \neq l$, and

$$c_1^{(1)} = 1\,, \qquad c_l^{(L)} = \prod_{s=1, s\neq l}^{L} \frac{\alpha_l}{y_l \alpha_s - y_s \alpha_l}\,, \qquad L > 1\,. \tag{48}$$

Using Eq. (47), the integral (44) can be written as a sum of simple integrals, and one obtains

$$\mathrm{Re}\mathcal{Z}_L^{\pm} = \frac{1}{2^{L-2}\rho_L} \sum_{l=1}^{L} a_l^{(L)}\, I_G^{\pm}(E_{0l};\mathcal{T})\,, \tag{49}$$

$$\mathrm{Im}\mathcal{Z}_L^{\pm} = \frac{\pi}{2^{L-2}\rho_L} \sum_{l=1}^{L} a_l^{(L)}\, G^{\pm}(E_{0l};\mathcal{T})\,. \tag{50}$$

The assumption is used that for every pair $E_{0s} \neq E_{0l}$ if $s \neq l$. Here

$$a_1^{(1)} = 1\,, \qquad a_l^{(L)} = \prod_{s=1, s\neq l}^{L} \frac{1}{E_{0l} - E_{0s}}\,, \qquad L > 1\,, \tag{51}$$

$$I_G^{\pm}(E_0;\mathcal{T}) = \mathcal{P}\int_m^{\Lambda_E} \mathrm{d}E\, \frac{pf(\pm E - \mu; T)}{E - E_0}\,, \tag{52}$$

and

$$G^{\pm}(E;\mathcal{T}) = p\, f(\pm E - \mu; T)\, \Theta(E - m)\, \Theta(\Lambda_E - E)\,. \tag{53}$$

Considering the corresponding integral of absolute value and separating the singularity, one can show that we must demand $E_{0l} < m$ or $E_{0l} > \Lambda_E$ to have a real part while the imaginary one exists for any E_{0l}.

For $L = 3$, one can find

$$a_1^{(3)} = \frac{1}{(E_{02} - E_{01})(E_{03} - E_{01})},$$
$$a_2^{(3)} = -\frac{1}{(E_{02} - E_{01})(E_{03} - E_{02})},$$
$$a_3^{(3)} = \frac{1}{(E_{03} - E_{01})(E_{03} - E_{02})}$$

which are connected by

$$a_1^{(3)} + a_2^{(3)} + a_3^{(3)} = 0\,.$$

One should keep in mind that application of Eq. (46) is a simplification since when all E_{0l} lie outside the integration interval, the integral is finite also if we have roots of any multiplicity. Therefore, the above consideration allowed us to prove that the conditions $E_{0l} \notin [m, \Lambda_E]$ are necessary and sufficient also for simple roots. This remark gives another view on the obtained condition for the existence of the real part (49). Namely, if the latter is fulfilled, the transition between cases of simple and multiple roots is direct and continuous.

As a result, if we expect that some E_{0l} coincide, it is better to calculate the real part in a straightforward way as

$$\mathrm{Re}\mathcal{Z}_L^{\pm} = \frac{1}{2^{L-2}\rho_L} \int\limits_m^{\Lambda_E} \mathrm{d}E\, p\, f(\pm E - \mu; T) \prod_{l=1}^{L} \frac{1}{E - E_{0l}}\,. \tag{54}$$

However, the assumption that $E_{0l} \neq E_{0s}$ is crucial for calculating and existence of the imaginary part. As a result, both parts are simultaneously unconditionally convergent only if $E_{0l} \neq E_{0s}$ and $E_{0l} \notin [m, \Lambda_E]$. As one can see, $\mathrm{Im}\mathcal{Z}_L^{\pm} = 0$ under this conditions. As we will see further, the imaginary part vanishes when the real part exists for any values of external momenta, $\boldsymbol{k}_l$, and any number of lines.

4.2. Calculation of $\mathcal{Z}_L^{\pm}$ for $\boldsymbol{k}_l \times \boldsymbol{k}_s = 0$

The condition $\boldsymbol{k}_l \times \boldsymbol{k}_s = 0$ for some l, s means that $\boldsymbol{k}_l$ and $\boldsymbol{k}_s$ are collinear if $k_l k_s \neq 0$. In this case, using the coordinates (17), we have

$$\begin{aligned} \cos\psi_{ls} &\equiv \eta_{ls} = \pm 1\,, \qquad && \sin\psi_{ls} = 0\,, \\ \sin\delta_l &= 0\,, && \cos\delta_l = \eta_{ls}\cos\delta_s = \pm 1\,. \end{aligned} \tag{55}$$

Another possibility is that $k_l = 0$ or/and $k_s = 0$.

Below, we assume $L > 1$ since we need two or more vectors to speak about collinearity[3]. Equation (12) can be written as

$$\begin{aligned}\mathcal{Z}_L^{\pm}(\mathcal{X}_1^+, \ldots, \mathcal{X}_L^+; \mathcal{T}) &= \frac{1}{2^L \pi} \lim_{\epsilon \to 0} \int\limits_m^{\Lambda_E} \mathrm{d}E\, p\, f(\pm E - \mu; T) \\ &\times \prod_{s=L-n+1}^{L} \frac{1}{\lambda_s (E - E_{0s} - i\epsilon)} \\ &\times \int\limits_{-1}^{1} \mathrm{d}x \int\limits_0^{2\pi} \mathrm{d}\phi \prod_{l=1}^{L-n} \frac{1}{\lambda_l (E - E_{0l} - i\epsilon) + xpk_l \cos\delta_l} ,\end{aligned}$$

where $n = \overline{0, L-1}$. If $n = 0$, the first product is absent.

Let us define

$$\omega_{ls} = k_l \lambda_s - k_s \lambda_l \eta_{ls} , \qquad \zeta_{ls} = \frac{k_l \varepsilon_s - k_s \varepsilon_l \eta_{ls}}{\omega_{ls}} . \tag{56}$$

One can see that if $k_s = 0$, $\zeta_{ls} = E_{0s}$. When $k_l k_s \neq 0$, due to $\eta_{ls}^2 = 1$, we have two nice properties

$$\omega_{ls} = -\omega_{sl} \eta_{ls} , \qquad \zeta_{ls} = \zeta_{sl} . \tag{57}$$

Collinearity of $\boldsymbol{k}_l$ allows us to decompose the product over x, according to Eq. (47), as

$$\begin{aligned}\prod_{l=1}^{L'} \frac{1}{\lambda_l (E - E_{0l} - i\epsilon) + xpk_l \cos\delta_l} &= \sum_{l=1}^{L'} \frac{1}{\lambda_l (E - E_{0l} - i\epsilon) + xpk_l \cos\delta_l} \\ &\times \prod_{s=1, s\neq l}^{L'} \frac{k_l}{(E - \zeta_{ls} - i\epsilon)\, \omega_{ls}} .\end{aligned} \tag{58}$$

Then we again apply Eq. (47) to the two-component E-product

$$\prod_{s=1, s\neq l}^{L'} \frac{k_l}{(E - \zeta_{ls} - i\epsilon)\, \omega_{ls}} \prod_{s=L'+1}^{L''} \frac{1}{\lambda_s (E - \zeta_{ls} - i\epsilon)} = k_l \sum_{s=1, s\neq l}^{L''} \frac{b_{ls}^{(L'')}}{E - \zeta_{ls} - i\epsilon} , \tag{59}$$

[3] The solution for the case of $L = 1$ can be found in [13]. We only mention that values of the momentum k_1 for which $\mathrm{Re}\mathcal{Z}_1^{\pm}$ exists are determined from the inequality $\Phi_1(E) \leq 0$.

where

$$b_{ls}^{(L'')} = \frac{d_{ls}^{(L'')}}{\omega_{ls}}\,, \qquad d_{12}^{(2)} = 1\,, \qquad d_{ls}^{(L'')} = \prod_{t=1,t\neq s,t\neq l}^{L''} \frac{k_l}{\omega_{lt}(\zeta_{ls}-\zeta_{lt})}$$

and, by definition, $L' < L''$. These coefficients have the following nice properties:

$$b_{ls}^{(L'')} = -b_{sl}^{(L'')}\eta_{ls}\,, \qquad d_{ls}^{(L'')} = d_{sl}^{(L'')}\,. \tag{60}$$

Combining the two above decompositions and using Eq. (19), one gets

$$\begin{aligned} &p \prod_{s=L-n+1}^{L} \frac{1}{\lambda_s(E-E_{0s}-i\epsilon)} \prod_{l=1}^{L-n} \frac{1}{\lambda_l(E-E_{0l}-i\epsilon)+xpk_l\cos\delta_l} \\ &= \sum_{l=1}^{L-n} \frac{pk_l}{\lambda_l(E-E_{0l}-i\epsilon)+xpk_l\cos\delta_l} \sum_{s=1,s\neq l}^{L} \frac{b_{ls}^{(L)}}{E-\zeta_{ls}-i\epsilon}\,. \end{aligned} \tag{61}$$

The ϕ-integration is trivial. Separating the real and imaginary parts, we obtain

$$\begin{aligned} \mathrm{Re}\mathcal{Z}_L^{\pm} &= \frac{1}{2^{L-1}} \sum_{l=1}^{L-n} \sum_{s=1,s\neq l}^{L} b_{ls}^{(L)} \int_m^{\Lambda_E} \mathrm{d}E\, f(\pm E-\mu) \\ &\times \int_{-1}^{1} \mathrm{d}x \left[\frac{1}{z_l + x\cos\delta_l} \frac{1}{E-\zeta_{ls}} - \pi^2 \mathrm{sgn}\lambda_l\, \delta(E-\zeta_{ls})\, \delta(z_l + x\cos\delta_l) \right], \end{aligned} \tag{62}$$

$$\begin{aligned} \mathrm{Im}\mathcal{Z}_L^{\pm} &= \frac{\pi}{2^{L-1}} \sum_{l=1}^{L-n} \sum_{s=1,s\neq l}^{L} b_{ls}^{(L)} \int_m^{\Lambda_E} \mathrm{d}E\, f(\pm E-\mu) \\ &\times \int_{-1}^{1} \mathrm{d}x \left[\frac{\delta(E-\zeta_{ls})}{z_l + x\cos\delta_l} + \frac{\mathrm{sgn}\lambda_l}{E-\zeta_{ls}}\, \delta(z_l + x\cos\delta_l) \right]. \end{aligned} \tag{63}$$

To define constraints on momenta, let us consider the integrals of the corresponding amplitudes. One can find that the first term of the real part exists if we have $1 - z_l^2 \leq 0$ for $E \in [m, \Lambda_E]$ and $\xi_{ls} \notin [m, \Lambda_E]$. As a consequence, the second term is equal to zero

$$\mathrm{Re}\mathcal{Z}_L^{\pm} = \frac{1}{2^{L-1}} \sum_{l=1}^{L-n} \sum_{s=1,s\neq l}^{L} b_{ls}^{(L)} I_J^{\pm}(\zeta_{ls}, k_l, \lambda_l, \varepsilon_l, \mathcal{T}) \tag{64}$$

with

$$I_J^{\pm}(E_0, k, \lambda, \varepsilon, \mathcal{T}) = \int\limits_m^{\Lambda_E} \mathrm{d}E \, \frac{f(\pm E - \mu)}{E - E_0} \ln \left| \frac{\lambda E - \varepsilon + k\,p}{\lambda E - \varepsilon - k\,p} \right| . \tag{65}$$

As was mentioned above, the inequality $1 - z_l^2 \leq 0$ can be satisfied only when $k_l \leq |\lambda_l|$.

The second term of the imaginary part exists when $\xi_{ls} \notin [m, \Lambda_E]$. Then the first term is zero if $|z_l(\xi_{ls})| > 1$ or, equivalently, $\Phi_l(\xi_{ls}) \leq 0$. Applying Eq. (41), we finally have

$$\mathrm{Im}\mathcal{Z}_L^{\pm} = \frac{\pi}{2^{L-1}} \sum_{l=1}^{L-n} \mathrm{sgn}\lambda_l \sum_{s=1, s\neq l}^{L} b_{ls}^{(L)} I_H^{\pm}(\zeta_{ls}, k_l, \lambda_l, \varepsilon_l; \mathcal{T}) , \tag{66}$$

$$I_H^{\pm}(E_0, k_l, \lambda_l, \varepsilon_l; \mathcal{T}) = \int\limits_m^{\Lambda_E} \mathrm{d}E \, \frac{f(\pm E - \mu)}{E - E_0} \, \Theta(\Phi_l(E)) . \tag{67}$$

Again, we find that $\mathrm{Im}\mathcal{Z}_L^{\pm} = 0$ when the real part exists.

Equations (64) and (66), in principle, are sufficient for numerical calculations, but it is useful to make some additional transformations which allow one to clarify the structure of the expressions. Using $\zeta_{ls} = \zeta_{sl}$, we can make resummation of the symmetric terms. Separating the part with $s > L - n$, one obtains

$$\begin{aligned} \mathrm{Re}\mathcal{Z}_L^{\pm} = & \frac{1}{2^{L-1}} \sum_{l=1}^{L-n-1} \sum_{s=l+1}^{L-n} b_{ls}^{(L)} \\ & \times \int\limits_m^{\Lambda_E} \mathrm{d}E \, \frac{f(\pm E - \mu)}{E - \zeta_{ls}} \ln \left| \frac{\lambda_l E - \varepsilon_l + k_l p}{\lambda_l E - \varepsilon_l - k_l p} \frac{\lambda_s E - \varepsilon_s - k_s \eta_{ls} p}{\lambda_s E - \varepsilon_s + k_s \eta_{ls} p} \right| \\ & + \frac{1}{2^{L-1}} \sum_{s=L-n+1}^{L} \sum_{l=1}^{L-n} b_{ls}^{(L)} I_J^{\pm}(E_{0s}, k_l, \lambda_l, \varepsilon_l, \mathcal{T}) , \end{aligned} \tag{68}$$

$$\begin{aligned} \mathrm{Im}\mathcal{Z}_L^{\pm} = & \frac{\pi}{2^{L-1}} \sum_{l=1}^{L-n-1} \sum_{s=l+1}^{L-n} b_{ls}^{(L)} \left[\mathrm{sgn}\lambda_s I_H^{\pm}(\zeta_{ls}, k_s, \lambda_s, \varepsilon_s; \mathcal{T}) \right. \\ & \left. + \mathrm{sgn}\lambda_l I_H^{\pm}(\zeta_{ls}, k_l, \lambda_l, \varepsilon_l; \mathcal{T}) \right] \\ & + \frac{\pi}{2^{L-1}} \sum_{s=L-n+1}^{L} \sum_{l=1}^{L-n} \mathrm{sgn}\lambda_l b_{ls}^{(L)} I_H^{\pm}(E_{0s}, k_l, \lambda_l, \varepsilon_l; \mathcal{T}) . \end{aligned} \tag{69}$$

5. Planar momenta

5.1. The configuration with $k_{1,2} > 0$, $k_3 = 0$, $\boldsymbol{k}_1 \times \boldsymbol{k}_2 \neq 0$

Below, we will work in the coordinates defined by Eq. (17). As a result, the condition of noncollinearity $\boldsymbol{k}_1 \times \boldsymbol{k}_2 \neq 0$ is equivalent to

$$\sin\psi_{12} = \sin\delta_2 > 0 \tag{70}$$

and we have

$$\begin{aligned}
&\mathcal{Z}_3^{\pm}(\mathcal{X}_1^+, \mathcal{X}_2^+, \mathcal{X}_3^+; \mathcal{T}) \\
&= \frac{1}{8\pi\lambda_3} \lim_{\epsilon\to 0} \int\limits_m^{\Lambda_E} \mathrm{d}E\, \frac{pf(\pm E-\mu)}{E-E_{03}-i\epsilon} \int\limits_{-1}^{1} \frac{\mathrm{d}x}{\lambda_1(E-E_{01}-i\epsilon)+pk_1\,x} \\
&\times \int\limits_0^{2\pi} \frac{\mathrm{d}\phi}{\lambda_2(E-E_{02}-i\epsilon)+pk_2(\sin\delta_2\cos\phi\sqrt{1-x^2}+x\cos\delta_2)}\,.
\end{aligned}$$

The ϕ-integration is made using Eq. (30), if the condition $a^2 > b^2 + c^2$ is assumed, and Eq. (42); meanwhile, integrals over x can be evaluated with the help of Eqs. (36), (41), and

$$\begin{aligned}
\int\limits_{-1}^{1} \mathrm{d}x \frac{\Theta(-\Delta(x,z,\cos\psi))}{(x+y)\sqrt{-\Delta(x,z,\cos\psi)}} &= \pi\,\frac{\Theta\big(1-z^2\big)}{\sqrt{\Delta_0(y,z,\cos\psi)}}\,\mathrm{sgn}(y-z\cos\psi)\,, \\
\Delta_0(y,z,\cos\psi) &> 0\,,
\end{aligned} \tag{71}$$

which was proved in [13] excluding that positivity of Δ_0 is a condition for the existence of the integral (71).

Finally, we find

$$\mathrm{Re}\mathcal{Z}_3^{\pm} = \frac{1}{4\lambda_3} \int\limits_m^{\Lambda_E} \mathrm{d}E\, \frac{f(\pm E-\mu)}{E-E_{03}} \frac{\Xi_{12}(E)}{\sqrt{\Phi_{12}(E)}}\,, \qquad \Phi_{1,2}(E) \leq 0\,, \tag{72}$$

$$\mathrm{Im}\mathcal{Z}_3^{\pm} = \frac{\pi}{4\lambda_3} \int\limits_m^{\Lambda_E} \mathrm{d}E \frac{f(\pm E-\mu)}{E-E_{03}} \frac{\Pi_{12}(E)}{\sqrt{\Phi_{12}(E)}}\,, \qquad \Phi_{12}(E) > 0\,, \quad \Phi_{1,2}(E_{03}) < 0\,, \tag{73}$$

with $E_{03} \notin [m, \Lambda_E]$ assumed in both expressions and defined for shortness

$$\begin{aligned} \Pi_{ls}(E) &= \mathrm{sgn}\lambda_l \,\mathrm{sgn}(z_s - z_l \cos\psi_{ls})\,\Theta(\Phi_l(E)) \\ &\quad + \mathrm{sgn}\lambda_s \,\mathrm{sgn}(z_l - z_s \cos\psi_{ls})\,\Theta(\Phi_s(E)) \\ &= \mathrm{sgn}\left[\lambda_l \omega_{ls}(E - \zeta_{ls})\right] \Theta(\Phi_l(E)) + \mathrm{sgn}\left[\lambda_s \omega_{sl}(E - \zeta_{ls})\right] \Theta(\Phi_s(E)) \,. \end{aligned} \tag{74}$$

5.2. *The case when* $k_l > 0$, $\boldsymbol{k}_l \times \boldsymbol{k}_s \neq 0$, $\boldsymbol{k}_1(\boldsymbol{k}_2 \times \boldsymbol{k}_3) = 0$

First of all, let us redefine this configuration into a more appropriate form. Three noncollinearity requirements give

$$\sin\delta_2,\ \sin\delta_3,\ \sin\psi_{23} > 0\,; \tag{75}$$

meanwhile, the coplanarity condition means that $\sin\phi_3 = 0$ and

$$\cos\phi_3 \equiv \eta_{23} = \pm 1. \tag{76}$$

As a result, we can write

$$\begin{aligned} \cos\psi_{23} &= \cos(\delta_2 - \eta_{23}\delta_3)\,, & \alpha_{23} &= \sin(\delta_2 - \eta_{23}\delta_3)\,, \\ \alpha_{32} &= -\eta_{23}\alpha_{23}\,, & \alpha_{23}^2 &= \alpha_{32}^2 = \sin^2\psi_{23}\,, \\ \beta_{ls} &= z_s \sin\delta_l - \eta_{23} z_l \sin\delta_s\,, & \beta_{32} &= -\eta_{23}\beta_{23}\,. \end{aligned} \tag{77}$$

Let us define the quadratic trinomial

$$\Psi_{ls}(x) \equiv (\beta_{ls} + \alpha_{ls}x)^2 + \Delta_l(x)\sin^2\delta_s \sin^2(\phi_l - \phi_s) \tag{78}$$

which enters into the picture for three nonzero momenta in the same way as $\Delta_l(x)$ for two ones. The symmetry $\Psi_{ls}(x) = \Psi_{sl}(x)$ can be easily checked. We also introduce[4] $\Delta_{123} = \Psi_{23}(-z_1)$. As one can see, the trinomial $\Phi_{123}(E) = p^2 k_1^2 k_2^2 k_3^2 \Delta_{123}$ is simplified in the considered case and can be written as

$$\Phi_{123}(E) = \left[\boldsymbol{b}^{(3)} E - \boldsymbol{a}^{(3)}\right]^2\,,$$

see Appendix B for details. It is useful to remain $\Phi_{123}(E)$ in the denominators of integrands in this quadratic form to make the connection with the 3D case more transparent.

Evaluation of $\mathcal{Z}_3^{\pm}$ for the considered case is very similar to the calculations of the three-line integral in Section 3. Equations (30) and (42) are used to perform ϕ-integration, Eqs. (36), (41), and (71) are sufficient for taking integrals over x.

[4] Compare with the definition of Δ_0.

To simplify the result, we need Eq. (39) and the following property:

$$\eta_{23}\Xi\left(\frac{\beta_{32}}{\alpha_{32}},z_3,\delta_3\right)-\Xi\left(\frac{\beta_{23}}{\alpha_{23}},z_2,\delta_2\right)=-\frac{\alpha_{23}}{\sin\psi_{23}}\,\Xi_{23}\,.$$

The result can be represented as a sum of functions

$$I^{(3)}_{K;ls}=\int\limits_m^{\Lambda_E}\mathrm{d}E\,f(\pm E-\mu)\frac{(\boldsymbol{k}_l\times\boldsymbol{k}_s)\left[\boldsymbol{b}^{(3)}E-\boldsymbol{a}^{(3)}\right]}{\Phi_{123}(E)}\frac{\Xi_{ls}(E)}{\sqrt{\Phi_{ls}(E)}}\,, \tag{79}$$

$$I^{(3)}_{L;ls}=\int\limits_m^{\Lambda_E}\mathrm{d}E\,f(\pm E-\mu)\frac{(\boldsymbol{k}_l\times\boldsymbol{k}_s)\left[\boldsymbol{b}^{(3)}E-\boldsymbol{a}^{(3)}\right]}{\Phi_{123}(E)}\frac{\Pi_{ls}(E)}{\sqrt{\Phi_{ls}(E)}}\,, \tag{80}$$

in the form of

$$\mathrm{Re}\mathcal{Z}^{\pm}_3=\frac{1}{4}\left[I^{(3)}_{K;12}+I^{(3)}_{K;23}+I^{(3)}_{K;31}\right]\,,\qquad \Phi_l(E)\le 0\,, \tag{81}$$

$$\mathrm{Im}\mathcal{Z}^{\pm}_3=\frac{\pi}{4}\left[I^{(3)}_{L;12}+I^{(3)}_{L;23}+I^{(3)}_{L;31}\right]\,,\qquad \Phi_{ls}(E)\ge 0\,, \tag{82}$$

where the root of $\Phi_{123}(E)$, $E_{123}=\boldsymbol{a}^{(3)}\boldsymbol{b}^{(3)}/[\boldsymbol{b}^{(3)}]^2$, must lie outside the interval $[m,\Lambda_E]$.

6. Box integral for three independent momenta

6.1. ϕ-integration

To calculate the box elementary function, it is needed to expand the product part of the integrand that can be explicitly written as

$$\frac{pk_1}{\lambda_1(E-E_{01}-i\epsilon)+pk_1x}$$
$$\frac{pk_2}{\lambda_2(E-E_{02}-i\epsilon)+pk_2(\cos\phi\sin\delta_2\sqrt{1-x^2}+x\cos\delta_2)}$$
$$\times\frac{pk_3}{\lambda_3(E-E_{03}-i\epsilon)+pk_3\left[(\cos\phi_3\cos\phi+\sin\phi_3\sin\phi)\sin\delta_3\sqrt{1-x^2}+x\cos\delta_3\right]}\,. \tag{83}$$

Using Eq. (19), one finds that the real part of the product is

$$
\begin{aligned}
&\frac{1}{(z_1+x)\left(z_2+x\cos\delta_2+\cos\phi\sin\delta_2\sqrt{1-x^2}\right)} \\
&\times\frac{1}{z_3+x\cos\delta_3+\cos\phi\sin\delta_3\cos\phi_3\sqrt{1-x^2}+\sin\phi\sin\delta_3\sin\phi_3\sqrt{1-x^2}} \\
&-\pi^2\,\mathrm{sgn}(\lambda_1\lambda_2) \\
&\times\frac{\delta\left(z_1+x\right)\,\delta\left(z_2+x\cos\delta_2+\cos\phi\sin\delta_2\sqrt{1-x^2}\right)}{z_3+x\cos\delta_3+\cos\phi\sin\delta_3\cos\phi_3\sqrt{1-x^2}+\sin\phi\sin\delta_3\sin\phi_3\sqrt{1-x^2}} \\
&-\pi^2\,\mathrm{sgn}(\lambda_1\lambda_3)\delta\left(z_1+x\right) \\
&\times\frac{\delta\left(z_3+x\cos\delta_3+\cos\phi\sin\delta_3\cos\phi_3\sqrt{1-x^2}+\sin\phi\sin\delta_3\sin\phi_3\sqrt{1-x^2}\right)}{z_2+x\cos\delta_2+\cos\phi\sin\delta_2\sqrt{1-x^2}} \\
&-\pi^2\,\mathrm{sgn}(\lambda_2\lambda_3)\frac{\delta\left(z_2+x\cos\delta_2+\cos\phi\sin\delta_2\sqrt{1-x^2}\right)}{z_1+x} \\
&\times\delta\left(z_3+x\cos\delta_3+\cos\phi\sin\delta_3\cos\phi_3\sqrt{1-x^2}+\sin\phi\sin\delta_3\sin\phi_3\sqrt{1-x^2}\right)
\end{aligned}
\tag{84}
$$

and the sum

$$
\begin{aligned}
&\pi\,\mathrm{sgn}\lambda_1\,\frac{\delta\left(z_1+x\right)}{z_2+x\cos\delta_2+\cos\phi\sin\delta_2\sqrt{1-x^2}} \\
&\times\frac{1}{z_3+x\cos\delta_3+\cos\phi\sin\delta_3\cos\phi_3\sqrt{1-x^2}+\sin\phi\sin\delta_3\sin\phi_3\sqrt{1-x^2}} \\
&+\pi\,\mathrm{sgn}\lambda_2 \\
&\times\frac{\delta\left(z_2+x\cos\delta_2+\cos\phi\sin\delta_2\sqrt{1-x^2}\right)}{(z_1+x)\left(z_3+x\cos\delta_3+\cos\phi\sin\delta_3\cos\phi_3\sqrt{1-x^2}+\sin\phi\sin\delta_3\sin\phi_3\sqrt{1-x^2}\right)} \\
&+\pi\,\mathrm{sgn}\lambda_3 \\
&\times\frac{\delta\left(z_3+x\cos\delta_3+\cos\phi\sin\delta_3\cos\phi_3\sqrt{1-x^2}+\sin\phi\sin\delta_3\sin\phi_3\sqrt{1-x^2}\right)}{(z_1+x)\left(z_2+x\cos\delta_2+\cos\phi\sin\delta_2\sqrt{1-x^2}\right)} \\
&-\pi^3\,\mathrm{sgn}(\lambda_1\lambda_2\lambda_3)\,\delta\left(z_1+x\right)\,\delta\left(z_2+x\cos\delta_2+\cos\phi\sin\delta_2\sqrt{1-x^2}\right) \\
&\times\delta\left(z_3+x\cos\delta_3+\cos\phi\sin\delta_3\cos\phi_3\sqrt{1-x^2}+\sin\phi\sin\delta_3\sin\phi_3\sqrt{1-x^2}\right)
\end{aligned}
\tag{85}
$$

determines the imaginary one.

As usual, we start by integrating over ϕ of the product under the integral. To do this, we need the following expression:

$$\begin{aligned}
&\int_0^{2\pi} \frac{\mathrm{d}\phi}{(a_1 + b_1 \cos\phi + c_1 \sin\phi)(a_2 + b_2 \cos\phi + c_2 \sin\phi)} \\
&= \frac{2\pi}{(a_1 b_2 a_2 b_1)^2 - (b_1 c_2 - b_2 c_1)^2 + (a_2 c_1 - a_1 c_2)^2} \\
&\times \left[\frac{a_2\left(b_1^2 + c_1^2\right) - a_1(b_1 b_2 + c_1 c_2)}{\sqrt{|a_1^2 - b_1^2 - c_1^2|}} \mathrm{sgn}(a_1)\, \Theta\big(a_1^2 - b_1^2 - c_1^2\big) \right. \\
&\left. + \frac{a_1\left(b_2^2 + c_2^2\right) - a_2(b_1 b_2 + c_1 c_2)}{\sqrt{|a_2^2 - b_2^2 - c_2^2|}} \mathrm{sgn}(a_2)\, \Theta\big(a_2^2 - b_2^2 - c_2^2\big) \right]
\end{aligned} \tag{86}$$

which is unconditionally convergent when $a_1^2-b_1^2-c_1^2 > 0$ and $a_2^2-b_2^2-c_2^2 > 0$. As one can see, Eq. (30) is a limit of Eq. (86).

Equations (42) and (86) allow us to make integration over ϕ. To simplify the result, we also use the relation

$$\delta\left([g(x)]^2 - [h(x)]^2\right) = \frac{\delta(g(x) - h(x)) + \delta(g(x) + h(x))}{2\,|h(x)|} \tag{87}$$

together with Eq. (42) and find

$$\begin{aligned}
&\int_0^{2\pi} \mathrm{d}\phi\, \delta\left(z_2 + x\cos\delta_2 + \cos\phi \sin\delta_2 \sqrt{1-x^2}\right) \\
&\times \delta\left(z_3 + x\cos\delta_3 + \cos\phi \sin\delta_3 \cos\phi_3 \sqrt{1-x^2} + \sin\phi \sin\delta_3 \sin\phi_3 \sqrt{1-x^2}\right) \\
&= 2\sin\delta_2 \sin\delta_3\, |\sin\phi_3|\, \delta\left(\Psi_{23}(x)\right) .
\end{aligned} \tag{88}$$

Applying the above mentioned formulas, we obtain

$$\begin{aligned}
\mathrm{Re}\mathcal{Z}_3^{\pm} &= \frac{1}{4k_1 k_2 k_3} \int_m^{\Lambda_E} \mathrm{d}E\, \frac{f(\pm E - \mu)}{p^2} \\
&\quad \times \int_{-1}^{1} \mathrm{d}x \left\{ \sin\delta_2 \frac{\beta_{23} + x\alpha_{23}}{(z_1 + x)\Psi_{23}(x)\sqrt{\Delta_2(x)}} \mathrm{sgn}(z_2 + x\cos\delta_2) \right.
\end{aligned}$$

$$
\begin{aligned}
&+\sin\delta_3 \frac{\beta_{32}+x\alpha_{32}}{(z_1+x)\Psi_{23}(x)\sqrt{\Delta_3(x)}}\,\mathrm{sgn}(z_3+x\cos\delta_3) \\
&-\pi\,\mathrm{sgn}(\lambda_2\lambda_3)\frac{\sin\delta_2\sin\delta_3\,|\sin\phi_3|\,\delta\left(\Psi_{23}(x)\right)}{z_1+x}\Bigg\}\,, \\
1-z_2^2 \;\leq\; 0\,,\quad 1-z_3^2\leq 0\,, &\qquad (89)
\end{aligned}
$$

where the terms with $\mathrm{sgn}(\lambda_1\lambda_2)$ and $\mathrm{sgn}(\lambda_1\lambda_3)$ vanish due to the conditions for the existence of the integral, and

$$
\begin{aligned}
\mathrm{Im}\mathcal{Z}_3^{\pm} \;=\; & \frac{1}{4k_1k_2k_3}\int\limits_m^{\Lambda_E}\mathrm{d}E\,\frac{f(\pm E-\mu)}{p^2} \\
&\times\int\limits_{-1}^{1}\mathrm{d}x\Bigg\{\mathrm{sgn}\lambda_2\,\sin\delta_2\frac{\beta_{23}+x\alpha_{23}}{(z_1+x)\Psi_{23}(x)\sqrt{-\Delta_2(x)}}\,\Theta(-\Delta_2(x)) \\
&+\mathrm{sgn}\lambda_3\,\sin\delta_3\frac{\beta_{32}+x\alpha_{32}}{(z_1+x)\Psi_{23}(x)\sqrt{-\Delta_3(x)}}\,\Theta(-\Delta_3(x)) \\
&+\pi\,\mathrm{sgn}\lambda_1\sin\delta_2\,\delta\left(z_1+x\right)\,\frac{\beta_{23}+x\alpha_{23}}{\Psi_{23}(x)\sqrt{\Delta_2(x)}}\,\mathrm{sgn}(z_2+x\cos\delta_2) \\
&+\pi\,\mathrm{sgn}\lambda_1\sin\delta_3\,\delta\left(z_1+x\right)\,\frac{\beta_{32}+x\alpha_{32}}{\Psi_{23}(x)\sqrt{\Delta_3(x)}}\,\mathrm{sgn}(z_3+x\cos\delta_3) \\
&-\pi^2\,\mathrm{sgn}(\lambda_1\lambda_2\lambda_3)\,\delta\left(z_1+x\right)\,\sin\delta_2\sin\delta_3\,|\sin\phi_3|\,\delta\left(\Psi_{23}(x)\right)\Bigg\}\,, \\
\Delta_{12} \;\geq\; 0\,,\quad \Delta_{13}\geq 0\,. &\qquad (90)
\end{aligned}
$$

The terms proportional to $\mathrm{sgn}\,\lambda_1$ determine the conditions where the imaginary part is finite and real. Applying Eq. (86) leads to appearing of $\Delta_{2,3}(x)$, which must be positive only at $x=-z_1$ due to the presence of $\delta(z_1+x)$.

6.2. Two new definite integrals

To calculate the box integral, we need to evaluate the integral

$$
\int\limits_{-1}^{1}\mathrm{d}x\,\frac{K\,x+S}{R_1(x)\sqrt{\Delta(x,z,\cos\delta)}}\,\mathrm{sgn}(z+x\cos\delta)\,, \qquad (91)
$$

which is a generalization of Eq. (36) and the integral

$$\int_{-1}^{1} \mathrm{d}x \, \frac{K\,x+S}{R_1(x)\sqrt{-\Delta(x,z,\cos\delta)}}\,\Theta(-\Delta(x,z,\cos\delta)) \tag{92}$$

which is an analog of Eq. (71), where

$$R_1(x) = Cx^2 + 2Bx + A\,. \tag{93}$$

Both integrals can be evaluated analytically, *e.g.*, using [16].

Let us start with the consideration of the first one. In the general case, integration is quite difficult. Fortunately, we can consider only a much simpler case when trinomials $\Delta(x,z,\cos\delta)$ and $R_1(x)$ are connected by the relation

$$D = B^2 - AC = -\frac{k}{C-k}\,D_0\,, \qquad C > k \geq 0\,, \tag{94}$$

with

$$D_0 = u^2 - C(C-k)\,d(z,\cos\delta)\,, \qquad u = B - Cb\,, \qquad b = z\cos\delta\,. \tag{95}$$

It means that

$$R_1(x) = (C-k)\left(x + b + \frac{u}{C-k}\right)^2 + k\Delta(x,z,\cos\delta)\,. \tag{96}$$

One more simplification is a consequence of integration over ϕ, see Eq. (89). We obtain the constraint $d(z,\cos\delta) \leq 0$ that means $D_0 > 0$, *i.e.* the trinomial $R_1(x)$ has no zeros. Then one can prove

$$\begin{aligned}
&\int \mathrm{d}x \frac{K\,x+S}{R_1(x)\sqrt{\Delta(x,z,\cos\delta)}} \\
&= \frac{\mathcal{V}}{2\sqrt{D_0}}\,\ln\left|\frac{u(x+b)+(C-k)d+\sqrt{D_0\Delta(x,z,\cos\delta)}}{u(x+b)+(C-k)d-\sqrt{D_0\Delta(x,z,\cos\delta)}}\right| \\
&+\mathcal{U}\arctan\frac{(C-k)(x+b)+u}{\sqrt{(C-k)k\Delta(x,z,\cos\delta)}}
\end{aligned} \tag{97}$$

with

$$\begin{aligned}
\mathcal{V} &= (C-k)\frac{u(S-Kb)-K(C-k)d}{u^2-(C-k)^2 d}\,, \\
\mathcal{U} &= \sqrt{\frac{C-k}{k}}\,\frac{(S-Kb)(C-k)-Ku}{u^2-(C-k)^2 d}\,.
\end{aligned} \tag{98}$$

Applying Eq. (97), we find

$$\int_{-1}^{1} \mathrm{d}x\, \frac{K\,x+S}{R_1(x)\sqrt{\varDelta(x,z,\cos\delta)}}\,\mathrm{sgn}(z+x\cos\delta) = \frac{\mathcal{V}\mathscr{V}}{\sqrt{D_0}} + \mathcal{U}\mathscr{U}\,, \tag{99}$$

where

$$\begin{aligned} \mathscr{V} \;&=\; \frac{1}{2}\Bigg[\ln\left|\frac{u(1+z\cos\delta)+(C-k)d+(z+\cos\delta)\sqrt{D_0}}{u(1+z\cos\delta)+(C-k)d-(z+\cos\delta)\sqrt{D_0}}\right| \\ &\quad + \ln\left|\frac{u(1-z\cos\delta)-(C-k)d+(z-\cos\delta)\sqrt{D_0}}{u(1-z\cos\delta)-(C-k)d-(z-\cos\delta)\sqrt{D_0}}\right|\Bigg]\,, \end{aligned} \tag{100}$$

$$\mathscr{U} \;=\; \arctan\frac{u+(C-k)(1+z\cos\delta)}{(z+\cos\delta)\sqrt{(C-k)k}} - \arctan\frac{u-(C-k)(1-z\cos\delta)}{(z-\cos\delta)\sqrt{(C-k)k}}\,. \tag{101}$$

In a similar way, we find

$$\begin{aligned} &\Theta(d)\int \mathrm{d}x \frac{K\,x+S}{R_1(x)\sqrt{-\varDelta(x,z,\cos\delta)}} \\ &= \Theta(d)\,\Theta(D_0)\,\frac{\mathcal{V}}{\sqrt{D_0}}\;\arctan\frac{u(x+b)+(C-k)\,d}{\sqrt{-D_0\varDelta(x)}} \\ &-\Theta(d)\,\Theta(D_0)\,\frac{\mathcal{U}}{2}\;\ln\left|\frac{u+(x+b)\,(C-k)-\sqrt{-k(C-k)\varDelta(x,z,\cos\delta)}}{u+(x+b)\,(C-k)+\sqrt{-k(C-k)\varDelta(x,z,\cos\delta)}}\right| \\ &+\Theta(d)\,\Theta(-D_0)\,\frac{C}{4\sqrt{dD}}\Bigg[\frac{\tau^-\,f^- - \tau^+}{\sqrt{|f^-|}} \\ &\times\ln\left|\frac{(x-x^-)\sqrt{|f^-|}-\sqrt{-\varDelta(x,z,\cos\delta)}}{(x-x^-)\sqrt{|f^-|}+\sqrt{-\varDelta(x,z,\cos\delta)}}\right| \\ &-\frac{\tau^-\,f^+ - \tau^+}{\sqrt{|f^+|}}\ln\left|\frac{(x-x^-)\sqrt{|f^+|}-\sqrt{-\varDelta(x,z,\cos\delta)}}{(x-x^-)\sqrt{|f^+|}+\sqrt{-\varDelta(x,z,\cos\delta)}}\right|\Bigg]\,, \end{aligned} \tag{102}$$

where

$$f^{\pm} = -\left[\frac{\sqrt{k(C-k)\,d}\mp\sqrt{|D_0|}}{u-(C-k)\,\sqrt{d}}\right]^2\,, \qquad \sqrt{|f^{\pm}|} = \frac{\sqrt{-\varDelta(X^{\pm})}}{X^{\pm}-x^-}\,, \tag{103}$$

and $X^{\pm}$ are the roots of $R_1(x)$. Equation (96) implies that $\varDelta(X^{\pm},z,\cos\delta)<0$.

We have to consider the integral

$$\int \mathrm{d}x \frac{|K\,x + S|}{|R_1(x)|\sqrt{-\Delta(x,z,\cos\delta)}}$$

together with Eq. (102). One can show that if D_0 is negative, there is no integral of the amplitude due to additional poles at $x = X^{\pm}$. It is the reason why we do not need expressions for $\tau^{\pm}$.

Keeping in mind the property

$$\Theta(-\Delta(x,z,\cos\delta)) = \Theta\big(1 - z^2\big)\,\Theta\big(x^+ - x\big)\,\Theta\big(x - x^-\big)\ ,$$

and that the integrand has no additional poles for positive D_0, we see that the second term in r.h.s. of Eq. (102) vanishes. To find values of the first term at $x^{\pm}$, one should observe that $|u| \geq \sqrt{C(C-k)d} \geq (C-k)\sqrt{d}$ when $D_0 > 0$, and $|x^{\pm} + b| = \sqrt{d}$. As a result, we can see that

$$\Theta(d)\,\Theta(D_0)\,\mathrm{sgn}\left[u(x^{\pm} + b) + (C-k)d\right] = \pm\Theta(d)\,\Theta(D_0)\,\mathrm{sgn}\,u\,. \tag{104}$$

Combining all these remarks, one can perform integration over x only when $D_0 > 0$ with the help of

$$\int\limits_{-1}^{1} \mathrm{d}x\,\frac{K\,x + S}{R_1(x)\sqrt{-\Delta(x,z,\cos\delta)}}\,\Theta(-\Delta(x,z,\cos\delta)) = \pi\mathcal{V}\frac{\Theta\big(1-z^2\big)}{\sqrt{D_0}}\,\mathrm{sgn}\,u\,. \tag{105}$$

Let us take one more step to get expressions that can be directly applied in further evaluations. Namely, we shall deal with integrals like

$$\begin{aligned}&\int \mathrm{d}x \frac{Mx + N}{(x - x_0)R_1(x)\sqrt{\pm\Delta(x,z,\cos\delta)}}\\ &= \frac{1}{R_1(x_0)}\left[Q\int \frac{\mathrm{d}x}{(x-x_0)\sqrt{\pm\Delta(x,z,\cos\delta)}} + \int \mathrm{d}x \frac{S - CQ\,x}{R_1(x)\sqrt{\pm\Delta(x,z,\cos\delta)}}\right]\end{aligned} \tag{106}$$

with $Q = Mx_0 + N$, $S = MA - 2NB - NCx_0$. The integral with $\sqrt{-\Delta(x,z,\cos\delta)}$ in the denominator is taken using Eqs. (71) and (105). The result is

$$\begin{aligned}&\int\limits_{-1}^{1} \mathrm{d}x \frac{M\,x + N}{(x-x_0)R_1(x)\sqrt{-\Delta(x,z,\cos\delta)}}\,\Theta(-\Delta(x,z,\cos\delta)) = \frac{\pi\,\Theta\big(1-z^2\big)}{R_1(x_0)}\\ &\times\left[\mathrm{sgn}(u)\frac{\mathcal{V}'}{\sqrt{D_0}} + \mathrm{sgn}(-x_0 - z\cos\delta)\frac{Q}{\sqrt{\Delta_0(x_0,z,\cos\delta)}}\right]\,, \qquad D_0, \Delta_0 > 0\,.\end{aligned} \tag{107}$$

When we deal with $\sqrt{\mathit{\Delta}(x,z,\cos\delta)}$, the second term is evaluated with the help of Eq. (99); meanwhile, the first term is just Eq. (36). We derive

$$\int_{-1}^{1} \mathrm{d}x \frac{Mx+N}{(x-x_0)R_1(x)\sqrt{\mathit{\Delta}(x,z,\cos\delta)}}\,\mathrm{sgn}(z+x\cos\delta)$$
$$= \frac{1}{R_1(x_0)}\left[\frac{Q\,\mathit{\Xi}(x_0,z,\cos\delta)}{\sqrt{\mathit{\Delta}_0(x_0,z,\cos\delta)}} + \frac{\mathcal{V}'\mathscr{V}}{\sqrt{D_0}}\right], \qquad 1-z^2 \le 0\,, \quad 1-x_0^2 \le 0\,, \tag{108}$$

where

$$\mathcal{V}' = (C-k)\frac{u(S+CQb)+CQ(C-k)d}{u^2-(C-k)^2 d}\,. \tag{109}$$

6.3. The real and imaginary parts of the integral with four fermion lines

The expressions obtained in the previous paragraph are sufficient for our purpose. To move further, we need to know A, B, C, M, and N.

When we make x-integration, $\mathit{\Psi}_{23}(x)$ plays the role of $R_1(x)$. Taking into account the property

$$\alpha_{ls}^2 + \sin^2\delta_s \sin^2(\phi_l-\phi_s) = \sin^2\psi_{ls}\,, \tag{110}$$

we find

$$C = \sin^2\psi_{23}\,, \qquad D = -\sin^2\delta_2\sin^2\delta_3\sin^2\phi_3\mathit{\Delta}_{23}\,. \tag{111}$$

Other quantities in Eqs. (107) and (108), such as k, D_0, u, are not $(l \leftrightarrow s)$ symmetric since they depend on b. For integrals with $\mathit{\Delta}_l(x)$, $l=2,3$, we have

$$\begin{aligned} k_l &= \sin^2\delta_s\sin^2(\phi_l-\phi_s)\,,\\ D_{0l} &= \alpha_{ls}^2\sin^2\delta_l\mathit{\Delta}_{ls}\,,\\ u_l &= (\beta_{ls}-z_l\alpha_{ls}\cos\delta_l)\alpha_{ls}\,, \end{aligned} \tag{112}$$

and

$$M_l = \alpha_{ls}\,, \qquad N_l = \beta_{ls}\,. \tag{113}$$

These expressions allow us to find

$$\mathcal{V}'_l = \alpha_{ls}T_1\,, \tag{114}$$

where we used that $u = (N - Mb)M$ and $C - k_l = M^2$ which result in $\mathcal{V}' = -M(B + Cx_0)$, and we also introduced the quantities

$$T_l = z_l \sin^2 \psi_{st} - z_s(\cos\psi_{ls} - \cos\psi_{lt}\cos\psi_{st}) - z_t(\cos\psi_{lt} - \cos\psi_{ls}\cos\psi_{st}) \, , \quad t \neq l, s \, . \tag{115}$$

To simplify the result, we shall also use the following relations:

$$\alpha_{ls} \sin\delta_l = \cos\delta_s - \cos\delta_l \cos\psi_{ls} \tag{116}$$

and

$$\beta_{ls} - z_l \alpha_{ls} \cos\delta_l = (z_s - z_l \cos\psi_{ls}) \sin\delta_l \, . \tag{117}$$

Taking into account all the mentioned properties and applying Eqs. (41), (107), and (108), we obtain

$$\mathrm{Re} Z_3^{\pm} = \frac{1}{4}\left[I^{(3)}_{K;12} + I^{(3)}_{K;23} + I^{(3)}_{K;31}\right] , \qquad \varPhi_l(E) \leq 0 \, , \tag{118}$$

$$\mathrm{Im} Z_3^{\pm} = \frac{\pi}{4}\left[I^{(3)}_{L;12} + I^{(3)}_{L;23} + I^{(3)}_{L;31}\right] , \qquad \varPhi_{ls}(E) > 0 \, , \tag{119}$$

as in the planar case, and $\varPhi_{123}(E) \neq 0$ when $E \in [m, \varLambda_E]$. The separated terms $\mathscr{V}_{2,3}$ were joined together using the property

$$\Theta(\varDelta_{ls}) \left[\mathrm{sgn}(\alpha_{ls})\mathscr{V}_l + \mathrm{sgn}(\alpha_{sl})\mathscr{V}_s\right] = \varXi_{ls}\, \Theta(\varDelta_{ls}) \, .$$

At last, one should take into account the relation

$$pk_1k_2k_3k_sk_tT_l = \pm\left[\boldsymbol{b}^{(3)}E - \boldsymbol{a}^{(3)}\right](\boldsymbol{k}_s \times \boldsymbol{k}_t)$$

to reduce the formulas to the form similar to the *planar* configuration.

7. Some words about calculations of integrals with $L + 1$ lines

In Sections 3–6, we used Eq. (19) to expand the product under the integral. As one can explicitly check for three-line and box integrals, all terms containing two or more δ-functions vanish when the real or imaginary part exists. Generally, it is more convenient to use the only small parameter ϵ for all poles. In such a way, a simpler expression can be obtained for any L. Namely, transforming the product into a sum with the help of Eq. (47), applying Eq. (18), and making the inverse transformation to the product, one can find

$$\prod_{l=1}^{L} \frac{1}{\lambda_l(E - E_{0l} - i\epsilon) + \boldsymbol{p}\boldsymbol{k}_l} = \mathcal{P}\prod_{l=1}^{L} \frac{1}{\lambda_l(E - E_{0l}) + \boldsymbol{p}\boldsymbol{k}_l}$$
$$+ i\pi \sum_{l=1}^{L} \mathrm{sgn}(\lambda_l)\, \delta(\lambda_l(E - E_{0l}) + \boldsymbol{p}\boldsymbol{k}_l) \prod_{s=1, s\neq l}^{L} \frac{1}{\lambda_s(E - E_{0s}) + \boldsymbol{p}\boldsymbol{k}_s} \, . \tag{120}$$

As a result,

$$\mathrm{Re}\mathcal{Z}_L^{\pm}\left(\mathcal{X}_1^+,\ldots,\mathcal{X}_L^+;\mathcal{T}\right)=\frac{1}{2^L\pi}\,\mathcal{P}\int\limits_m^{\varLambda_E}\mathrm{d}E\,p\,f(\pm E-\mu)$$
$$\times\int\limits_{-1}^{1}\mathrm{d}x\int\limits_0^{2\pi}\mathrm{d}\phi\prod_{l=1}^{L}\frac{1}{\lambda_l(E-E_{0l})+\boldsymbol{pk}_l}\,, \tag{121}$$

$$\mathrm{Im}\mathcal{Z}_L^{\pm}\left(\mathcal{X}_1^+,\ldots,\mathcal{X}_L^+;\mathcal{T}\right)=\frac{1}{2^L}\sum_{l=1}^{L}\mathrm{sgn}(\lambda_l)\int\limits_m^{\varLambda_E}\mathrm{d}E\,p\,f(\pm E-\mu)$$
$$\times\int\limits_{-1}^{1}\mathrm{d}x\int\limits_0^{2\pi}\mathrm{d}\phi\,\delta(\lambda_l(E-E_{0l})+\boldsymbol{pk}_l)\prod_{s=1,s\neq l}^{L}\frac{1}{\lambda_s(E-E_{0s})+\boldsymbol{pk}_s}\,, \tag{122}$$

where we can see only the terms without δ-functions or with a single one.

Equation (122) can be simplified by performing integration in the imaginary part

$$\mathrm{Im}\mathcal{Z}_L^{\pm}\left(\mathcal{X}_1^+,\ldots,\mathcal{X}_L^+;\mathcal{T}\right)$$
$$=\frac{1}{2^L}\frac{1}{\prod_{s=1}^{L}k_s}\sum_{l=1}^{L}\mathrm{sgn}(\lambda_l)\int\limits_m^{\varLambda_E}\mathrm{d}E\,\frac{f(\pm E-\mu)}{p^{L-1}}\,\varPi_l(E)\Theta\left(\varPhi_l(E)\right)\,, \tag{123}$$

with

$$\varPi_1(E)=\int\limits_0^{2\pi}\mathrm{d}\phi\prod_{s=2}^{L}\frac{1}{z_s-z_1\cos\delta_s+\cos(\phi-\phi_s)\sin\delta_s\sqrt{1-z_1^2}}\,, \tag{124}$$

and

$$\begin{aligned}\varPi_l(E)\;&=\;\int\limits_{-1}^{1}\mathrm{d}x\frac{\Theta(-\varDelta_l(x))}{(z_1+x)\sqrt{-\varDelta_l(x)}}\prod_{s=2,s\neq l}^{L}\frac{\sin\delta_l}{\varPsi_{ls}(x)}\\ &\quad\times\Bigg[\prod_{s=2,s\neq l}^{L}\left[\beta_{ls}+x\alpha_{ls}-\sin(\phi_l-\phi_s)\sin\delta_s\sqrt{-\varDelta_l(x)}\right]\\ &\quad+\prod_{s=2,s\neq l}^{L}\left[\beta_{ls}+x\alpha_{ls}+\sin(\phi_l-\phi_s)\sin\delta_s\sqrt{-\varDelta_l(x)}\right]\Bigg]\end{aligned} \tag{125}$$

for $l>1$.

The real part can be considered only when $\Phi_l(E) \leq 0$, as follows from Section 4.2. We can also see that the integrand in the definition of $\Pi_1(E)$ has no singularities when $\Delta_{1s} > 0$. For symmetry reasons, one should conclude that the imaginary part exists only if $\Phi_{ls}(E) > 0$.

The derived conditions for the existence of the real part have an important consequence. Equation (123) immediately leads to the conclusion that the imaginary part is zero if the real part exists. In contrary, the imaginary part is nonzero only when the real part does not exist.

8. Conclusions

In the article, we have given the formulas required for numerical evaluation of the four fermion line integral at finite temperature/density that is often needed for theoretical calculations such as within the NJL model. Both the real and imaginary parts of these functions are presented separately. It was proved that the integrals do not exist for some values of the parameters. We give the corresponding constraints for all considered cases.

Derivation of the necessary expressions obliged us to correct the results obtained in [13] for the integral with three lines and, moreover, to refine the approach used in the cited paper. It was shown that careful consideration of the conditions for the existence of elementary integrals plays a crucial role in obtaining valid results. Particularly, it was proved that the imaginary part vanishes if the real part exists.

We have also found, as a by-product, expressions for integrals with any number of lines when momenta are in simple "dot" or "line" configurations. For a general configuration, the corresponding expressions are presented as triple and double integrals allowing numerical evaluation.

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Appendix A

Summation of inverse trigonometric functions

In this Appendix we just cite some beautiful and useful formulas from [16]. The sum and difference of two arccosines are

$$\begin{aligned} &\arccos X + \arccos Y \\ &= 2\pi\Theta(-X - Y) + \operatorname{sgn}(X + Y) \arccos\left(XY - \sqrt{1 - X^2}\sqrt{1 - Y^2}\right) , \end{aligned} \tag{A.1}$$

$$\arccos X - \arccos Y = \mathrm{sgn}(Y-X) \arccos\left(XY + \sqrt{1-X^2}\sqrt{1-Y^2}\right) . \tag{A.2}$$

For arctangents, we have

$$\begin{aligned} \arctan X + \arctan Y &= \arctan \frac{X+Y}{1-XY} + \pi\,\mathrm{sgn}(X+Y)\Theta(XY-1) \\ &= \Theta(|X+Y|) \arctan \frac{XY-1}{X+Y} + \frac{\pi}{2}\mathrm{sgn}(X+Y) , \end{aligned} \tag{A.3}$$

where the second line is obtained with the help of

$$\arctan X + \arctan \frac{1}{X} = \frac{\pi}{2}\,\mathrm{sgn} X . \tag{A.4}$$

Appendix B

Some properties of quadratic trinomials arising during the evaluation of integrals

When we make integration over angles, it is convenient to exclude the variable E using z_l. But after that, in the final expressions, we should make the inverse transition as a result of which the following functions arise:

$$\Phi_l(E) \equiv \Phi(E; k_l, \lambda_l, \varepsilon_l, m) = p^2 k_l^2 \left(1 - z_l^2\right) , \tag{B.1}$$

$$\Phi_{ls}(E) = p^2 k_l^2 k_s^2 \Delta_{ls} , \tag{B.2}$$

$$\Phi_{123}(E) = p^2 k_1^2 k_2^2 k_3^2 \Delta_{123} . \tag{B.3}$$

Recognizing that these functions are just trinomials, let us introduce the generalized trinomial

$$\begin{aligned} \Omega_L(E; \mathcal{X}_1^+, \ldots, \mathcal{X}_L^+, m) &= C_\Omega(\mathcal{X}_1^+, \ldots, \mathcal{X}_L^+, m)\, E^2 \\ &\quad -2B_\Omega(\mathcal{X}_1^+, \ldots, \mathcal{X}_L^+, m)\, E + A_\Omega(\mathcal{X}_1^+, \ldots, \mathcal{X}_L^+, m) \\ &= \left[\boldsymbol{b}^{(L)} E - \boldsymbol{a}^{(L)}\right]^2 - \left[w^{(L)}\right]^2 p^2 , \end{aligned} \tag{B.4}$$

where we demand $|C_\Omega| \neq 0$. The coefficients are

$$\begin{aligned} A_\Omega &= \left[\boldsymbol{a}^{(L)}\right]^2 + m^2 \left[w^{(L)}\right]^2 , \\ B_\Omega &= \boldsymbol{a}^{(L)} \boldsymbol{b}^{(L)} , \\ C_\Omega &= \left[\boldsymbol{b}^{(L)}\right]^2 - \left[w^{(L)}\right]^2 . \end{aligned} \tag{B.5}$$

As we can easily see, the determinant is

$$D_\Omega = \left[w^{(L)}\right]^2 \left\{ A_\Omega - m^2 \left[\boldsymbol{b}^{(L)}\right]^2 - \frac{\left[\boldsymbol{a}^{(L)} \times \boldsymbol{b}^{(L)}\right]^2}{[w^{(L)}]^2} \right\} , \tag{B.6}$$

where we used that

$$\boldsymbol{a}^2 \boldsymbol{b}^2 = (\boldsymbol{a} \times \boldsymbol{b})^2 + (\boldsymbol{a}\boldsymbol{b})^2 \ . \tag{B.7}$$

Using the above properties, one can prove that the magnitude of both roots of $\Omega_L(E)$ is not less than m. As we have seen, the conditons for the existence of the considered improper integrals have the form of $\Omega_L(E) > 0$ for any $E \in [m, \Lambda_E]$. It means that the sign of $\Omega_L(E)$ in the integration interval is fixed which is possible only if both roots lie outside. Then we can prove

$$\Theta(\Omega_L(E)) = \Theta(C_\Omega)\,\Theta(\Omega_L(E)) \tag{B.8}$$

and

$$\Theta(-\Omega_L(E)) = \Theta(D_\Omega)\,\Theta(-C_\Omega)\,\Theta(-\Omega_L(E)) \ . \tag{B.9}$$

The vectors $\boldsymbol{a}^{(L)}$ and $\boldsymbol{b}^{(L)}$ can be expressed as linear combinations of a set of vectors $\mathbf{q}_l$ for any L:

$$\boldsymbol{a}^{(L)} = \sum_{l=1}^{L} \varepsilon_l \mathbf{q}_l \,, \qquad \boldsymbol{b}^{(L)} = \sum_{l=1}^{L} \lambda_l \mathbf{q}_l \,, \tag{B.10}$$

and $\mathbf{q}_l = \mathbf{q}_l\left(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_L\right)$.

Using the generalized trinomial, one can write

$$\Phi_l(E) = -\Omega_1\left(E; \mathcal{X}_l^+, m\right) \tag{B.11}$$

with

$$\boldsymbol{a}_l^{(1)} = \varepsilon_l \frac{\boldsymbol{k}}{k} \,, \qquad \boldsymbol{b}_l^{(1)} = \lambda_l \frac{\boldsymbol{k}}{k} \,, \qquad w_l^{(1)} = k_l \,, \tag{B.12}$$

and $\boldsymbol{k}$ is an arbitrary vector.

For $L = 2$, we have

$$\Phi_{ls}(E) \ = \ \Omega_2\left(E; \mathcal{X}_l^+, \mathcal{X}_s^+, m\right) \,, \tag{B.13}$$

$$\boldsymbol{a}_{ls}^{(2)} \ = \ \varepsilon_s \boldsymbol{k}_l - \varepsilon_l \boldsymbol{k}_s \,, \qquad \boldsymbol{b}_{ls}^{(2)} = \lambda_s \boldsymbol{k}_l - \lambda_l \boldsymbol{k}_s \,, \qquad w_{ls}^{(2)} = |\boldsymbol{k}_l \times \boldsymbol{k}_s| \ . \tag{B.14}$$

We also need the coefficients for $L = 3$

$$\begin{aligned} \boldsymbol{a}^{(3)} &= \varepsilon_1 \boldsymbol{k}_2 \times \boldsymbol{k}_3 + \varepsilon_2 \boldsymbol{k}_3 \times \boldsymbol{k}_1 + \varepsilon_3 \boldsymbol{k}_1 \times \boldsymbol{k}_2 \,, \\ \boldsymbol{b}^{(3)} &= \lambda_1 \boldsymbol{k}_2 \times \boldsymbol{k}_3 + \lambda_2 \boldsymbol{k}_3 \times \boldsymbol{k}_1 + \lambda_3 \boldsymbol{k}_1 \times \boldsymbol{k}_2 \,, \qquad \text{(B.15)} \\ w^{(3)} &= \boldsymbol{k}_1 (\boldsymbol{k}_2 \times \boldsymbol{k}_3) \,. \qquad \text{(B.16)} \end{aligned}$$

One can easily prove

$$\begin{aligned} \boldsymbol{a}^{(2)}_{ls} \times \boldsymbol{b}^{(2)}_{ls} &= \varrho_{ls} (\boldsymbol{k}_l \times \boldsymbol{k}_s) \,, \qquad \text{(B.17)} \\ \boldsymbol{a}^{(3)} \times \boldsymbol{b}^{(3)} &= w^{(3)} \left(\varrho_{12} \boldsymbol{k}_3 - \varrho_{13} \boldsymbol{k}_2 + \varrho_{23} \boldsymbol{k}_1 \right) \,, \qquad \text{(B.18)} \end{aligned}$$

where

$$\varrho_{ls} = \lambda_s \, \varepsilon_l - \lambda_l \varepsilon_s = \lambda_l \lambda_s (E_{0l} - E_{0s}) \,. \qquad \text{(B.19)}$$

Then, using

$$\begin{aligned} \boldsymbol{a}^{(3)} &= \varepsilon_t (\boldsymbol{k}_l \times \boldsymbol{k}_s) - \boldsymbol{a}^{(2)}_{ls} \times \boldsymbol{k}_t \,, \\ \boldsymbol{b}^{(3)} &= \lambda_t (\boldsymbol{k}_l \times \boldsymbol{k}_s) - \boldsymbol{b}^{(2)}_{ls} \times \boldsymbol{k}_t \qquad \text{(B.20)} \end{aligned}$$

and Eq. (B.7), we find

$$\varPhi_{123}(E) \, (\boldsymbol{k}_l \times \boldsymbol{k}_s)^2 = \left\{ \left[\boldsymbol{b}^{(3)} E - \boldsymbol{a}^{(3)} \right] (\boldsymbol{k}_l \times \boldsymbol{k}_s) \right\}^2 + \left[w^{(3)} \right]^2 \varPhi_{ls}(E) \qquad \text{(B.21)}$$

from which one can see that $\varPhi_{123}(E) \geq 0$ if $\varPhi_{ls}(E) \geq 0$.

Similar expressions are also valid for $\varPhi_{ls}(E)$. The relation

$$\varPhi_{ls}(E) k_s^2 = \left\{ \left[\boldsymbol{b}^{(2)} E - \boldsymbol{a}^{(2)} \right] \boldsymbol{k}_s \right\}^2 - \left[w^{(2)} \right]^2 \varPhi_l(E) \qquad \text{(B.22)}$$

results in non-negativity of $\varPhi_{ls}(E)$ if $\varPhi_l(E) \leq 0$.

The generalized trinomial $\varOmega_L$ allows one to deal in the same way with the constraints on k_l and integrands containing $\Theta(\pm\varOmega_L)$ or $\varOmega_L^{-1}$ like $I^{\pm}_{H,K,L}$.

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