SOLUTIONS OF THE BOSONIC MASTER-FIELD EQUATION FROM A SUPERSYMMETRIC MATRIX MODEL*

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It has been argued that the bosonic large-N master field of the IIB matrix model can give rise to an emergent classical spacetime. In a recent paper, we have obtained solutions of a *simplified* bosonic master-field equation from a related matrix model. In this simplified equation, the effects of dynamic fermions were removed. We now consider the *full* bosonic master-field equation from a related supersymmetric matrix model for dimensionality $D = 3$ and matrix size $N = 3$. In this last equation, the effects of dynamic fermions are included. With an explicit realization of the random constants entering this algebraic equation, we establish the existence of nontrivial solutions. The small matrix size, however, does not allow us to make a definitive statement as to the appearance of a diagonal/band-diagonal structure in the obtained matrices.

1. Introduction

The IIB matrix model [1, 2] has been proposed as a nonperturbative formulation of type-IIB superstring theory. Recently, the corresponding large-N bosonic master field [3, 4] has been suggested as the possible source of an emerging classical spacetime [5] (see also Ref. [6] for a follow-up paper on cosmology and Ref. [7] for a review).

The task, now, is to solve the relevant bosonic master-field equation. Preliminary results have shown that there may appear a bosonic master-field solution, whose matrices have an approximate diagonal/band-diagonal structure [8]. These results were, however, obtained from a *simplified* bosonic master-field equation with dynamical effects of the fermions removed altogether.

In the present paper, we will consider the *full* bosonic master-field equation with dynamical effects of the fermions included. There are, then, two crucial questions. First, does the full bosonic master-field equation still have solutions? Second, assuming the existence of a solution, do the fermion effects preserve the diagonal/band-diagonal structure found from the previous simplified equation?

It is well known that the fermions of the IIB matrix model give rise to a Pfaffian, which is extremely difficult to calculate symbolically [9–11]. As a first step towards answering the two questions of the previous paragraph, we consider a low dimensionality $D = 3$ and a small matrix size $N = 3$. Then, we can give a clear affirmative answer to the first question on the existence of solutions, but are not yet able to give a definitive answer to the second question on a possible diagonal/band-diagonal structure (even though, the $D = N = 3$ results are somewhat encouraging).

2. Supersymmetric matrix model

2.1. General case

Let us briefly review the IIB matrix model [1, 2], first allowing for a different number of spacetime dimensions than ten. We essentially take over the conventions and notation of Ref. [9], except that we write A^μ for the bosonic matrices, with a directional index μ running over $\{1, \ldots, D\}$. These bosonic matrices, as well as the fermionic matrices, are $N \times N$ traceless Hermitian matrices. The partition function for $D > 2$ and $N \geq 2$ is then defined as follows [1, 2, 9]:

$$
Z_{D,N}^F = \int \prod_{I=1}^{g} \prod_{\mu=1}^{D} \frac{\mathrm{d}A_\mu^I}{\sqrt{2\pi}} \, \mathrm{e}^{-S_{\mathrm{bos}}[A]}
$$

$$
\times \left(\int \prod_{I=1}^{g} \prod_{\alpha=1}^{\mathcal{N}} \mathrm{d}\Psi_\alpha^I \, \mathrm{e}^{-S_{\mathrm{ferm}}[A, \Psi]} \right)^F , \qquad (2.1a)
$$

$$
S_{\mathrm{bos}}[A] = -\frac{1}{2} \mathrm{Tr}\left(\left[A^\mu, A^\nu \right] \left[A^\mu, A^\nu \right] \right), \qquad (2.1b)
$$

$$
S_{\mathrm{ferm}}[A, \Psi] = -\mathrm{Tr}\left(\bar{\Psi}_\alpha \, \Gamma^\mu_{\alpha\beta} \left[A^\mu, \Psi_\beta \right] \right), \qquad (2.1c)
$$

$$
A_\mu = A_\mu^I T_I , \qquad \Psi_\alpha = \Psi_\alpha^I T_I , \qquad (2.1d)
$$

$$
\mathrm{Tr}\left(T_I \cdot T_J \right) = \frac{1}{2} \delta_{IJ} , \qquad (2.1e)
$$

$$g \equiv N^2 - 1 \,, \tag{2.1f}$$

$$\mathcal{N} \equiv 2^{[D/2]} \times \begin{cases} 1\,, & \text{for odd } D \,, \\ 1/2\,, & \text{for even } D \,, \end{cases} \tag{2.1g}$$

$$F \in \{0, 1\} \,, \tag{2.1h}$$

where repeated Greek indices are summed over (corresponding to an implicit Euclidean "metric") and F is an on/off parameter to include ($F = 1$) or exclude ($F = 0$) dynamic-fermion effects. The square bracket in the exponent of (2.1g) stands for the Entier/Floor function and the factor $1/2$ corresponds to the Weyl projection [for $D = 10$, there is also a reality (Majorana) condition on the fermions]. The commutators entering the action terms (2.1b) and (2.1c) are defined by $[X, Y] \equiv X \cdot Y - Y \cdot X$ for square matrices X and Y of equal dimension.

The matrix model (2.1) with $F = 1$ is supersymmetric for dimensionality

$$D = 3,\, 4,\, 6,\, 10 \,, \tag{2.2}$$

where the field transformations have, for example, been given by Eq. (2) in Ref. [9]. Expansions (2.1d), for real coefficients A_μ^I and Grassmannian coefficients Ψ_α^I, use the $N \times N$ traceless Hermitian SU(N) generators T_I with normalization (2.1e). Remark also that we have set the model length scale ℓ to unity, so that the coefficients A_μ^I and Ψ_α^I are dimensionless.

The Gaussian integrals over the Grassmann variables Ψ_α^I in (2.1a) can be performed analytically, so that the partition function reduces to a purely bosonic integral,

$$Z_{D,N}^F = \int \prod_{I=1}^{g} \prod_{\mu=1}^{D} \frac{\mathrm{d}A_\mu^I}{\sqrt{2\pi}} \left(\mathcal{P}_{D,N}[A] \right)^F \mathrm{e}^{-S_{\mathrm{bos}}[A]}$$

$$= \int \prod_{I=1}^{g} \prod_{\mu=1}^{D} \frac{\mathrm{d}A_\mu^I}{\sqrt{2\pi}} \, \mathrm{e}^{-S_{\mathrm{eff},\,D,\,N}^F[A]} \,, \tag{2.3a}$$

$$S_{\mathrm{eff},\,D,\,N}^F[A] = S_{\mathrm{bos}}[A] - F \log \mathcal{P}_{D,N}[A] \,. \tag{2.3b}$$

The Pfaffian $\mathcal{P}_{D,N}[A]$, which can be absorbed into the effective action $S_{\mathrm{eff}}[A]$, is given explicitly by a sum over permutations [9] or by a sum involving the Levi-Civita symbol. Concretely, the Pfaffian $\mathcal{P}_{D,N}[A]$ is a homogenous polynomial in the bosonic coefficients A_μ^I, where the order K, for the special dimensions (2.2), is given by the following expression [9, 10]:

$$K = (D - 2)\left(N^2 - 1\right) \,. \tag{2.4}$$

An explicit example of the Pfaffian will be given in Sec. 2.2. Further discussion of the Pfaffian appears in, *e.g.*, Refs. [9–11].

The partition function of the genuine IIB matrix model [1, 2] has the following parameters in (2.1):

$$\{D,\, N,\, F\} = \{10,\, \infty,\, 1\}\,, \tag{2.5}$$

and there is a second supersymmetry transformation in addition to the one mentioned below (2.2). The large-N limit may require further discussion, but, at this moment, we just consider N to be large and finite (for exploratory numerical results, see, *e.g.*, Refs. [12–14] and references therein).

2.2. Particular case

Now consider the matrix model (2.1) with the particular parameters

$$\{D,\, N,\, F\} = \{3,\, 3,\, 1\}\,, \tag{2.6}$$

for which the model has a supersymmetry invariance, as mentioned in the second paragraph of Sec. 2.1. The eight generators T_I are proportional to the 3×3 Gell-Mann matrices λ_I used in elementary particle physics. Remarkably, there is an explicit result for the Pfaffian [9]:

$$\begin{aligned}
\mathcal{P}_{3,3}[A] =\ & -\frac{3}{4} \operatorname{Tr}\left([A^\mu, A^\nu]\{A^\rho, A^\sigma\}\right) \operatorname{Tr}\left([A^\mu, A^\nu]\{A^\rho, A^\sigma\}\right) \\
& + \frac{6}{5} \operatorname{Tr}\left(A^\mu [A^\nu, A^\rho]\right) \operatorname{Tr}\left(A^\mu [\{A^\nu, A^\sigma\}, \{A^\rho, A^\sigma\}]\right),
\end{aligned} \tag{2.7}$$

which corresponds to a homogenous eighth-order polynomial in the bosonic coefficients A_μ^I. Expression (2.7) contains, in addition to commutators, also anticommutators, defined by $\{X,\, Y\} \equiv X \cdot Y + Y \cdot X$ for square matrices X and Y of equal dimension.

Using expression (2.7) for the Pfaffian, the effective action is given by

$$S^F_{\mathrm{eff},\,3,\,3}[A] = S_{\mathrm{bos},\,3,\,3}[A] - F \,\log \mathcal{P}_{3,3}[A]\,, \tag{2.8}$$

where $F = 0$ removes the effects of dynamic fermions and $F = 1$ includes them. In a previous paper [8], we had simply removed the fermion term in the effective action but, here, we intend to study it carefully. Incidentally, the integrals in (2.3), for parameters (2.6), may have convergence problems [11], but our focus will be solely on a type of saddle-point equation obtained from the effective action (2.8).

3. Bosonic master field

3.1. Bosonic observables and master field

As our main interest is in the possible recovery of an emerging classical spacetime [5–7], we primarily consider the bosonic observable

$$w^{\mu_1 \cdots \mu_m} \equiv \frac{1}{N} \, \mathrm{Tr} \left(A^{\mu_1} \cdots A^{\mu_m} \right), \tag{3.1}$$

where the $1/N$ prefactor on the right-hand side is only for convenience. Now, arbitrary strings of these bosonic observables have expectation values

$$\langle w^{\mu_1 \cdots \mu_m} \, w^{\nu_1 \cdots \nu_n} \, \cdots \, w^{\omega_1 \cdots \omega_z} \rangle^F_{D, N}$$
$$= \frac{1}{Z^F_{D, N}} \int \mathrm{d}A \left(w^{\mu_1 \cdots \mu_m} \, w^{\nu_1 \cdots \nu_n} \, \cdots \, w^{\omega_1 \cdots \omega_z} \right) \mathrm{e}^{- S^F_{\mathrm{eff}, D, N}}, \tag{3.2}$$

where "$\mathrm{d}A$" is a short-hand notation of the measure appearing in (2.1a) and $Z^F_{D, N}$ is defined by the integral (2.3).

These expectation values, at large values of N, have a remarkable factorization property:

$$\langle w^{\mu_1 \cdots \mu_m} \, w^{\nu_1 \cdots \nu_n} \, \cdots \, w^{\omega_1 \cdots \omega_z} \rangle^F_{D, N} \overset{N}{=} \langle w^{\mu_1 \cdots \mu_m} \rangle^F_{D, N} \, \langle w^{\nu_1 \cdots \nu_n} \rangle^F_{D, N}$$
$$\cdots \langle w^{\omega_1 \cdots \omega_z} \rangle^F_{D, N}, \tag{3.3}$$

where the equality holds to leading order in N. According to Witten [3], the factorization (3.3) implies that the path integrals (3.2) are saturated by a single configuration, which has been called the "master field" and whose matrices will be denoted by $\widehat{A}^\mu$. To leading order in N, the expectation values (3.2) are then given by the bosonic master-field matrices $\widehat{A}^\mu$ in the following way:

$$\langle w^{\mu_1 \cdots \mu_m} \, w^{\nu_1 \cdots \nu_n} \, \cdots \, w^{\omega_1 \cdots \omega_z} \rangle^F_{D, N} \overset{N}{=} \widehat{w}^{\mu_1 \cdots \mu_m} \, \widehat{w}^{\nu_1 \cdots \nu_n} \, \cdots \, \widehat{w}^{\omega_1 \cdots \omega_z}, \tag{3.4a}$$

$$\widehat{w}^{\mu_1 \cdots \mu_m} \equiv \frac{1}{N} \, \mathrm{Tr} \left(\widehat{A}^{\mu_1} \cdots \widehat{A}^{\mu_m} \right), \tag{3.4b}$$

where the master-field matrices $\widehat{A}^\mu$ have an implicit dependence on the model parameters D, N, and F. See Refs. [5, 7] for further discussion and references.

3.2. Bosonic master-field equation

Introducing N random constants $\widehat{p}_k$ and the $N \times N$ diagonal matrix

$$D_{(\widehat{p})}(\tau) \equiv \mathrm{diag}\left(e^{i\,\widehat{p}_1\,\tau},\ \ldots,\ e^{i\,\widehat{p}_N\,\tau}\right), \qquad (3.5)$$

the bosonic master-field matrices take the following "quenched" form [4, 5]:

$$\widehat{A}^{\,\rho}\ =\ D_{(\widehat{p})}\left(\tau_{\mathrm{eq}}\right)\cdot\widehat{a}^{\,\rho}\cdot D_{(\widehat{p})}^{-1}\left(\tau_{\mathrm{eq}}\right), \qquad (3.6a)$$

for a sufficiently large value of τ_{eq} (see below for further explanations). The τ-independent matrix $\widehat{a}^{\,\rho}$ in (3.6a) is determined by the algebraic equation [5]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[D_{(\widehat{p})}(\tau)\cdot\widehat{a}^{\,\rho}\cdot D_{(\widehat{p})}^{-1}(\tau)\right]_{\tau=0}\ =\ -\frac{\delta S_{\mathrm{eff},\,D,\,N}^{F}\left[\widehat{a}\right]}{\delta\,\widehat{a}_{\rho}}\ +\widehat{\eta}^{\,\rho}. \quad (3.6b)$$

All matrix indices have been suppressed in the three equations above [the notation with a functional derivative on the right-hand side of (3.6b) is purely symbolic] and $S_{\mathrm{eff},\,D,\,N}^{F}[\widehat{a}]$ is given by (2.3b) or by (2.8) for the particular case considered. The left-hand side of (3.6b), with matrix indices $\{k,\,l\}$ added, reads $i\,(\widehat{p}_k-\widehat{p}_l)\,\widehat{a}^{\,\rho}_{\,kl}$ and the equation is manifestly algebraic (see below for further comments on its basic structure).

The algebraic equation (3.6b) has two types of constants: the master momenta $\widehat{p}_k$ (uniform random numbers) and the master noise matrices $\widehat{\eta}^{\,\rho}_{\,kl}$ (Gaussian random numbers). Very briefly, the meaning of these two types of random numbers is as follows. The dimensionless time τ is the fictitious Langevin time of stochastic quantization, with a Gaussian noise term η in the differential equation (its basic structure is as follows: $\mathrm{d}A/\mathrm{d}\tau = -\delta S_{\mathrm{eff}}/\delta A + \eta$). The τ evolution drives the system to equilibrium at $\tau = \tau_{\mathrm{eq}}$ and the resulting configuration $A^{\rho}(\tau_{\mathrm{eq}})$ corresponds to the master field $\widehat{A}^{\,\rho}$. For large N, the τ-dependence of the bosonic variable $A^{\rho}_{\,kl}(\tau)$ and the Langevin noise matrix $\eta^{\rho}_{\,kl}(\tau)$ is quenched by use of the uniform random momenta $\widehat{p}_k$. Note that all master variables and master constants are denoted by a caret. See Refs. [4, 5] for further discussion and references.

We now have three technical remarks on the obtained master-field equations (3.6a) and (3.6b). First, the algebraic equation (3.6b) for $F = 0$ reproduces the simplified equation (3.1) of Ref. [8], up to an irrelevant minus sign of the double-commutator there. Second, the derivative term on the right-hand side of (3.6b) for $F = 1$ involves not only the derivative of the Pfaffian (which has been studied in, *e.g.*, Ref. [10]) but also the inverse of the Pfaffian. Third, as the Pfaffian is a Kth order polynomial, denoted symbolically by $P_K[A]$ with K given by (2.4), the basic structure of the

algebraic equation (3.6b) is as follows:

$$P_1^{(\widehat{p})}\left[\widehat{a}\right] \;=\; P_3\left[\widehat{a}\right] + F\,\frac{P_{K-1}\left[\widehat{a}\right]}{P_K\left[\widehat{a}\right]} + P_0^{(\widehat{\eta})}\left[\widehat{a}\right]\,, \tag{3.7}$$

where only the on/off constant F is shown explicitly and where the suffixes on P_1 and P_0 indicate their respective dependence on the master momenta $\widehat{p}_k$ and the master noise $\widehat{\eta}_{kl}^{\rho}$. If we multiply (3.7) by $P_K\left[\widehat{a}\right]$, we get a polynomial equation of the order of $K+3$.

In order to obtain the component equations [labeled by an index I running over $1,\ \ldots\ ,(N^2-1)$ and an index ρ running over $1,\ \ldots\ ,D$], we matrix multiply (3.6b) by T_I, take the trace, and multiply the result by two. There are then $D\,g = D\left(N^2-1\right)$ coupled algebraic equations for an equal number of unknowns $\{\widehat{a}_1^{\,1},\ \ldots\ ,\widehat{a}_D^{\,g}\}$. It appears impossible to obtain a general solution of these algebraic equations. We will look for solutions of these coupled algebraic equations with an explicit realization of the random constants $\widehat{p}_k$ and $\widehat{\eta}_{kl}^{\,\rho}$. This is still a formidable problem for large values of N. Only for very small values of N are we, at this moment, able to get an explicit result.

4. Solutions for $D=3$ and $N=3$

4.1. Setup and method

We will now obtain, for the particular case (2.6), several solutions of the algebraic equation (3.6b), which corresponds to 24 real algebraic equations for 24 real unknowns $\{\widehat{a}_1^{\,1},\ \ldots\ ,\widehat{a}_3^{\,8}\}$.

For the constants entering these 24 algebraic equations, we take pseudorandom rational numbers with a range $[-1/2,\,1/2]$ for the master momenta $\widehat{p}_k$ and pseudorandom rational numbers with a range $[-1,\,1]$ for the master noise coefficients $\widehat{\eta}_{\,\rho}^{\,I}$. Specifically, we restrict to rational numbers of the form of $n/1000$, for $n \in \mathbb{Z}$, and take the $\widehat{p}_k$ numbers from a uniform distribution with a range $[-1/2,\,1/2]$ and the $\widehat{\eta}_{\,\rho}^{\,I}$ numbers from a truncated Gaussian distribution (with spread $\sigma = 2$ and cut-off value $x_{\mathrm{trunc}} = 1$ in the notation of Sec. III B of Ref. [8]; ultimately we must take $x_{\mathrm{trunc}} \gg \sigma$).

Explicitly, we use the following realization (labeled α) of the pseudorandom constants:

$$\widehat{p}_{\;\alpha\text{-realization}} = \left\{-\frac{27}{100},\,\frac{257}{1000},\,\frac{121}{1000}\right\}\,, \tag{4.1a}$$

$$\widehat{\eta}^{\,1}_{\alpha\text{-realization}} =$$

$$
\begin{pmatrix}
\dfrac{547}{2000} + \dfrac{13\sqrt{3}}{400} & \dfrac{36}{125} - \dfrac{38\,i}{125} & \dfrac{103}{250} - \dfrac{131\,i}{2000} \\[3mm]
\dfrac{36}{125} + \dfrac{38\,i}{125} & -\dfrac{547}{2000} + \dfrac{13\sqrt{3}}{400} & -\dfrac{247}{2000} + \dfrac{41\,i}{250} \\[3mm]
\dfrac{103}{250} + \dfrac{131\,i}{2000} & -\dfrac{247}{2000} - \dfrac{41\,i}{250} & \dfrac{-13\sqrt{3}}{200}
\end{pmatrix}, \qquad (4.1b)
$$

$$\widehat{\eta}^{\,2}_{\alpha\text{-realization}} =$$

$$
\begin{pmatrix}
\dfrac{319}{2000} - \dfrac{467}{2000\sqrt{3}} & -\dfrac{921}{2000} + \dfrac{43\,i}{400} & \dfrac{163}{1000} + \dfrac{97\,i}{400} \\[3mm]
-\dfrac{921}{2000} - \dfrac{43\,i}{400} & -\dfrac{319}{2000} - \dfrac{467}{2000\sqrt{3}} & \dfrac{951}{2000} + \dfrac{419\,i}{2000} \\[3mm]
\dfrac{163}{1000} - \dfrac{97\,i}{400} & \dfrac{951}{2000} - \dfrac{419\,i}{2000} & \dfrac{467}{1000\sqrt{3}}
\end{pmatrix}, \quad (4.1c)
$$

$$\widehat{\eta}^{\,3}_{\alpha\text{-realization}} =$$

$$
\begin{pmatrix}
\dfrac{28}{125} + \dfrac{989}{2000\sqrt{3}} & -\dfrac{419}{1000} + \dfrac{27\,i}{1000} & \dfrac{169}{500} + \dfrac{13\,i}{40} \\[3mm]
-\dfrac{419}{1000} - \dfrac{27\,i}{1000} & -\dfrac{28}{125} + \dfrac{989}{2000\sqrt{3}} & \dfrac{219}{500} - \dfrac{241\,i}{1000} \\[3mm]
\dfrac{169}{500} - \dfrac{13\,i}{40} & \dfrac{219}{500} + \dfrac{241\,i}{1000} & \dfrac{-989}{1000\sqrt{3}}
\end{pmatrix}. \quad (4.1d)
$$

The tracelessness of the $\widehat{\eta}^{\,\rho}$ matrices in (4.1) is manifest. Other realizations (labeled β, γ, ...) have given similar results.

The 3 solution matrices $\widehat{a}^{\,\rho}_{\alpha\text{-sol}}$ are determined by 24 coefficients $\big(\widehat{a}^{\,\rho}_{\alpha\text{-sol}}\big)^I$. Before we present these coefficients, which are obtained from the 24 algebraic equations mentioned above, let us briefly describe the method used. Strictly speaking, it does not matter how the 24 real numbers $\big(\widehat{a}^{\,\rho}_{\alpha\text{-sol}}\big)^I$ are obtained, as long as they solve the 24 algebraic equations. Here, we obtain these 24 real numbers with the numerical minimization routine FindMinimum from Mathematica 12.1 (*cf.* Ref. [15]). The minimization operates on a penalty function, which consists of a sum of 24 squares, each square containing one

of the real components of the algebraic equation without further overall numerical factor. (This penalty function has a size of approximately 29 MB, as further simplifications are hard to obtain.) For all calculations of the present paper, we use a 36-digit working precision. The accuracy of the obtained 24 real numbers $\left(\widehat{a}^{\,\rho}_{\alpha\text{-sol}}\right)^I$ can, in principle, be increased arbitrarily. Given the exact (pseudorandom) rational constants $\widehat{p}_k$ and $\widehat{\eta}^{\,I}_\rho$ from (4.1), the obtained matrices may, therefore, be called "quasi-exact."

4.2. Solution without dynamic fermions $(F = 0)$

We, first, get a solution of the $D = N = 3$ algebraic master-field equation (3.6b), where dynamic-fermion effects have been excluded by setting $F = 0$ in the effective action (2.8). Then, the resulting 24 coupled algebraic equations, with constants (4.1), have the following solution (for display and readability reasons, we split each matrix into the sum of a matrix with real entries and a matrix with imaginary entries):

$$
\widehat{a}^{\,1}_{\alpha\text{-sol}}\Big|^{(F=0)} = \begin{pmatrix} 0.186159 & 0.073147 & 0.562726 \\ 0.073147 & -0.281384 & -0.393265 \\ 0.562726 & -0.393265 & 0.0952246 \end{pmatrix}
$$

$$
+ \begin{pmatrix} 0 & 0.401829\,i & -0.217741\,i \\ -0.401829\,i & 0 & -0.100372\,i \\ 0.217741\,i & 0.100372\,i & 0 \end{pmatrix}, \quad (4.2a)
$$

$$
\widehat{a}^{\,2}_{\alpha\text{-sol}}\Big|^{(F=0)} = \begin{pmatrix} -0.194103 & 0.008581 & -0.580188 \\ 0.008581 & 0.131909 & 0.546198 \\ -0.580188 & 0.546198 & 0.0621938 \end{pmatrix}
$$

$$
+ \begin{pmatrix} 0 & -0.360539\,i & 0.482773\,i \\ 0.360539\,i & 0 & 0.389392\,i \\ -0.482773\,i & -0.389392\,i & 0 \end{pmatrix}, \quad (4.2b)
$$

$$\left.\widehat{a}_{\alpha\text{-sol}}^{\ 3}\right|^{(F=0)} = \begin{pmatrix} 0.130861 & -0.201837 & -0.1076245 \\ -0.201837 & 0.304673 & 0.277152 \\ -0.1076245 & 0.277152 & -0.435534 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -0.581365\,i & -0.0760187\,i \\ 0.581365\,i & 0 & -0.281138\,i \\ 0.0760187\,i & 0.281138\,i & 0 \end{pmatrix}, \quad (4.2c)$$

where up to 6 significant digits are shown (a 36-digit working precision is used). The apparent violation of tracelessness is solely due to roundoff errors and is absent in the true solution, which is given by an expansion in terms of traceless generators, as in (2.1d).

A cursory inspection of the matrices (4.2) shows that the far-off-diagonal entries [1, 3] and [3, 1] are not all really small. Following the discussion of our previous paper [8], we will consider the absolute values of the entries in the $\rho = 1$ matrix (4.2a), calculate the average band-diagonal value from 2+3+2 entries, the average off-band-diagonal value from 1+1 entries, and the ratio R_1 of the average band-diagonal value over the average off-band-diagonal value. For the $\rho = 2$ and $\rho = 3$ matrices, we follow the same procedure and get the ratios R_2 and R_3. In order to avoid any confusion, we give the general definition of this ratio R for a symmetric 3×3 matrix M with nonnegative entries $m[i, j]$:

$$R_M \equiv \frac{1}{7}\left(\sum_{i=1}^{3} m[i, i] + 2\sum_{j=1}^{2} m[j, j+1]\right)\frac{1}{m[1, 3]}, \quad (4.3)$$

where the symmetry of M has been used to simplify the expression.

From the matrices (4.2), we then get the following ratio values:

$$\left\{R_1,\, R_2,\, R_3\right\}_{\text{Abs}\left[\widehat{a}_{\alpha\text{-sol}}^{\ \rho}\right]}^{(F=0)} = \{0.519,\, 0.464,\, 3.13\}, \quad (4.4)$$

where two ratios lie below unity and one above. The somewhat large value of the last ratio in (4.4) is due to the rather small [1, 3] and [3, 1] entries in the matrix (4.2c).

Next, we diagonalize one of the matrices, while ordering the eigenvalues, and look at the other two matrices to see if they have a band-diagonal structure (even for the very small value of N we are considering). If we diagonalize and order $\widehat{a}_{\alpha\text{-sol}}^{\ 1}$ (the new matrices are denoted by a prime), we get

$$\widehat{a}'^{1\ (F=0)}_{\alpha\text{-sol}} = S_1 \cdot \widehat{a}^{1\ (F=0)}_{\alpha\text{-sol}} \cdot S_1^{-1}$$

$$= \mathrm{diag}\big(-0.760473,\ -0.188386,\ 0.948859\big), \tag{4.5a}$$

$$\widehat{a}'^{2\ (F=0)}_{\alpha\text{-sol}} = S_1 \cdot \widehat{a}^{2\ (F=0)}_{\alpha\text{-sol}} \cdot S_1^{-1}$$

$$= \begin{pmatrix} 0.666791 & -0.102279 & -0.0133662 \\ -0.102279 & 0.516194 & 0.066637 \\ -0.0133662 & 0.066637 & -1.18298 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -0.255347\,i & 0.0232436\,i \\ 0.255347\,i & 0 & 0.207603\,i \\ -0.0232436\,i & -0.207603\,i & 0 \end{pmatrix}, \tag{4.5b}$$

$$\widehat{a}'^{3\ (F=0)}_{\alpha\text{-sol}} = S_1 \cdot \widehat{a}^{3\ (F=0)}_{\alpha\text{-sol}} \cdot S_1^{-1}$$

$$= \begin{pmatrix} 0.727690 & 0.273460 & -0.0776651 \\ 0.273460 & -0.347068 & 0.136913 \\ -0.0776651 & 0.136913 & -0.380622 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0.440485\,i & 0.0239532\,i \\ -0.440485\,i & 0 & 0.100836\,i \\ -0.0239532\,i & -0.100836\,i & 0 \end{pmatrix}. \tag{4.5c}$$

The last two matrices in (4.5) have a rather clear band-diagonal structure. This can, again, be quantified by the ratios R_ρ of the average band-diagonal value over the average off-band-diagonal value,

$$\left\{ R_1,\ R_2,\ R_3 \right\}^{(F=0)}_{\mathrm{Abs}\big[\widehat{a}'^{\rho}_{\alpha\text{-sol}}\big]} = \left\{ \infty,\ 17.9,\ 4.98 \right\}, \tag{4.6}$$

where the first ratio is simply infinite because the $[1, 3]$ entry of the matrix (4.5a) vanishes. All three ratios from (4.6) are larger than 1, just as seen in our previous results [8].

4.3. Solutions with dynamic fermions $(F = 1)$

We, next, get solutions of the $D = N = 3$ algebraic master-field equation (3.6b), where dynamic-fermion effects have been included by setting $F = 1$ in the effective action (2.8). The resulting 24 coupled algebraic equations, with constants (4.1), have then the following two solutions.

4.3.1. First solution

Starting the minimization procedure from the configuration (4.2) obtained in Sec. 4.2, we find

$$
\widehat{a}^{\;1}_{\;\alpha\text{-sol}}\Big|^{(F=1)} = \begin{pmatrix} 0.481805 & 0.325813 & 0.384395 \\ 0.325813 & -0.682181 & -0.183233 \\ 0.384395 & -0.183233 & 0.200375 \end{pmatrix}
$$

$$
+ \begin{pmatrix} 0 & 0.576164\,i & -0.050542\,i \\ -0.576164\,i & 0 & 0.593735\,i \\ 0.050542\,i & -0.593735\,i & 0 \end{pmatrix} , \quad (4.7a)
$$

$$
\widehat{a}^{\;2}_{\;\alpha\text{-sol}}\Big|^{(F=1)} = \begin{pmatrix} -0.222436 & -0.0458796 & 0.048921 \\ -0.0458796 & 0.000171969 & 0.287881 \\ 0.048921 & 0.287881 & 0.222264 \end{pmatrix}
$$

$$
+ \begin{pmatrix} 0 & -0.1004043\,i & 0.503341\,i \\ 0.1004043\,i & 0 & 0.093830\,i \\ -0.503341\,i & -0.093830\,i & 0 \end{pmatrix} , \quad (4.7b)
$$

$$
\widehat{a}^{\;3}_{\;\alpha\text{-sol}}\Big|^{(F=1)} = \begin{pmatrix} 0.0389571 & -0.304651 & -0.194354 \\ -0.304651 & 0.858818 & 0.777316 \\ -0.194354 & 0.777316 & -0.897775 \end{pmatrix}
$$

$$
+ \begin{pmatrix} 0 & -0.920567\,i & 0.239810\,i \\ 0.920567\,i & 0 & -0.977785\,i \\ -0.239810\,i & 0.977785\,i & 0 \end{pmatrix} , \quad (4.7c)
$$

where up to 6 significant digits are shown (as mentioned before, the apparent violation of tracelessness is due to roundoff errors). The most important result of the present paper is the fact that we were able to find a nontrivial solution (4.7) for the case of dynamic fermions (another solution will be given in Sec. 4.3.2).

There is perhaps a slight resemblance between the $F = 0$ matrices (4.2) and the $F = 1$ matrices (4.7), if, for example, the signs of the diagonal elements are considered. Anyway, it is clear that the far-off-diagonal entries [1, 3] and [3, 1] of the matrices (4.7) are not really small. Again, we will consider the absolute values of the entries in the matrices (4.7) and calculate, for each matrix, the ratio R of the average band-diagonal value over the average off-band-diagonal value, according to the general definition (4.3). We then get the following ratio values:

$$\left\{R_1,\, R_2,\, R_3\right\}^{(F=1)}_{\mathrm{Abs}\left[\widehat{a}\,^{\rho}_{\alpha\text{-sol}}\right]} = \{1.45,\, 0.359,\, 2.88\}\,, \tag{4.8}$$

where two ratios lie above unity and one below.

Next, we diagonalize and order $\widehat{a}\,^{1\,(F=1)}_{\alpha\text{-sol}}$ (the new matrices are denoted by a prime) and get

$$\widehat{a}\,'^{1\,(F=1)}_{\alpha\text{-sol}} = \widetilde{S}_1 \cdot \widehat{a}\,^{1\,(F=1)}_{\alpha\text{-sol}} \cdot \widetilde{S}_1^{-1}$$
$$= \mathrm{diag}\!\left(-1.32720,\, 0.514388,\, 0.812811\right), \tag{4.9a}$$

$$\widehat{a}\,'^{2\,(F=1)}_{\alpha\text{-sol}} = \widetilde{S}_1 \cdot \widehat{a}\,^{2\,(F=1)}_{\alpha\text{-sol}} \cdot \widetilde{S}_1^{-1}$$
$$= \begin{pmatrix} 0.0919594 & 0.1408718 & -0.177064 \\ 0.1408718 & 0.0633399 & 0.179233 \\ -0.17706 & 0.179233 & -0.155299 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & -0.0118767\,i & -0.134760\,i \\ 0.0118767\,i & 0 & -0.537712\,i \\ 0.134760\,i & 0.537712\,i & 0 \end{pmatrix}, \tag{4.9b}$$

$$\widehat{a}'^{3\,(F=1)}_{\alpha\text{-sol}} \;=\; \widetilde{S}_1 \cdot \widehat{a}^{3\,(F=1)}_{\alpha\text{-sol}} \cdot \widetilde{S}_1^{-1}$$

$$= \begin{pmatrix} 1.91375 & 0.245877 & 0.022834 \\ 0.245877 & -1.57339 & 0.102361 \\ 0.022834 & 0.102361 & -0.340362 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0.219132\,i & 0.184308\,i \\ -0.219132\,i & 0 & -0.296644\,i \\ -0.184308\,i & 0.296644\,i & 0 \end{pmatrix}. \qquad (4.9c)$$

The last two matrices in (4.9) have a mild band-diagonal structure, quantified by the following ratios R_ρ of the average band-diagonal value over the average off-band-diagonal value:

$$\left\{ R_1,\, R_2,\, R_3 \right\}^{(F=1)}_{\mathrm{Abs}\left[\widehat{a}'^{\,\rho}_{\alpha\text{-sol}}\right]} = \left\{ \infty,\, 1.11,\, 3.93 \right\}. \qquad (4.10)$$

All three values in (4.10) lie above unity, but the second not by much.

4.3.2. Second solution

With a different start configuration, we obtain another dynamic-fermion solution (denoted by an underline):

$$\widehat{\underline{a}}^{1}_{\alpha\text{-sol}}\Big|^{(F=1)} = \begin{pmatrix} 0.125763 & 0.412706 & 0.233008 \\ 0.412706 & -0.817158 & 0.126363 \\ 0.233008 & 0.126363 & 0.691395 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -0.424807\,i & -0.106924\,i \\ 0.424807\,i & 0 & -0.774279\,i \\ 0.106924\,i & 0.774279\,i & 0 \end{pmatrix}, \quad (4.11a)$$

$$\widehat{\underline{a}}^{2}_{\alpha\text{-sol}}\Big|^{(F=1)} = \begin{pmatrix} -0.0889768 & -0.256429 & 0.164899 \\ -0.256429 & -0.299159 & -0.078519 \\ 0.164899 & -0.078519 & 0.388136 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -0.199471\,i & 0.170884\,i \\ 0.199471\,i & 0 & -0.153213\,i \\ -0.170884\,i & 0.153213\,i & 0 \end{pmatrix}, \quad (4.11b)$$

$$\widehat{\underline{a}}\,{}^{3}_{\alpha\text{-sol}}\Big|^{(F=1)} = \begin{pmatrix} 0.823634 & -0.164815 & -0.353134 \\ -0.164815 & -0.365779 & -0.158931 \\ -0.353134 & -0.158931 & -0.457855 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -0.354320\,i & 0.320626\,i \\ 0.354320\,i & 0 & -0.293612\,i \\ -0.320626\,i & 0.293612\,i & 0 \end{pmatrix}, \quad (4.11c)$$

where up to 6 significant digits are shown (the apparent violation of trace-lessness is due to roundoff errors).

Inspection of the matrices (4.11) shows that the far-off-diagonal entries [1, 3] and [3, 1] are not really small. Considering the absolute values of the entries in the matrices (4.11) and calculating, for each matrix, the ratio R of the average band-diagonal value over the average off-band-diagonal value, according to the general definition (4.3), we get the following ratio values:

$$\left\{R_1,\ R_2,\ R_3\right\}^{(F=1)}_{\text{Abs}\left[\widehat{\underline{a}}\,^{\rho}_{\alpha\text{-sol}}\right]} = \{2.45,\ 1.06,\ 0.927\}\,, \quad (4.12)$$

where all three ratios are of the order of unity.

Next, we diagonalize and order $\widehat{\underline{a}}\,{}^{1}_{\alpha\text{-sol}}$ (the new matrices are denoted by a further prime) and get

$$\widehat{\underline{a}}\,'^{1\,(F=1)}_{\alpha\text{-sol}} = \underline{S}_1 \cdot \widehat{\underline{a}}\,{}^{1\,(F=1)}_{\alpha\text{-sol}} \cdot \underline{S}_1^{-1}$$
$$= \text{diag}\big(-1.37609,\ 0.249211,\ 1.12688\big)\,, \quad (4.13a)$$

$$\widehat{\underline{a}}\,'^{2\,(F=1)}_{\alpha\text{-sol}} = \underline{S}_1 \cdot \widehat{\underline{a}}\,{}^{2\,(F=1)}_{\alpha\text{-sol}} \cdot \underline{S}_1^{-1}$$

$$= \begin{pmatrix} -0.346970 & -0.204388 & 0.0067429 \\ -0.204388 & 0.172384 & 0.049531 \\ 0.0067429 & 0.049531 & 0.174586 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0.161013\,i & 0.1331020\,i \\ -0.161013\,i & 0 & 0.370677\,i \\ -0.1331020\,i & -0.370677\,i & 0 \end{pmatrix}, \quad (4.13b)$$

$$\widehat{\underline{a}}_{\alpha\text{-sol}}^{\prime\,3\,(F=1)} = \underline{S}_1 \cdot \widehat{\underline{a}}_{\alpha\text{-sol}}^{3\,(F=1)} \cdot \underline{S}_1^{-1}$$

$$= \begin{pmatrix} -0.491709 & -0.366405 & -0.0482257 \\ -0.366405 & 0.909724 & -0.428317 \\ -0.0482257 & -0.428317 & -0.418015 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -0.207799\,i & -0.0663827\,i \\ 0.207799\,i & 0 & -0.113580\,i \\ 0.0663827\,i & 0.113580\,i & 0 \end{pmatrix}. \quad (4.13\text{c})$$

The last two matrices in (4.13) have a mild band-diagonal structure. This can, again, be quantified by the ratios R_ρ of the average band-diagonal value over the average off-band-diagonal value:

$$\left\{ R_1,\, R_2,\, R_3 \right\}_{\mathrm{Abs}\left[\widehat{\underline{a}}_{\alpha\text{-sol}}^{\prime\,\rho}\right]}^{(F=1)} = \left\{ \infty,\, 2.10,\, 6.18 \right\}. \quad (4.14)$$

All three values in (4.14) lie above unity, the last two values being somewhat larger than those of the first $F = 1$ solution in (4.10).

5. Discussion

In the present article, we have obtained, for the first time, solutions of the full bosonic master-field equation from the supersymmetric matrix model (2.1) with dimensionality $D = 3$, matrix size $N = 3$, and fermion-inclusion parameter $F = 1$. These particular values of D and N are, of course, far below the values (2.5) needed for the IIB matrix model [1, 2]. Still, it is an important point of principle to have established the existence of solutions with full fermion dynamics, even for small values of D and N. Let us turn the argument around: assume that there were no such solutions for $D = 3$ and $N = 3$, then it would be hard to believe that there could be solutions for $D = 10$ and $N \gg 1$, as needed for the IIB matrix model. For completeness, we mention that the bosonic master-field equation (3.6b) is purely algebraic, with a basic structure clarified in (3.7) for K as given by (2.4) .

The matrices of the obtained bosonic $D = N = 3$ solutions feel the induction effects of the dynamic fermions ($F = 1$). The comparison between the solution without dynamic fermions (Sec. 4.2) and the corresponding solution with dynamic fermions (Sec. 4.3.1) suggests that the fermions reduce somewhat the strength of the diagonal/band-diagonal structure residing in the obtained matrices [specifically, compare the ratios in (4.6) with

those in (4.10)]. But there is, at least, one other dynamic-fermion solution (Sec. 4.3.2), which has a somewhat more pronounced diagonal/band-diagonal structure, as shown by the ratios in (4.14).

At this moment, we would like to proceed to larger values of (D, N), in order to ultimately reach the parameter values $D = 10$ and $N \gg 1$ of the genuine IIB matrix model [1, 2]. But this will be difficult. Already the matrix model with modest values $(D, N) = (4, 4)$ has a Pfaffian given as the determinant of a 30×30 complex matrix [9]. And the matrix model with $(D, N) = (10, 4)$ has a Pfaffian of a 240×240 skew-symmetric matrix. In both cases, it will be hard to evaluate the Pfaffian symbolically.

Instead of searching for a direct algebraic solution of the full bosonic master-field equation, an indirect numerical approach may be called for or even a reliable approximation method, if at all available.

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