# A SUPERSPACE DIRAC OPERATOR IN NCG AND THE "FACTORIZATION" OF THE ORDINARY DIRAC OPERATOR

Dominik Ciurla[†], Leszek Hadasz[‡], Thomas E. Williams[§]

Institute of Theoretical Physics, Jagiellonian University
S. Łojasiewicza 11, 30-348 Kraków, Poland

*(Received October 5, 2021; accepted October 27, 2021)*

We review a procedure of factorizing the Minkowski space Dirac equation over a suitable superspace, discuss its Euclidean space version and apply the worked out formalism in the case of an almost-commutative Dirac operator. The presented framework is an attempt to reconcile noncommutative geometry and supersymmetry.

DOI:10.5506/APhysPolB.52.1449

## 1. Introduction

In 1928 P.A.M. Dirac reported his now famous procedure for deriving an equation governing the quantum mechanical properties for particles with half-integer spin [1]. The process he pioneered may be essentially described as taking the "square root" of the Klein–Gordon equation.

The natural question whether this process is iterable was posed and solved by the use of superspace coordinates and their (first order) derivatives [2]. A series of papers followed, studying the free and interacting forms of the resulting equations acting on (super)spaces of superfields [3–7]. Suitably modified, Szwed's idea is also the first ingredient we use in the present note. The second one is the framework of noncommutative geometry [8].

Noncommutative geometry, pioneered by Connes in the 1980s and 1990s, is a profoundly deep branch of mathematics with roots and branches stretching in many directions. The mathematical depth of this subject belies its usefulness in theoretical physics, where it has found application in reproducing the Standard Model of particle physics coupled with gravity. Specifically, it is within a certain subclass of noncommutative geometries known as

† `dominik.ciurla@doctoral.uj.edu.pl`
‡ `leszek.hadasz@uj.edu.pl`
§ `t.e.williams@doctoral.uj.edu.pl`

almost-commutative (AC) geometries, in which such physical models may be described. This adaptation was pioneered in [9], but for the working physicist, we also recommend the presentation in [10]. Of central importance to this framework is the notion of a suitable Dirac operator. Given that the natural geometric setting for supersymmetry is superspace [11, 12], we expect that any Dirac operator which is claimed to govern the dynamics of particles in a supersymmetric model, should, in an essential way, take into account superspace coordinates and their derivatives.

One possibility would be to construct a superspace Dirac operator associated with the underlying superspace spin bundle. This would be a sort of "inside-out" approach where the fundamental space under consideration is a superspace exhibiting supersymmetry through infinitesimal global translations of its coordinates. Considered in this way, supersymmetry is an explicit, unavoidable property of the model. We postpone further discussion of this interpretation for future work.

Alternatively, one may consider an "outside-in" approach. This time, the basic ingredients are those of the usual AC-geometry approach for obtaining physical models from NCG, *i.e.* the underlying space is an ordinary Riemannian spin manifold and the Dirac operator is the spin connection acting fiberwise on square integrable sections of the spin bundle. This leads to an "on-shell" reconciliation of supersymmetry and noncommutative geometry: given the data of the (unfluctuated!) total space Dirac operator, we are led to a set of "equations of motion" (59) on a restricted space of superfield spinors, which are compatible with the supersymmetry transformations and whose form is determined by the assumed AC-geometry. We view their construction as the main result of the present note.

## 2. Factorization of the Dirac operator

### *2.1. Minkowski space — the Szwed approach*

Using two-component spinor notation (ofttimes referred to as Van der Waerden notation) and the chiral representation for the Dirac matrices (for the conventions see [13]), one can write the Dirac equation in 4-dimensional Minkowski space as

$$-\begin{pmatrix} i\bar{\sigma}^{\mu\,\dot{\alpha}\beta}\partial_\mu & m\delta^{\dot{\alpha}}_{\dot{\beta}} \\ m\delta^{\beta}_{\alpha} & i\sigma^{\mu}_{\alpha\dot{\beta}}\partial_\mu \end{pmatrix}\begin{pmatrix} \psi_\beta \\ \bar{\chi}^{\dot{\beta}} \end{pmatrix} \equiv \mathcal{D}\begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix} = 0\,. \tag{1}$$

Taking a "square root" of the Dirac operator corresponds to the construction of an operator $A$, which satisfies

$$A^\dagger A = \mathcal{D}\,. \tag{2}$$

If one requires $A$ to be a local operator and to contain space-time derivatives, then, since there is no second order derivative in the Dirac operator, one is compelled to assume that the coefficients of $\partial_\mu$ in $A$ are nilpotent. Therefore, one is lead to consider the operator $A$ as acting on a superspace with the coordinates $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$.

There are several first order differential operators which can be defined on this space. In particular, the spinorial ones,

$$\begin{aligned} D_\alpha &= \partial/\partial\theta^\alpha + i\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\,, \\ \bar{D}_{\dot{\alpha}} &= -\partial/\partial\bar{\theta}^{\dot{\alpha}} - i\theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu\,, \end{aligned} \tag{3}$$

satisfy an algebra with relations given by

$$\begin{aligned} \{D_\alpha, D_\beta\} &= \left\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\right\} = 0\,, \\ \left\{D_\alpha, \bar{D}_{\dot{\beta}}\right\} &= -2i\sigma^\mu{}_{\alpha\dot{\beta}}\partial_\mu\,. \end{aligned} \tag{4}$$

If we now define $2 \times 2$ matrices

$$A_{\beta\alpha} = \begin{pmatrix} D^\beta & -\bar{D}_{\dot{\beta}} \\ \bar{D}^{\dot{\alpha}} & D_\alpha \end{pmatrix}, \tag{5}$$

then

$$(A_{\alpha\beta})^\dagger A_{\beta\alpha} = \begin{pmatrix} \left\{D^\beta, \bar{D}^{\dot{\alpha}}\right\} & \bar{D}_{\dot{\beta}}\bar{D}^{\dot{\alpha}} + D^\beta D_\alpha \\ \bar{D}_{\dot{\beta}}\bar{D}^{\dot{\alpha}} + D^\beta D_\alpha & \left\{D_\alpha, \bar{D}_{\dot{\beta}}\right\} \end{pmatrix}. \tag{6}$$

In particular,

$$(A_{\alpha\alpha})^\dagger A_{\alpha\alpha} = -2\begin{pmatrix} i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu & M \\ M & i\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu \end{pmatrix} \tag{7}$$

with

$$M = -\frac{1}{4}\left(DD + \bar{D}\bar{D}\right) \equiv -\frac{1}{4}\left(\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}} + D^\alpha D_\alpha\right). \tag{8}$$

In effect, even if an operator $A$ satisfying (2) actually does not exist, the equality (7) was the motivation in [2, 5] for postulating the following set of equations as a "square root" of the Dirac equation:

$$D^\alpha\psi_\alpha - \bar{D}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = 0\,, \qquad \bar{D}^{\dot{\alpha}}\psi_\alpha + D_\alpha\bar{\chi}^{\dot{\alpha}} = 0\,, \tag{9}$$

in which the spinors $\psi_\alpha$ and $\bar{\chi}^{\dot{\alpha}}$ are considered to be functions of the superspace coordinates $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, and are subject to the additional constraint

$$\left(DD + \bar{D}\bar{D}\right)\psi_\alpha + 4m\psi_\alpha = \left(DD + \bar{D}\bar{D}\right)\bar{\chi}^{\dot{\alpha}} + 4m\bar{\chi}^{\dot{\alpha}} = 0\,. \tag{10}$$

The solution set of these equations turned out to be nonempty and interesting. In particular, a simple case in which $\psi_\alpha = \chi_\alpha$ corresponds to the Maxwell superfield [5].

### *2.2. 4d Euclidean space*

It is essential to the noncommutative methods, which we intend to employ in Section 3, that the "total-space" Dirac operator is Hermitian. Therefore, we proceed in a Riemannian signature and for simplicity choose to work in 4-dimensional Euclidean space.

In particular, in this setting, the Lorentz transformations are the 4-dimensional rotations characterized by the symmetry group SO(4). Their spin representation is given by the universal covering Lie group, $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$, and the corresponding Clifford algebra is isomorphic to the Lie algebra of infinitesimal generators, $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

After defining

$$\sigma^m \equiv (i\tau_1, i\tau_2, i\tau_3, \mathbf{1}_2) \quad \text{and} \quad \tilde{\sigma}^m \equiv (-i\tau_1, -i\tau_2, -i\tau_3, \mathbf{1}_2)\,, \tag{11}$$

where $\tau_i$ are the Pauli matrices, it is immediate to check that the Hermitian matrices

$$\gamma_{\mathrm{E}}^m \equiv \begin{pmatrix} 0 & \sigma^m \\ \tilde{\sigma}^m & 0 \end{pmatrix} \tag{12}$$

generate the Clifford algebra of 4-dimensional Euclidean space,

$$\{\gamma_{\mathrm{E}}^m, \gamma_{\mathrm{E}}^n\} = 2\delta^{mn}\mathbf{1}_4\,. \tag{13}$$

Furthermore, this algebra possesses a natural grading induced by the operator

$$\gamma_{\mathrm{E}}^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}\,. \tag{14}$$

The Euclidean Dirac operator has the form of

$$\mathcal{D} = i\gamma_{\mathrm{E}}^m\partial_m + m\mathbf{1}_4 = \begin{pmatrix} m\mathbf{1}_2 & i\sigma^m\partial_m \\ i\tilde{\sigma}^m\partial_m & m\mathbf{1}_2 \end{pmatrix} \tag{15}$$

and acts on a bispinor

$$\Psi = \begin{pmatrix} \psi \\ \tilde{\chi} \end{pmatrix}\,. \tag{16}$$

As for the spinorial indices, we declare

$$\begin{aligned} \psi &= (\psi_\alpha)\,, & \tilde{\chi} &= \left(\tilde{\chi}^{\dot{\alpha}}\right)\,, \\ \tilde{\sigma}^m &= \left(\tilde{\sigma}^{m\dot{\alpha}\alpha}\right)\,, & \sigma^m &= \left(\sigma^m_{\alpha\dot{\alpha}}\right) \end{aligned} \tag{17}$$

which allows us to present the Dirac equation as

$$\begin{aligned} i\tilde{\sigma}^{m\dot{\alpha}\alpha}\partial_m\psi_\alpha + m\tilde{\chi}^{\dot{\alpha}} &= 0\,, \\ i\sigma^m_{\alpha\dot{\alpha}}\partial_m\tilde{\chi}^{\dot{\alpha}} + m\psi_\alpha &= 0\,. \end{aligned} \tag{18}$$

Unlike the Minkowski case, the spinors $\psi$ and $\tilde{\chi}$ transform independently under the action of Spin(4). Indeed, if we parameterize a matrix $L \in \mathrm{SO}(4)$ as $L = \exp\omega$ (with $\omega_{mn} = -\omega_{nm}$), then

$$\psi'_\alpha(x) = M_\alpha{}^\beta \psi_\beta\left(L^{-1}x\right), \qquad \tilde{\chi}'^{\dot\alpha} = W^{\dot\alpha}{}_{\dot\beta}\tilde{\chi}^{\dot\beta}\left(L^{-1}x\right), \tag{19}$$

where

$$\begin{aligned} M(L) &= \exp\left(\tfrac{1}{8}\omega_{mn}(\sigma^m\tilde{\sigma}^n - \sigma^n\tilde{\sigma}^m)\right), \\ W(L) &= \exp\left(\tfrac{1}{8}\omega_{mn}(\tilde{\sigma}^m\sigma^n - \tilde{\sigma}^n\sigma^m)\right) \end{aligned} \tag{20}$$

are distinct operators, *i.e.* $M(L)$ depends on six independent parameters $\omega_{mn}$ only through a (three-parameter) combination $\sum_{k=1}^{3}\sum_{l=1}^{k-1}\epsilon_{jkl}\omega_{kl}+\omega_{j4}$, while $W(L)$ depends on $\omega_{mn}$ through $\sum_{k=1}^{3}\sum_{l=1}^{k-1}\epsilon_{jkl}\omega_{kl}-\omega_{j4}$.

In order to construct a relevant superspace, we introduce two constant (anticommuting) spinors $\xi_\alpha$ and $\tilde{\zeta}^{\dot\alpha}$. By construction, under the action of Spin(4)

$$\xi_\alpha \to M_\alpha{}^\beta\xi_\beta, \qquad \tilde{\zeta}^{\dot\alpha} \to W^{\dot\alpha}{}_{\dot\beta}\tilde{\zeta}^{\dot\beta}, \tag{21}$$

and thus $\xi_\alpha$ and $\tilde{\zeta}^{\dot\alpha}$ are necessarily complex, *i.e.* we may treat $\xi_\alpha$ and $\bar{\xi}^\beta = (\xi_\beta)^\dagger$, as well as $\tilde{\zeta}^{\dot\alpha}$ and $\bar{\tilde{\zeta}}_{\dot\beta} = \left(\tilde{\zeta}^{\dot\beta}\right)^\dagger$, as independent Grassmann variables.

For the Levi-Civita tensor, we adapt the convention $\varepsilon_{12} = \varepsilon_{\dot1\dot2} = \varepsilon^{21} = \varepsilon^{\dot2\dot1} = 1$. In effect,

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^\alpha_\gamma, \qquad \epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \delta^{\dot\gamma}_{\dot\alpha}$$

and

$$\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^m_{\beta\dot\beta} = \tilde{\sigma}^{m\dot\alpha\alpha}. \tag{22}$$

Let us now define the spinorial derivatives

$$D^\alpha = \frac{\partial}{\partial\xi_\alpha} + i\,\bar{\tilde{\zeta}}_{\dot\beta}\,\tilde{\sigma}^{m\dot\beta\alpha}\,\partial_m, \qquad \tilde{D}_{\dot\alpha} = \frac{\partial}{\partial\tilde{\zeta}^{\dot\alpha}} + i\,\bar{\xi}^\beta\,\sigma^m_{\beta\dot\alpha}\,\partial_m, \tag{23}$$

and consequently

$$\bar{D}_\alpha = \frac{\partial}{\partial\bar{\xi}^\alpha} + i\,\sigma^m_{\alpha\dot\beta}\,\tilde{\zeta}^{\dot\beta}\,\partial_m, \qquad \bar{\tilde{D}}^{\dot\alpha} = \frac{\partial}{\partial\bar{\tilde{\zeta}}_{\dot\alpha}} + i\,\tilde{\sigma}^{m\dot\alpha\beta}\,\xi_\beta\,\partial_m. \tag{24}$$

They satisfy an algebra

$$\left\{D^\alpha, \bar{\tilde{D}}^{\dot\alpha}\right\} = 2i\,\tilde{\sigma}^{m\dot\alpha\alpha}\,\partial_m, \qquad \left\{\bar{D}_\alpha, \tilde{D}_{\dot\alpha}\right\} = 2i\,\sigma^m_{\alpha\dot\alpha}\,\partial_m, \tag{25}$$

with all the remaining anticommutators vanishing. Moreover, if we define

$$Q^{\alpha} = \frac{\partial}{\partial \xi_{\alpha}} - i\,\bar{\tilde{\zeta}}_{\dot\beta}\,\tilde{\sigma}^{m\dot\beta\alpha}\,\partial_m\,, \qquad \tilde{Q}_{\dot\alpha} = \frac{\partial}{\partial \tilde{\zeta}^{\dot\alpha}} - i\,\bar{\xi}^{\beta}\,\sigma^m_{\beta\dot\alpha}\,\partial_m\,, \tag{26}$$

and

$$\bar{Q}_{\alpha} = \frac{\partial}{\partial \bar{\xi}^{\alpha}} - i\,\sigma^m_{\alpha\dot\beta}\,\tilde{\zeta}^{\dot\beta}\,\partial_m\,, \qquad \bar{\tilde{Q}}^{\dot\alpha} = \frac{\partial}{\partial \bar{\tilde{\zeta}}_{\dot\alpha}} - i\,\tilde{\sigma}^{m\dot\alpha\beta}\,\xi_{\beta}\,\partial_m\,, \tag{27}$$

then it is immediate to check that all of the anticommutators involving one of the operators (23) or (24), and one of the operators (26) or (27), vanish. In effect, all equations formulated in terms of derivatives (23) and (24) are invariant under the (supersymmetry) transformations generated by (26) and (27).

We next promote $\psi_\alpha$ and $\tilde{\chi}^{\dot\alpha}$ to spinor valued functions on the Euclidian superspace with coordinates $(x, \xi_\alpha, \bar{\xi}^\alpha, \tilde{\zeta}^{\dot\alpha}, \bar{\tilde{\zeta}}_{\dot\alpha})$ and, guided by (9), subject them to the following set of equations:

$$D^{\alpha}\psi_{\alpha} + \tilde{D}_{\dot\alpha}\tilde{\chi}^{\dot\alpha} = 0 \tag{28}$$

and

$$\bar{\tilde{D}}^{\dot\alpha}\psi_{\alpha} + \bar{D}_{\alpha}\tilde{\chi}^{\dot\alpha} = 0\,. \tag{29}$$

In (28), the indices are summed over (so that the l.h.s. is a scalar), while (29) is a vanishing condition for a certain tensor, and thus also has an invariant meaning.

From (25), (28), and (29), we get

$$i\tilde{\sigma}^{m\dot\alpha\alpha}\partial_m\psi_\alpha + \tilde{M}^{\dot\alpha}{}_{\dot\beta}\tilde{\chi}^{\dot\beta} \;=\; 0\,, \tag{30}$$

$$i\sigma^m_{\alpha\dot\alpha}\partial_m\tilde{\chi}^{\dot\alpha} + M_{\alpha}{}^{\beta}\psi_\beta \;=\; 0\,, \tag{31}$$

where

$$M_{\alpha}{}^{\beta} = \frac{1}{2}\left(\delta^{\beta}_{\alpha}\,\tilde{D}_{\dot\alpha}\bar{\tilde{D}}^{\dot\alpha} + \bar{D}_{\alpha}D^{\beta}\right)\,, \qquad \tilde{M}^{\dot\alpha}{}_{\dot\beta} = \frac{1}{2}\left(\delta^{\dot\alpha}_{\dot\beta}\,D^{\alpha}\bar{D}_{\alpha} + \bar{\tilde{D}}^{\dot\alpha}\tilde{D}_{\dot\beta}\right)\,. \tag{32}$$

We conclude that (28) and (29) imply Dirac equations for the (super) spinors $\psi_\alpha$ and $\tilde{\chi}^{\dot\alpha}$ on the subspace of superfields satisfying

$$M_{\alpha}{}^{\beta}\psi_\beta = m\psi_\alpha\,, \qquad \tilde{M}^{\dot\alpha}{}_{\dot\beta}\tilde{\chi}^{\dot\beta} = m\tilde{\chi}^{\dot\alpha}\,. \tag{33}$$

To see that there exist nontrivial solutions of the set of equations (30), (31), and (33), we consider a simple case

$$\tilde{\chi}^{\dot\alpha} = \frac{\partial\psi_\alpha}{\partial\bar{\xi}^{\beta}} = \frac{\partial\psi_\alpha}{\partial\tilde{\zeta}^{\dot\beta}} = 0\,. \tag{34}$$

Equation

$$\bar{\tilde{D}}^{\dot\alpha}\psi_\alpha = 0 \tag{35}$$

then implies that $\psi_\alpha$ depends on $\bar{\tilde{\zeta}}_{\dot\alpha}$ only through a combination of the form of

$$y^m = x^m - i\bar{\tilde{\zeta}}_{\dot\alpha}\tilde\sigma^{m\dot\alpha\alpha}\xi_\alpha\,. \tag{36}$$

If we take

$$\psi_\alpha\left(x, \xi_\alpha, \tilde\zeta^{\dot\alpha}\right) = \lambda_\alpha(y) + F_{mn}(y)\left(\sigma^m\tilde\sigma^n\right)_\alpha{}^\beta\,\xi_\beta\,, \tag{37}$$

then, since

$$D^\alpha y^m = 2i\bar{\tilde{\zeta}}_{\dot\alpha}\tilde\sigma^{m\dot\alpha\alpha}\,, \tag{38}$$

we get

$$\begin{aligned} D^\alpha\psi_\alpha \;&=\; \mathrm{Tr}(\sigma^m\tilde\sigma^n)\,F_{mn}(y) + 2i\bar{\tilde{\zeta}}_{\dot\alpha}\tilde\sigma^{m\dot\alpha\alpha}\partial_m\lambda_\alpha(y) \\ &\quad + 2i\bar{\tilde{\zeta}}_{\dot\alpha}\xi_\beta(\tilde\sigma^p\sigma^m\tilde\sigma^n)^{\dot\alpha\beta}\partial_p F_{mn}(y)\,. \end{aligned} \tag{39}$$

Vanishing of the second term on the r.h.s. of formula (39) implies that $\lambda_\alpha$ satisfies the massless Dirac equation,

$$i\tilde\sigma^{m\dot\alpha\alpha}\partial_m\lambda_\alpha = 0\,, \tag{40}$$

meanwhile, vanishing of the first term implies that the tensor $F_{mn}$ is antisymmetric, and consequently the identity

$$\tilde\sigma^p\sigma^m\tilde\sigma^n = \epsilon^{pmnr}\,\tilde\sigma^r + \delta^{mp}\tilde\sigma^n + \delta^{mn}\tilde\sigma^p - \delta^{np}\tilde\sigma^m, \qquad \epsilon^{1234} = 1, \tag{41}$$

applied to the last term, gives

$$\epsilon^{rpmn}\partial_p F_{mn} = 0\,, \qquad \partial^m F_{mn} = 0\,. \tag{42}$$

We conclude that a particular solution of the postulated set of equations is a spinor superfield with component fields consisting of a massless spinor field and a Maxwell gauge field.

Since the matrices (20) are unitary with unit determinant, spinors $\xi^\alpha \equiv \epsilon^{\alpha\beta}\xi_\beta$ and $\bar\xi^\alpha$ (as well as $\tilde\zeta_{\dot\alpha} \equiv \epsilon_{\dot\alpha\dot\beta}\tilde\zeta^{\dot\beta}$ and $\bar{\tilde{\zeta}}_{\dot\alpha}$) transform in the same way under Spin(4). We can therefore construct spinorial derivatives

$$D^\alpha = \frac{\partial}{\partial\xi_\alpha} + i\,\tilde\zeta_{\dot\beta}\,\tilde\sigma^{m\dot\beta\alpha}\,\partial_m\,, \qquad \tilde D_{\dot\alpha} = \frac{\partial}{\partial\tilde\zeta^{\dot\alpha}} + i\,\xi^\beta\,\sigma^m_{\beta\dot\alpha}\,\partial_m\,, \tag{43}$$

and corresponding supercharges

$$Q^\alpha = \frac{\partial}{\partial\xi_\alpha} - i\,\tilde\zeta_{\dot\beta}\,\tilde\sigma^{m\dot\beta\alpha}\,\partial_m\,, \qquad \tilde Q_{\dot\alpha} = \frac{\partial}{\partial\tilde\zeta^{\dot\alpha}} - i\,\xi^\beta\,\sigma^m_{\beta\dot\alpha}\,\partial_m\,, \tag{44}$$

without invoking conjugated Grassmann variables. Then the set of equations

$$D^{\alpha}\psi_{\alpha} + \tilde{D}_{\dot{\alpha}}\tilde{\chi}^{\dot{\alpha}} = 0\,, \tag{45}$$

and

$$\tilde{D}^{\dot{\alpha}}\psi_{\alpha} + D_{\alpha}\tilde{\chi}^{\dot{\alpha}} = 0\,, \tag{46}$$

where $D_{\alpha} = D^{\beta}\epsilon_{\beta\alpha}$ and $\tilde{D}^{\dot{\alpha}} = \tilde{D}_{\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}$, imposed on "analytic", spinorial superfields

$$\psi_{\alpha} = \psi_{\alpha}\left(x, \xi, \tilde{\zeta}\right)\,, \qquad \tilde{\chi}^{\dot{\alpha}} = \tilde{\chi}^{\dot{\alpha}}\left(x, \xi, \tilde{\zeta}\right)\,, \tag{47}$$

is invariant with respect to both Spin(4) and supersymmetric transformations (generated by (44)) and implies the Dirac equation (18) on a subspace satisfying the "mass" constraints

$$\begin{aligned}\frac{1}{4}\left(\delta_{\alpha}^{\beta}\epsilon^{\dot{\gamma}\dot{\alpha}}\left[\tilde{D}_{\dot{\alpha}}, \tilde{D}_{\dot{\gamma}}\right] + \epsilon_{\gamma\alpha}\left[D^{\gamma}, D^{\beta}\right]\right)\psi_{\beta} \;&=\; m\psi_{\alpha}\,,\\ \frac{1}{4}\left(\delta_{\dot{\beta}}^{\dot{\alpha}}\epsilon_{\gamma\alpha}\left[D^{\alpha}, D^{\gamma}\right] + \epsilon^{\dot{\gamma}\dot{\alpha}}\left[\tilde{D}_{\dot{\gamma}}, \tilde{D}_{\dot{\beta}}\right]\right)\tilde{\chi}^{\dot{\beta}} \;&=\; m\tilde{\chi}^{\dot{\alpha}}\,.\end{aligned} \tag{48}$$

Nontrivial solutions of (45), (46), and (48) with $m = 0$ can be found (even if by "brute force", *i.e.* expanding $\psi_{\alpha}$ and $\tilde{\chi}^{\dot{\alpha}}$ in a series of nonvanishing powers of $\xi$ and $\tilde{\zeta}$ and then working out and solving the resulting differential equations for the coefficient functions). Notice that necessarily both $\psi_{\alpha}$ and $\tilde{\chi}^{\dot{\alpha}}$ are nonzero. Indeed, for $\tilde{\chi}^{\dot{\alpha}} = 0$ equations (45), (46) imply

$$D^{\alpha}\psi_{\alpha} = 0\,, \qquad \tilde{D}_{\dot{\beta}}\psi_{\alpha} = 0\,, \tag{49}$$

which is inconsistent since the anticommutator $\{D^{\alpha}, \tilde{D}_{\dot{\beta}}\}$ does not vanish.

## 3. Almost-commutative geometry

From a physicists point of view, the critical aspects of noncommutative geometry which make it so elegantly suited for the business of model building are the following:

1. All physically relevant information pertaining to a manifold may be distilled within a short list of algebraic quantities, known as a spectral triple, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Speaking informally, $\mathcal{A}$ is a (commutative) algebra of functions continuously defined on the manifold and which is faithfully represented as operators on a Hilbert space, $\mathcal{H}$, and $\mathcal{D} : \mathcal{H} \to \mathcal{H}$ is a Dirac operator.

2. Conversely, and under certain conditions, geometric information may be reconstructed from the data of an *a priori* given spectral triple. In particular, by allowing the algebra $\mathcal{A}$ to be noncommutative, one recovers "noncommutative" geometric information which is said to describe a "noncommutative" manifold.

3. An action functional (and hence, Lagrangian) is then obtained by applying techniques of spectral theory (utilizing a heat kernel expansion) to the Dirac operator. This stresses the importance of the role which the Dirac operator plays in this story, it essentially encodes the metric data of the model.

A particularly interesting class of noncommutative geometries for physicists, due to their being a natural setting for the construction of gauge theories, are the so-called almost-commutative or (AC) manifolds, a detailed description of which can be found in [10]. Here, we find it sufficient to comment that such a manifold is actually described by the product of two spectral triples, the product being again a spectral triple. The first, loosely described above, and the second being a "finite" spectral triple, $(\mathcal{A}_{\mathrm{F}}, \mathcal{H}_{\mathrm{F}}, D_{\mathrm{F}})$ consisting of a finite-dimensional algebra, $\mathcal{A}_{\mathrm{F}}$, represented on a finite-dimensional Hilbert space, $\mathcal{H}_{\mathrm{F}}$, and a symmetric matrix operator $D_{\mathrm{F}}$, are combined into the "total-space" spectral triple

$$(\mathcal{A} \otimes \mathcal{A}_{\mathrm{F}}, \mathcal{H} \otimes \mathcal{H}_{\mathrm{F}}, D_{\mathrm{AC}})\,. \tag{50}$$

With such an AC-geometry approach applied to our 4-dimensional Euclidean space, we have a total space Dirac operator of the form of

$$D_{\mathrm{AC}} = \mathcal{D} \otimes \mathbf{1}_N + \gamma^5_{\mathrm{E}} \otimes D_{\mathrm{F}}\,, \tag{51}$$

where $\mathcal{D}$ is the Euclidean Dirac operator defined in (15), $\gamma^5_{\mathrm{E}}$ is of the form given in (14), and $D_{\mathrm{F}}$ is a finite Dirac operator on $\mathbb{C}^N$, *i.e.* a Hermitian $N \times N$ matrix. Therefore, $D_{\mathrm{AC}}$ can be explicitly written as a $4N \times 4N$ matrix, acting on bispinors of the form of

$$\Psi = \begin{pmatrix} \psi \\ \tilde{\chi} \end{pmatrix}\,, \tag{52}$$

where

$$\psi = (\psi_{i\alpha})\,, \qquad \tilde{\chi} = \left(\tilde{\chi}^{\dot{\alpha}}_i\right)\,, \qquad i = 1, \dots, N\,, \tag{53}$$

and the Dirac equation can be written in the form of

$$\begin{aligned} i\,\tilde{\sigma}^{m\dot{\alpha}\alpha}\partial_m\psi_{i\alpha} + m\tilde{\chi}^{\dot{\alpha}}_i + (D_{\mathrm{F}})_{ij}\,\tilde{\chi}^{\dot{\alpha}}_j &= 0\,, \\ i\,\sigma^m_{\alpha\dot{\alpha}}\partial\tilde{\chi}^{\dot{\alpha}}_i + m\psi_{i\alpha} - (D_{\mathrm{F}})_{ij}\,\psi_{j\alpha} &= 0\,. \end{aligned} \tag{54}$$

Consider now the algebra

$$\begin{aligned}\left\{D_i^\alpha, D_j^\beta\right\} &= 2\epsilon^{\alpha\beta} Z_{ij}\,, \qquad Z_{ij} = -Z_{ji}\,,\\ \left\{\tilde{D}_{i\dot\alpha}, \tilde{D}_{j\dot\beta}\right\} &= 2\epsilon_{\dot\alpha\dot\beta}\tilde{Z}_{ij}\,, \qquad \tilde{Z}_{ij} = -\tilde{Z}_{ji}\,,\end{aligned} \tag{55}$$

together with

$$\left\{D_i^\alpha, \tilde{D}_j^{\dot\alpha}\right\} = 2i\,\delta_{ij}\tilde{\sigma}^{m\dot\alpha\alpha}\,\partial_m\,, \qquad \left\{D_{i\alpha}, \tilde{D}_{j\dot\alpha}\right\} = 2i\,\delta_{ij}\sigma^m_{\alpha\dot\alpha}\,\partial_m\,, \tag{56}$$

where $D_{i\alpha} = D_i^\beta \epsilon_{\beta\alpha}$ and $\tilde{D}_j^{\dot\alpha} = \tilde{D}_{j\dot\beta}\epsilon^{\dot\beta\dot\alpha}$. It can be realized as an algebra of differential operators on a superspace with coordinates $(x^m, \xi_{i\alpha}, \tilde{\zeta}_i^{\dot\alpha})$

$$\begin{aligned}D_i^\alpha &= \frac{\partial}{\partial\xi_{i\alpha}} + i\tilde{\zeta}^{i\dot\alpha}\tilde{\sigma}^{m\dot\alpha\alpha}\partial_m + Z_{ij}\xi_j^\alpha\,,\\ \tilde{D}_{i\dot\alpha} &= \frac{\partial}{\partial\tilde{\zeta}_j^{\dot\alpha}} + i\xi_i^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m + \tilde{Z}_{ij}\tilde{\zeta}_{j\dot\alpha}\,.\end{aligned} \tag{57}$$

The corresponding supercharges, anticommuting with derivatives (57), have the form of

$$\begin{aligned}Q_i^\alpha &= \frac{\partial}{\partial\xi_{i\alpha}} - i\tilde{\zeta}^{i\dot\alpha}\tilde{\sigma}^{m\dot\alpha\alpha}\partial_m - Z_{ij}\xi_j^\alpha\,,\\ \tilde{Q}_{i\dot\alpha} &= \frac{\partial}{\partial\tilde{\zeta}_j^{\dot\alpha}} - i\xi_i^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m - \tilde{Z}_{ij}\tilde{\zeta}_{j\dot\alpha}\,.\end{aligned} \tag{58}$$

If we postulate equations of the form of

$$\begin{aligned}D_i^\alpha\psi_{j\alpha} + \tilde{D}_{j\dot\alpha}\tilde{\chi}_i^{\dot\alpha} &= 0\,,\\ \tilde{D}_i^{\dot\beta}\psi_{i\alpha} + D_{i\alpha}\tilde{\chi}_i^{\dot\beta} &= 0\,,\end{aligned} \tag{59}$$

then, using (56), we can conclude that solutions of (59) satisfy the Dirac equation (54) provided that the "mass" conditions

$$\begin{aligned}\left(m\delta_{ij} + (D_{\rm F})_{ij}\right)\tilde{\chi}_j^{\dot\alpha} &= \frac{1}{2}\left(\delta^{\dot\alpha}_{\dot\beta}D_i^\alpha D_{j\alpha} + \delta_{ij}\tilde{D}_k^{\dot\alpha}\tilde{D}_{k\dot\beta}\right)\tilde{\chi}_j^{\dot\beta}\,,\\ \left(m\delta_{ij} - (D_{\rm F})_{ij}\right)\psi_{j\alpha} &= \frac{1}{2}\left(\delta_\alpha^\beta\tilde{D}_{i\dot\alpha}\tilde{D}_j^{\dot\alpha} + \delta_{ij}D_{k\alpha}D_k^\beta\right)\psi_{j\beta}\end{aligned} \tag{60}$$

are satisfied. With the help of (55), equation (60) can be alternatively presented as

$$\left(m\delta_{ij} + (D_{\rm F})_{ij} - Z_{ij}\right)\tilde{\chi}_j^{\dot\alpha} = \frac{1}{4}\left(\delta^{\dot\alpha}_{\dot\beta}\epsilon_{\beta\alpha}\left[D_i^\alpha, D_j^\beta\right] + \delta_{ij}\epsilon^{\dot\gamma\dot\alpha}\left[\tilde{D}_{k\dot\gamma}, \tilde{D}_{k\dot\beta}\right]\right)\tilde{\chi}_j^{\dot\beta} \tag{61}$$

and

$$\left(m\delta_{ij} - (D_{\mathrm{F}})_{ij} - \tilde{Z}_{ij}\right)\psi_{j\alpha} = \frac{1}{4}\left(\delta^{\beta}_{\alpha}\epsilon^{\dot{\beta}\dot{\alpha}}\left[\tilde{D}_{i\dot{\alpha}}, \tilde{D}_{j\dot{\beta}}\right] + \delta_{ij}\epsilon_{\gamma\alpha}\left[D^{\gamma}_{k}, D^{\beta}_{k}\right]\right)\psi_{j\beta}\,. \tag{62}$$

The simplest solutions of these equations (and, most likely, the only consistent with (59), although the general proof of this claim is still missing) correspond to a situation in which both the l.h.s. and the r.h.s. of (61) and (62) vanish. This implies that the constructed framework, which reconciles noncommutative geometry with supersymmetry in a simple setting, requires the finite part of the Dirac operator (51) to be antisymmetric and expressible through central charges of the algebra (55), (56) as

$$(D_{\mathrm{F}})_{ij} = Z_{ij} = -\tilde{Z}_{ij}\,. \tag{63}$$

It is worth mentioning that in the usual development via the AC-geometry approach to noncommutative geometry, the finite spectral triple only contains data pertaining to the fermionic particle content of the model. The bosonic content of the theory, or gauge fields, are then given by the inner fluctuations which arise through consideration of Morita equivalences of the algebra. The Morita (self-)equivalent total-space spectral triple is then comprised of the algebra, Hilbert space, and the "fluctuated" Dirac operator taking into account the gauge fields. While we have seen that gauge fields arise naturally through the "factorization" procedure which we have herein described, one could also consider the implications of factorizing the fluctuated Dirac operator. This possibility is almost certainly worthy of further investigation.

## REFERENCES

[1] P.A.M. Dirac, R.H. Fowler, «The quantum theory of the electron», *Proc. R. Soc. Lond. A* **117**, 610 (1928).

[2] J. Szwed, «The "square root" of the Dirac equation within superymmetry», *Phys. Lett. B* **181**, 305 (1986).

[3] S.V. Ketov, Y.S. Prager, «On the "Square Root" of the Dirac Equation within Extended Supersymmetry», *Acta Phys. Pol. B* **21**, 463 (1990).

[4] S.V. Ketov, Y.S. Prager, «"Square root" of the Dirac equation in extended supersymmetry», *Russ. Phys J.* **33**, 207 (1990).

[5] A. Bzdak, L. Hadasz, «The square root of the Dirac operator on superspace and the Maxwell equations», *Phys. Lett. B* **582**, 113 (2004).

[6] A. Bzdak, J. Szwed, «The "square root" of the interacting Dirac equation», *Europhys. Lett.* **69**, 189 (2005).

[7] J. Szwed, «The "Square Root" of the Dirac Equation and Solutions on Superspace», *Acta Phys. Pol. B* **37**, 455 (2006).

[8] A. Connes, «Noncommutative Geometry», *Academic Press, Inc.*, San Diego 1994, https://www.alainconnes.org/docs/book94bigpdf.pdf

[9] A. Connes, J. Lott, «Particle models and noncommutative geometry», *Nucl. Phys. B Proc. Suppl.* **18**, 29 (1991).

[10] W.D. van Sujlekom, «Noncommutative Geometry and Particle Physics», *World Scientific*, Singapore 2007.

[11] S.P. Martin, «A Supersymmetry Primer», *Adv. Ser. Dir. High Energy Phys.* **18**, 1 (1997).

[12] S.J. Gates, M.T. Grisaru, M. Rocek, W. Siegel, «Superspace or One Thousand and One Lessons in Supersymmetry», *Front. Phys.* **58**, 1 (1983), arXiv:hep-th/0108200.

[13] J. Wess, J. Bagger, «Supersymmetry and Supergravity», *Princeton University Press*, 1983.