

A LOCAL REDUCTION OF THE DIRAC EQUATION APPLIED TO THE STUDY OF QUARK INTERACTIONS*

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(Received January 24, 2021; accepted March 10, 2021)

A general procedure of local reduction for the Dirac equation is introduced to study one- and n -body interacting systems. In the one-body case, we show that the reduction allows for an approximate solution of the Dirac equation, correlating the upper and the lower components of the wave function. The two-body case is studied in more detail. We show that the method prevents from introducing spurious, unphysical states. The reduction is also applied to another relativistic equation. Finally, the method is used to construct a specific model in order to study the charmonium spectrum.

DOI:10.5506/APhysPolB.52.125

1. Introduction

Relativistic wave equations for bound systems represent a relevant, controversial and extremely extended area of investigation in theoretical physics. The aim of this work is to propose a three-dimensional reduction of the Dirac and Breit equations in order to describe in a relativistic way, avoiding some known inconsistencies, the dynamics of spin $1/2$ particles bound states.

The article is organized as follows: in the remainder of the introduction, in Subsection 1.1, we contextualize the present study in the field of the relativistic equations and define the aim of the work; in Subsection 1.2, we introduce the symbols and the notation of the article; then, in Section 2, we study an approximate formal method of solution of the one-body Dirac equation; taking advantage of the obtained results, in Section 3, we discuss reduction procedure starting from the one-body case; in Section 4, we generalize the reduction procedure to the two-body case that will be analyzed in

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more detail, and to the n -body case that will be introduced at formal level; in Section 5, we apply our reduction procedure to the modified Dirac equation proposed by Mandezweig and Wallace; in Section 6, we introduce, as an example, an interaction model for the study of the charmonium spectrum; the numerical method of solution for the equation is briefly discussed in Section 7; in Section 8, we discuss the results and make a comparison with the experimental data for the charmonium spectrum; finally, some conclusions are drawn in Section 9. Appendix A is devoted to analyze the technical details of some three-dimensional relativistic equations, particularly relevant for this work. The reduction of the one- and two-body interaction is analyzed in the Appendixes B and C, respectively.

1.1. Context and aim of the work

In this subsection, before defining the objective of the work, we try to contextualize the content of the present paper in the framework of the three-dimensional relativistic wave equations (TDRWEs), with no attempt to cover the whole subject. For a concise description of the technical details about the TDRWEs related to this work, the interested reader is referred to Appendix A.

We start by recalling that Dirac equation represents the basic element for the study of all the field theories, as QED, electroweak theory and QCD. Historically, that equation, considered as a *one-body* relativistic equation for a *bound* electron in an external potential, has been successfully used to determine the fine structure effects of the hydrogen atom spectrum.

On the other hand, the study of relativistic equations for two- and n -body *bound* systems (that is very relevant for atomic, nuclear and, particularly, for hadronic physics) is much more complex and many different strategies have been proposed.

In principle, it is possible to construct the Hamiltonian for two- and n -body systems as a straightforward sum of the one-body Dirac Hamiltonians. This procedure gives rise to the Breit equation [1–3] that will be conventionally denoted in the present paper as *Dirac-like equation* (DLE) in order to emphasize its relationship with the original Dirac theory.

The DLE is a TDRWE in Hamiltonian form. A specific, relevant advantage of this equation consists in its *full locality*, if a local interaction is taken.

In recent years, the derivation of the DLE has been formally revised considering direct interactions between spin 1/2 particles [4]. Furthermore, this equation has been deeply analyzed and successfully used to calculate finite size *perturbative corrections* in the hydrogen atom [5] and in mesonic atoms [6].

A semi-analytic study has been performed for some bound states with a Coulomb potential [7]. A calculation of the spectra of quarkonia has been also developed [8].

However, for that equation a relevant formal difficulty has been found: the so-called *continuum dissolution problem* (CDP) [9]. Essentially, it is related to the presence of spurious null mass solutions for the case of non-interacting particles; in more detail, considering a two-body system, it is possible to have a positive energy solution for one particle and a negative energy solution for the other particle; in consequence, in the rest frame, a zero total mass is obtained. Equivalently, these null mass free solutions correspond to unphysical poles in the three-dimensional Green function, as shown in Eq. (A.20). The CDP is related to the difficulty of treating the negative energy states generated by the one-body Dirac terms of the Hamiltonian.

As a consequence, in *nonperturbative* calculations, the reliability of the solutions of the DLE is strongly questioned, while it can be safely used in the perturbative ones.

Many other different methods have been followed to study relativistic bound states. Among them, we only quote the *Dirac's constraint dynamics* [10–12] and the *relativistic path integral Hamiltonian approach* [13–15].

We now discuss some models, more strictly related to the present work, that have been derived from the Bethe–Salpeter equation (BSE) [16, 17]; for this equation, an extensive didactic exposition can be found in Ref. [18].

The BSE is an explicitly covariant four-dimensional formalism that, *in principle*, allows to sum up the infinite series of all the Feynman graphs for two interacting particles, reproducing completely the dynamics of the bound system. However, this procedure would require to introduce in the interaction kernel *all* the corresponding *irreducible* Feynman graphs. Unfortunately, this task cannot be accomplished: only the *tree-level* boson exchange graph is usually considered for the kernel. In this way, the BSE could only reproduce the series of the *ladder* graphs. However, at this point, also another problem is found: the tree-level boson exchange graph, due to its singularities, gives rise to abnormal (unphysical) solutions [19]; for this reason, one is forced to assume an *instantaneous* tree-level interaction. With this assumption, the BSE is reduced to the three-dimensional Salpeter instantaneous equation (SIE) [18, 20]. (In this concern, we recall that for the electromagnetic interaction in the Coulomb gauge, the Coulomb term is instantaneous.) The SIE is, in any case, a *nonlocal* equation that is practically written as an integral equation by means of the Green function that propagates the $++$ and the $--$ states *but not* the $+-$ and $-+$ states that are excluded from the model. For this reason, the Green function of this model is not invertible. The technical details about this point are given in

Appendix A; in particular, see Eq. (A.21). Incidentally, we recall that a comparison between the numerical solutions of the SIE and of the DLE has been performed, finding for the DLE equation unphysical effects related to the CDP [21].

Many efforts have been devoted to improve the SIE trying to incorporate, with some approximation, the *crossed* graphs in order to go beyond the *ladder* approximation for the full series of the equation.

This objective has been achieved in part by putting *on-shell* one fermion, in the so-called *Relativistic Spectator Formalism*, originally developed for nuclear systems and also applied to quark bound states; see, for example Refs. [22–25].

In another approach, that is the Mandelzweig and Wallace equation (MWE) [26–28], the crossed graphs are taken into account, in the eikonal approximation, by means of a suitable definition of the Green function. In this way, the Green function is that of the SIE, *plus* the contributions of the $+-$ and $-+$ states, as shown in Eq. (A.22). As a result, one obtains an *invertible* Green function that finally gives rise to a modified Dirac equation for the bound state. This equation has not the form of an eigenvalue equation for the total energy of the system; the noninteracting term depends on nonlocal operators but the *interaction term has a local form*. Due to its structure, the MWE is free from the CDP and represents an interesting improvement with respect to the SIE. However, when applied to an effective gluon exchange interaction, it should be carefully reexamined considering in particular the noncommutativity of the interaction vertices for the crossed graphs.

The MWE has been also applied to study relativistic corrections for few-body nuclear systems, taking into account, in that case, the non-Abelian character of the one-pion exchange interaction [29–31].

Given the structure of the MWE and, in particular, due to the locality of the interaction term, we shall apply also to this model the reduction procedure developed in the present work.

A common problem of the DLE and MWE equations is the lack of explicit relativistic covariance. This problem is standardly solved by defining in a covariant way the variables of the center of mass (CM), where these equations are originally derived [27, 30, 32].

Finally, we note that the contribution of the $++$ states to the Green function is *the same* for the DLE, SIE and MWE.

For this reason, and also considering the dynamical uncertainties discussed above and the difficulties of the numerical solutions, a possible starting point for a relativistic study of the bound systems consists in projecting any TRWE, that is the DLE, SIE or MWE, *only* onto the positive energy states, excluding completely the $+-$, $-+$ and $--$ states. In this way, the

equation shown in Eqs. (A.32) and (A.35) has been obtained. We denote this equation as *positive energy state equation* (PESE). Moreover, it is possible to introduce into this equation some retardation contributions without formal difficulties. This equation has been used to study the spectra of heavy quarkonia [33, 34].

The previous discussion shows that the problem of the relativistic equations for bound states is still an open issue, with different levels of complexity.

From the numerical point of view, in the two-body case, for the DLE and MWE, one has to solve a coupled equation for *four* two-component spinors. For the DLE, one has an eigenvalue equation with standard differential operators in the coordinate space. For the MWE, the noninteracting term is energy-dependent and nonlocal; in consequence, a specific strategy should be studied. For the SIE, due to its nonlocal form, one has to solve a coupled integral equation for *two* two-body spinors, corresponding to the $++$ and $--$ components of the wave function. In the case of Eq. (A.35) for $++$ states only, one has an integral equation for *one* two-body spinor.

From the dynamical point of view, we note that, in any case, starting from a hypothetical exact theory, many (and not completely justified) approximations are required to define a specific model.

In particular, assuming that the BSE represents the correct starting point, one has to take into account that the crossed graphs are not included or approximately included in the SIE and MWE, respectively. On the other hand, the DLE can be considered an equation based on *first principles* but the unphysical singularities of the Green function must be removed.

This situation has motivated the development of the present study.

In particular, the aim is to find a *reduction* of the DLE equation by establishing a relationship, or *correlation*, between the lower and upper components of the Dirac spinors of each interacting fermion. This correlation has the same structure of the solutions of the one-body Dirac equation in the “spin-symmetry” case [35, 36]. We recall that a simplified Dirac–Coulomb equation was proposed for atomic systems [37]. A Dirac harmonic oscillator shell model with spin-symmetry was also used to study quark–antiquark spectroscopy [38].

The reduction of this work is mainly oriented to the study of *few-body hadronic systems* and avoids from the beginning the CDP. An energy-dependent, three-dimensional, completely *local* equation is obtained; consequently, a relatively simple numerical solution is achievable in the coordinate space. For the two-body case, one has to handle only *one* two-body spinor.

Our reduction can be considered equivalent to the standard PESE when applied to the scattering of on-shell particles but includes, for bound states, some contributions of the negative energy states. More details are given in Eqs. (A.38), (A.39).

After studying the reduction procedure for the DLE, we also apply it to the MWE. In this case, for the reduced noninteracting term of the equation, we obtain a nonlocal operator but, for the reduced interaction operator, we have the same local form obtained for the DLE. In consequence, it is possible to use for the variational solution, the harmonic oscillator (HO) basis that admits an analytic Fourier transform. We shall use the momentum space HO wave functions for the noninteracting term, while the coordinate space wave functions will be used for the interaction term, as discussed in Section 7.

As an example of application, we use our reduced equation to study the charmonium spectrum by means of a standard interaction given by a vector and a scalar term. The results given in Section 8 show that a good quality reproduction of the spectrum can be obtained.

1.2. Symbols and notation

In the present work, we use the gamma matrices γ^μ in the standard representation. Given that we shall employ the *Hamiltonian formalism* for the Dirac equation, we also introduce $\beta = \gamma^0$ and the matrices $\gamma^0\gamma^\mu = (\mathcal{I}, \boldsymbol{\alpha})$, where $\mathcal{I}$ represents the identity matrix in the 4×4 Dirac space.

In the two- and n -body cases, for the matrices and the operators, a *particle* lower index $i = 1, 2, \dots, n$ is introduced; but for the one-body case, the index 1 is omitted.

The following shorthand notation is used: $O_i = O(m_i, \mathbf{p}_i; \boldsymbol{\alpha}_i, \dots)$ where the generic one-body operator $O(m, \mathbf{p}; \boldsymbol{\alpha}, \dots)$ is calculated for the i^{th} particle.

An operator specifically introduced for a n -body system will be denoted by the subscript (n) , with parentheses.

The letter Ψ denotes the complete Dirac wave functions for the n -body system. The letter Φ is used for the spinorial (reduced) wave functions.

For the general equations, we use the *bra-ket* Dirac notation $|\Psi\rangle$, $|\Phi\rangle$.

The interaction operators $W_{(n)}$ are referred to the Hamiltonian formalism, that is $\Psi^\dagger W_{(n)} \Psi = \bar{\Psi} \gamma_1^0 \dots \gamma_n^0 W_{(n)} \Psi$.

The *reduced* operators derived in the present work will be denoted by a *hat*.

The spin indices will be generally omitted. Only in Section 7 the spin quantum numbers are explicitly indicated for the variational wave functions.

Finally, throughout the work, we use the so-called natural units, that is $\hbar = c = 1$.

2. The Dirac equation and its approximate solution

In order to introduce the reduction procedure, we previously analyze a solution method for the Dirac equation. Then, the reduction procedure of the one-body case will be studied in Section 3 and then generalized to two- and n -body DLE and to MWE.

We write the Dirac equation in the Hamiltonian form of

$$\left(H^{\text{free}} - E + W_{(1)}\right) |\Psi\rangle = 0, \quad (1)$$

where H^{free} represents the standard one-body, free, Dirac Hamiltonian, that is

$$H^{\text{free}} = H^{\text{free}}(m; \mathbf{p}, \boldsymbol{\alpha}, \beta) = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m \quad (2)$$

with the Dirac matrices recalled in Subsection 1.2; $\mathbf{p}$ and m , respectively, represent the particle momentum and mass; furthermore, in Eq. (1), E is the energy eigenvalue and $W_{(1)}$ represents the one-body interaction with an external field. Finally, $|\Psi\rangle$ represents the four-component Dirac spinor that in the coordinate representation reads $\Psi(\mathbf{r}) = \langle \mathbf{r} | \Psi \rangle$.

In view of the formal development of the work, we also introduce here the Dirac operator in the form of

$$D = D(m, E; \mathbf{p}, \boldsymbol{\alpha}, \beta) = H^{\text{free}} - E. \quad (3)$$

We take, for the following introductory discussion, a specific interaction with a scalar field $V_s(\mathbf{r})$ and the time component of a vector field $V_v^0(\mathbf{r})$; in this way the one-body interaction has the form of

$$W_{(1)} = \beta V_s(\mathbf{r}) + V_v^0(\mathbf{r}). \quad (4)$$

We split the four-component one-body Dirac spinor into two two-component spinors, for the *upper* and the *lower* components

$$|\Psi\rangle = \begin{pmatrix} |\Phi_U\rangle \\ |\Phi_L\rangle \end{pmatrix}. \quad (5)$$

For the interaction fields, we introduce the shorthand notation

$$\begin{aligned} V_U(\mathbf{r}) &= V_v^0(\mathbf{r}) + V_s(\mathbf{r}), \\ V_L(\mathbf{r}) &= V_v^0(\mathbf{r}) - V_s(\mathbf{r}). \end{aligned} \quad (6)$$

With this notation, the total interaction of Eq. (4) can be written as

$$W_{(1)} = \frac{1}{2}\beta[V_U(\mathbf{r}) - V_L(\mathbf{r})] + \frac{1}{2}[V_U(\mathbf{r}) + V_L(\mathbf{r})]. \quad (7)$$

In this way, the Dirac equation (1) can be conveniently written as a matrix equation in the form of

$$\begin{pmatrix} m - E + V_U(\mathbf{r}) & \boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -(m + E) + V_L(\mathbf{r}) \end{pmatrix} \begin{pmatrix} |\Phi_U\rangle \\ |\Phi_L\rangle \end{pmatrix} = 0 \quad (8)$$

that represents a *coupled equation* for the two spinors $|\Phi_U\rangle$ and $|\Phi_L\rangle$.

Assuming that the quantity $m + E - V_L(\mathbf{r})$ is nonvanishing, one can express $|\Phi_L\rangle$ by means of $|\Phi_U\rangle$ in the form of

$$|\Phi_L\rangle = [m + E - V_L(\mathbf{r})]^{-1} \boldsymbol{\sigma} \cdot \mathbf{p} |\Phi_U\rangle. \quad (9)$$

In this way, one *correlates* exactly the upper and the lower components of the Dirac state $|\Psi\rangle$.

Then, the equation for $|\Phi_U\rangle$ can be written *exactly* as

$$\left[m - E + V_U(\mathbf{r}) + \boldsymbol{\sigma} \cdot \mathbf{p} (m + E - V_L(\mathbf{r}))^{-1} \boldsymbol{\sigma} \cdot \mathbf{p} \right] |\Phi_U\rangle = 0. \quad (10)$$

The last term of this equation, for a central interaction $V_L = V_L(\mathbf{r})$, can be rewritten by using the transformation given in Eq. (B.7) of Appendix B.

In order to introduce our reduction technique, we factorize the constant factor $(m + E)^{-1}$. With standard algebra we write

$$[m + E - V_L(\mathbf{r})]^{-1} = [1 + B(E; \mathbf{r})] \cdot \frac{1}{m + E} \quad (11)$$

with

$$B(E; \mathbf{r}) = F(E; \mathbf{r}) \cdot \frac{1}{1 - F(E; \mathbf{r})} \quad (12)$$

and

$$F(E; \mathbf{r}) = \frac{V_L(\mathbf{r})}{m + E}. \quad (13)$$

Replacing Eq. (11) in Eq. (10), one obtains

$$\left[m - E + \frac{\mathbf{p}^2}{m + E} + V_U(\mathbf{r}) + \frac{1}{m + E} \boldsymbol{\sigma} \cdot \mathbf{p} B(E; \mathbf{r}) \boldsymbol{\sigma} \cdot \mathbf{p} \right] |\Phi_U\rangle = 0, \quad (14)$$

where one has to remember the definitions of Eqs. (12) and (13) for $B(E; \mathbf{r})$. Equation (14) is an energy-dependent, still *exact* equation for $|\Phi_U\rangle$; $|\Phi_L\rangle$ can be reconstructed by means of Eq. (9). With respect to Eq. (10), the previous transformations have allowed to isolate the energy-dependent *pseudo-kinetic* term, that is

$$T_K(E) = \frac{\mathbf{p}^2}{m + E}. \quad (15)$$

For the last term in the parenthesis of Eq. (14), analogously to what observed for Eq. (10), one can use the transformation given in Eq. (B.7) of Appendix B. In this way, a momentum-dependent term and the spin-orbit interaction are obtained.

A case of special interest is when

$$V_L(\mathbf{r}) = 0. \quad (16)$$

In this case, due to the definition of Eq. (6), one has $V_v^0(\mathbf{r}) = V_s(\mathbf{r})$. Furthermore, Eq. (9) does not depend on $V_L(\mathbf{r})$ and, in consequence, we also have

$$B(E; \mathbf{r}) = F(E; \mathbf{r}) = 0. \quad (17)$$

This case, traditionally denoted as *spin-symmetry* case [35, 36], allows for simple solutions of the Dirac equation, in which the spin-orbit interaction is absent and the orbital angular momentum and the spin are *decoupled*.

We now consider the case in which the absolute values of the matrix elements of the adimensional quantity $F(E; \mathbf{r})$ are small. In this case, $B(E; \mathbf{r})$, defined in Eq. (12), can be expanded in a power series of $F(E; \mathbf{r})$

$$B(E; \mathbf{r}) = \sum_{k=1}^{\infty} [F(E; \mathbf{r})]^k. \quad (18)$$

At the first order ($k = 1$), one simply has

$$B(E; \mathbf{r}) \simeq F(E; \mathbf{r}). \quad (19)$$

One can replace this relation in Eq. (14) obtaining an approximated equation for $|\Phi_U\rangle$.

3. The one-body reduction

Let us now study the formal reduction procedure suitable for the generalization to the two- and n -body cases.

In the first place, we write the four-component Dirac spinor that represents the *correlated* (approximate) solution, in the form of

$$|\Psi_{\text{corr}}\rangle = N_{(1)} \cdot K \cdot |\Phi\rangle, \quad (20)$$

where $N_{(1)}$ represents the one-body numerical normalization constant, to be discussed in the following, K is the local reduction operator that transforms the (reduced) spinor $|\Phi\rangle$ into a four-component Dirac spinor; it is defined as

$$K = K(m, E; \mathbf{p}, \boldsymbol{\sigma}) = \begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{m+E} \end{pmatrix}. \quad (21)$$

In Eq. (20), we have taken as approximate solution a Dirac spinor that represents the exact solution in the case of Eq. (16), that is when $V_L(\mathbf{r}) = 0$.

The operator K of the last equation defines the *correlation* between the upper and lower components of $|\Psi_{\text{corr}}\rangle$.

We now replace in the original Dirac equation (1) (written by means of the Dirac operator D of Eq. (3)) the exact solution $|\Psi\rangle$ with $|\Psi_{\text{corr}}\rangle$; furthermore, in order to obtain an Hermitean reduced operator acting on $|\Phi\rangle$, we also multiply the same equation from the left by $K^\dagger$.

In consequence, the reduced (approximated) equation for $|\Phi\rangle$ is formally written in the form of

$$K^\dagger [D + W_{(1)}] K |\Phi\rangle = 0. \quad (22)$$

For the Dirac operator D of Eq. (3), with standard calculations, one finds the corresponding reduced *noninteracting* operator $\hat{D}$ in the form of

$$\hat{D} = \hat{D}(m, E; \mathbf{p}) = K^\dagger D K = m - E + \frac{\mathbf{p}^2}{m + E}. \quad (23)$$

The one-body reduced interaction is written, in general, in the form of

$$K^\dagger W_{(1)} K = \hat{W}_{(1)}. \quad (24)$$

For the specific Dirac interaction of Eq. (7), the one-body reduced interaction takes the form of

$$\begin{aligned} \hat{W}_{(1)} &= V_U(\mathbf{r}) + \frac{1}{m + E} \boldsymbol{\sigma} \cdot \mathbf{p} F(E; \mathbf{r}) \boldsymbol{\sigma} \cdot \mathbf{p} \\ &= V_U(\mathbf{r}) + \frac{1}{(m + E)^2} \boldsymbol{\sigma} \cdot \mathbf{p} V_L(\mathbf{r}) \boldsymbol{\sigma} \cdot \mathbf{p}. \end{aligned} \quad (25)$$

The reduction procedure can be generalized to any interaction. The whole Appendix B is devoted to calculate the reduction of the one-particle interaction with external scalar and vector fields. At the end of that Appendix, the corresponding transformation equations are also given.

Note that $\hat{W}_1$, in the previous equation, and also, in the following, the two- and n -body reduced interactions $\hat{W}_{(2)}$, $\hat{W}_{(n)}$ are all *energy-dependent* operators.

Considering Eqs. (23) and (24), we can write the one-body Dirac reduced equation in the form of

$$[\hat{D} + \hat{W}_{(1)}] |\Phi\rangle = 0. \quad (26)$$

We have obtained for $|\Phi\rangle$ the same equation derived for $|\Phi_U\rangle$, see Eq. (14), with $B(E; \mathbf{r})$ expanded up to the order of $k = 1$, as given in Eq. (19).

We now introduce

$$\hat{Q} = \hat{Q}(m, E; \mathbf{p}) = K^\dagger K = 1 + \frac{\mathbf{p}^2}{(m + E)^2}. \quad (27)$$

By means of this operator, we can define the one-body normalization constant, $N_{(1)}$, that is irrelevant for obtaining the energy eigenvalue E but is necessary to determine in a complete way the correlated Dirac spinor and to calculate the matrix elements of any (other) Dirac operator. The normalization constant $N_{(1)}$ can be obtained by requiring that, for a bound state, the correlated Dirac spinor of Eq. (20) is normalized to unity. By using Eq. (27), one has the following implicit definition:

$$1 = N_{(1)}^2 \langle \Phi | \hat{Q} | \Phi \rangle = N_{(1)}^2 \int d^3 r \Phi^\dagger(\mathbf{r}) \hat{Q} \Phi(\mathbf{r}) \quad (28)$$

from which one can immediately obtain $N_{(1)}$. We consider $N_{(1)}$ as a numerical constant, not included in the definition of K , in order to have a *local* reduced Dirac equation. Otherwise, one could introduce the normalized, nonlocal, reduction operator, in the form of

$$K_{\text{norm}} = K \cdot \left[1 + \frac{\mathbf{p}^2}{(m + E)^2} \right]^{-1/2}. \quad (29)$$

This choice will not be used in the present work because we prefer to obtain a local equation.

Finally, we anticipate that $N_{(2)}$ and $N_{(n)}$ that respectively represent the two-body and the n -body normalization constants, will be determined with an analogous procedure in Section 4.

We also note that *exact* equation (14), without expansion of $B(E; \mathbf{r})$, can be recovered if in the interaction (see Eq. (25)) one replaces $V_L(\mathbf{r})$ with $V_L^{\text{eff}}(\mathbf{r})$ defined as

$$V_L^{\text{eff}}(\mathbf{r}) = V_L(\mathbf{r}) \cdot \frac{1}{1 - F(E; \mathbf{r})}. \quad (30)$$

Otherwise, if the fundamental interaction is not known, one can construct a phenomenological model for $W_{(1)}$, by using a suitable parametrization and then fitting the results to the experimental data.

4. Two- and n -body reduction of the DLE

We introduce here the generalization of our model to the two- and n -body case. We start analyzing in detail the (relatively simple) two-body case. The DLE is formally written in the form of

$$[D_1 + D_2 + W_{(2)}] |\Psi\rangle = 0, \quad (31)$$

where we have used, for each particle ($i = 1, 2$), the standard one-body Dirac operator defined in Eq. (3) with the shorthand notation introduced in Subsection 1.2. Furthermore, $W_{(2)}$ represents the Dirac interaction operator for the two-body case and $|\Psi\rangle$ is the Dirac state of the system. Finally, the total energy is $E_T = E_1 + E_2$.

In a relativistic context, the separation of variables into CM and relative variables is a difficult problem that will not be studied here in detail. In order to calculate the mass M of the two-body bound system, it is sufficient to study the problem in the CM reference frame, where $E_T = M$ and the total momentum $\mathbf{P}$ is vanishing. In this respect, *without introducing a new notation*, we assume in the following that all the states we use (*i.e.*, Dirac states, correlated Dirac states and reduced states) satisfy the condition of vanishing momentum:

$$\mathbf{P}|\Psi\rangle = 0, \quad \mathbf{P}|\Psi_{\text{corr}}\rangle = 0, \quad \mathbf{P}|\Phi\rangle = 0. \quad (32)$$

In order to define the relative variables, we shall focus our attention on a specific, relatively simple case that corresponds directly to the very relevant physical systems of the $q\bar{q}$ mesons. (However, as we shall see in the following, the formal reduction procedure of our model is quite general and does not depend on the specific choice of the CM and relative variables.)

Now, we consider two equal mass particles

$$m_1 = m_2 = m, \quad (33)$$

furthermore, we assume that, in the CM, the two particles have the same energy

$$E_1 = E_2 = \frac{E_T}{2} = \frac{M}{2}. \quad (34)$$

The momentum operators of the two particles are given by

$$\mathbf{p}_1 = -\mathbf{p}, \quad \mathbf{p}_2 = \mathbf{p}, \quad (35)$$

where $\mathbf{p}$ represents the relative momentum operator (in the CM reference frame), canonically conjugated to the relative distance vector

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1. \quad (36)$$

In this way, we can introduce, in that frame, the Dirac wave function $\Psi(\mathbf{r}) = \langle \mathbf{r} | \Psi \rangle$; furthermore, in a local model, the interaction operator depends on the spatial variable $\mathbf{r}$, that is $W_{(2)} = W_{(2)}(\mathbf{r})$.

We construct the reduced equation by introducing, for the Dirac correlated wave function, the following expression:

$$|\Psi_{\text{corr}}\rangle = N_{(2)} \cdot K_1 \cdot K_2 \cdot |\Phi\rangle, \quad (37)$$

where K_i represents the one-particle reduction operator of the i^{th} particle ($i = 1, 2$), as given in Eq. (21). Specifically, for the *arguments* of these operators (and of all the other operators in the following), the definitions of Eqs. (33)–(35) are used. Finally, in Eq. (37), $|\Phi\rangle$ is the two-particle reduced state, and $N_{(2)}$ is the numerical two-body normalization constant. This last quantity is implicitly defined by normalizing $|\Psi_{\text{corr}}\rangle$ to unity, that is

$$1 = N_{(2)}^2 \langle \Phi | \hat{Q}_1 \hat{Q}_2 | \Phi \rangle = N_{(2)}^2 \int d^3r \Phi^\dagger(\mathbf{r}) \hat{Q}_1 \hat{Q}_2 \Phi(\mathbf{r}). \quad (38)$$

By using Eqs. (33)–(35), for a two-body, equal mass system, one has $\hat{Q}_1 = \hat{Q}_2$ and

$$\hat{Q}_1 \cdot \hat{Q}_2 = \left[1 + \frac{\mathbf{p}^2}{(E_T/2 + m)^2} \right]^2. \quad (39)$$

As in the one-body case, after replacing $|\Psi_{\text{corr}}\rangle$ in Eq. (31), we multiply from the left the same equation by $K_1^\dagger \cdot K_2^\dagger$ in order to obtain an Hermitean reduced operator. We have

$$K_1^\dagger \cdot K_2^\dagger (D_1 + D_2 + W_{(2)}) K_1 \cdot K_2 |\Phi\rangle = 0. \quad (40)$$

Using for the one-body operators $\hat{Q}_i$ and $\hat{D}_i$ their definitions of Eqs. (27) and (23) respectively, the previous equation can be rewritten as

$$\left[\hat{Q}_2 \hat{D}_1 + \hat{Q}_1 \hat{D}_2 + \hat{W}_{(2)} \right] |\Phi\rangle = 0, \quad (41)$$

where the two-body reduced interaction is

$$\hat{W}_{(2)} = K_1^\dagger \cdot K_2^\dagger W_{(2)} K_1 \cdot K_2. \quad (42)$$

The reduction of a scalar and vector two-body interaction is studied in detail in Appendix C.

With the specific definitions for the arguments of the operators, given in Eqs. (33)–(35), one has $\hat{Q}_1 = \hat{Q}_2$ and $\hat{D}_1 = \hat{D}_2$; in consequence, the explicit reduction of the noninteracting operator, in the CM, gives

$$\begin{aligned} \hat{Q}_2 \hat{D}_1 + \hat{Q}_1 \hat{D}_2 &= -\hat{G}_{(2)\text{D}}^{-1}(E_T) \\ &= \left[1 + \frac{\mathbf{p}^2}{(E_T/2 + m)^2} \right] \left(\frac{2\mathbf{p}^2}{E_T/2 + m} + 2m - E_T \right), \end{aligned} \quad (43)$$

where, analogously to Eq. (A.26), we have introduced the shorthand notation $\hat{G}_{(2)\text{D}}^{-1}(E_T)$ for the reduced operator, inverse of the Green function. In this way, the reduced equation can be written as

$$\left[-\hat{G}_{(2)\text{D}}^{-1}(E_T) + \hat{W}_{(2)} \right] |\Phi\rangle = 0. \quad (44)$$

The explicit expression of Eq. (43) clearly shows that our model does not admit any free solution with $E_T = 0$, avoiding the CDP. Note also that both the reduced interaction of Eq. (42) and the operator of Eq. (43) are *local* quantities.

From the previous discussion, one can easily find the generalization to the case of a system with n constituents. The DLE has the form of

$$\left[\sum_{i=1}^n D_i + W_{(n)} \right] |\Psi\rangle = 0. \quad (45)$$

In the CM frame, one has to introduce as spatial variables the set of $n-1$ Jacobi variables, collectively denoted as $\{\mathbf{r}\}$ and their conjugated Jacobi momenta $\{\mathbf{p}\}$. Furthermore, one has to express the particle momenta $\mathbf{p}_i$ in terms of the Jacobi momenta. All the states satisfy, in the CM, the vanishing momentum condition (32). The Dirac correlated state is defined as

$$|\Psi_{\text{corr}}\rangle = \prod_{j=1}^n K_j \cdot |\Phi\rangle. \quad (46)$$

Then, the reduction of the DLE is performed analogously to Eq. (40), giving

$$\prod_{i=1}^n K_i^\dagger \cdot \left[\sum_{k=1}^n D_k + W_{(n)} \right] \cdot \prod_{j=1}^n K_j \cdot |\Phi\rangle = 0. \quad (47)$$

In consequence, the reduced equation, that generalizes Eq. (41), takes the form

$$\left[\sum_{i=1}^n \left(\prod_{j \neq i}^n \hat{Q}_j \right) \hat{D}_i + \hat{W}_{(n)} \right] |\Phi\rangle = 0, \quad (48)$$

where the product is performed over all the n particles, excluding the i^{th} one. The reduced interaction is

$$\hat{W}_{(n)} = \prod_{i=1}^n K_i^\dagger \cdot W_{(n)} \cdot \prod_{j=1}^n K_j. \quad (49)$$

Finally, the implicit normalization condition for the reduced wave function is

$$1 = N_{(n)}^2 \langle \Phi | \prod_{i=1}^n \hat{Q}_i | \Phi \rangle = N_{(n)}^2 \int d^3\{\mathbf{r}\} \Phi^\dagger(\{\mathbf{r}\}) \prod_{i=1}^n \hat{Q}_i \Phi(\{\mathbf{r}\}) \quad (50)$$

that generalizes the two-body case of Eq. (38).

Obviously, all the expressions for the n -body reduced operators become increasingly more complex as n increases.

5. Reduction of the MWE

The present reduction procedure can be also applied to the two-body MWE. We recall that this equation avoids from the beginning the CDP by including in the definition of the Green function the so-called *crossed graphs*, as discussed in Appendix A. For the two-body case, the MWE takes, in our notation, the following form:

$$[D_1 S_2 + D_2 S_1 + W_{(2)}] |\Psi\rangle = 0, \quad (51)$$

where the first two terms represent the noninteracting operator given by $-G_{(2)M}^{-1}$ of Eq. (A.27). The MWE should be compared with the DLE of Eq. (31), analyzed in Appendix A. In particular, the difference with respect to that equation consists in the insertion of the energy-sign operators S_i (denoted as $\hat{\rho}_i$ in the original paper [26]). These operators are introduced in Eq. (A.6) and are calculated here for the i^{th} particle. When applied to the free Dirac spinors, they give the energy sign of the free particle, as shown in Eq. (A.7). Note that, due to the presence of the S_i , in the MWE it is not possible to introduce an Hamiltonian operator.

The reduction of the MWE (51) is performed with the same technique used for the DLE in Section 4. The vanishing momentum condition (32) is used. Also, the same definitions of Eqs. (33)–(36) for the two-body equal mass problem are used here. The correlated Dirac state is given by Eq. (37). Analogously to Eq. (40), we have

$$K_1^\dagger \cdot K_2^\dagger [D_1 S_2 + D_2 S_1 + W_{(2)}] K_1 \cdot K_2 |\Phi\rangle = 0. \quad (52)$$

This reduced equation can be rewritten as

$$[\hat{D}_1 \hat{S}_2 + \hat{D}_2 \hat{S}_1 + \hat{W}_{(2)}] |\Phi\rangle = 0 \quad (53)$$

that replaces the Dirac-like reduced equation (41). In the previous equation, we have introduced the reduced $\hat{S}_i$ operators. The reduced $\hat{S}$ operator has the general form

$$\hat{S} = \hat{S}(m, E; \mathbf{p}) = K^\dagger S K = \frac{1}{\varepsilon} [\hat{D} + E \hat{Q}] = \frac{1}{\varepsilon} \left[m + \frac{\mathbf{p}^2}{m + E} + \frac{\mathbf{p}^2 E}{(m + E)^2} \right]. \quad (54)$$

With the definitions of Eqs. (33)–(35) one has $\hat{S}_1 = \hat{S}_2$ and $\hat{D}_1 = \hat{D}_2$. The reduction of the noninteracting operator that appears in Eq. (53), takes the form

$$\begin{aligned} \hat{D}_1 \hat{S}_2 + \hat{D}_2 \hat{S}_1 &= -\hat{G}_{(2)M}^{-1}(E_T) \\ &= \frac{1}{\varepsilon} \left[m + \frac{\mathbf{p}^2}{m + E_T/2} + \frac{\mathbf{p}^2 E_T/2}{(m + E_T/2)^2} \right] \left(\frac{2\mathbf{p}^2}{E_T/2 + m} + 2m - E_T \right), \end{aligned} \quad (55)$$

where, in analogy with Eq. (43), we have introduced the shorthand notation $\hat{G}_{(2)M}^{-1}(E_T)$. Inserting Eq. (55) into Eq. (53), one can write the complete reduced equation as

$$\left[-\hat{G}_{(2)M}^{-1}(E_T) + \hat{W}_{(2)} \right] |\Phi\rangle = 0. \quad (56)$$

Finally, the reduced interaction $\hat{W}_{(2)}$ is the same as that of the DLE, given in Eq. (42); the same implicit normalization condition of Eq. (38) is used here.

6. Model of $q\bar{q}$ interaction for the charmonium spectrum

We shall apply our reduction to the study of charmonium spectrum. In the present work, we consider only a relatively simple effective interaction model, following the standard prescriptions used for the study of heavy quarkonia. A study of different possible interactions with an analysis of the physical meaning of the various terms must be performed in a work apart.

In the present model, for the two-body interaction $W_{(2)}$, we take the sum of a vector and scalar term that is

$$W_{(2)} = W_{(2)}^v + W_{(2)}^s. \quad (57)$$

For the vector interaction, we take the following standard expression:

$$W_{(2)}^v = V_{(2)}^v(r) \gamma_1^0 \gamma_2^0 \cdot \gamma_1^\mu \gamma_2^\nu g_{\mu\nu} \quad (58)$$

with the Dirac matrices recalled in Subsection 1.2. The potential function $V_{(2)}^v(r)$ will be discussed in the following.

In order to have a local interaction operator, we have not included the retardation contributions. This approximated choice can be considered consistent with Eq. (34): we make the hypothesis that the quark energies are *fixed*; consequently, the quarks do not interchange energy with the effective gluonic field that mediates the interaction.

For the scalar interaction, we take the expression

$$W_{(2)}^s = V_{(2)}^s(r) \gamma_1^0 \gamma_2^0. \quad (59)$$

We now discuss the spatial potential functions $V_{(2)}^v(r)$ and $V_{(2)}^s(r)$ of the model. For the vector potential function of Eq. (58), we take the following effective, regularized, expression:

$$V_{(2)}^v(r) = \bar{V}_v - \frac{4}{3} \cdot \frac{\alpha_v}{r} \cdot F_v(r), \quad (60)$$

where $4/3$ is the color factor and $\alpha_v \equiv \alpha_{\text{strong}}$ represents the effective strong coupling constant; we use the subscript “v” (that denotes the vector interaction) to avoid confusion with the scalar terms.

The regularization for $r \rightarrow 0$ is performed having in mind a nonpoint-like chromo-electric charge distribution of the quarks that gives rise to the additive energy constant $\bar{V}_v$ and to the regularization function $F_v(r)$, as shown in Ref. [39]. A detailed study of the relationship between these two quantities must be done in a different work. Here, we recall that $\bar{V}_v$ is introduced also to reproduce phenomenologically the quark confinement. As for $F_v(r)$, we choose

$$F_v(r) = \text{erf} \left(\frac{r}{d_v} \right) \quad (61)$$

being d_v the regularization range. Note that $F_v(\infty) = 1$, not altering the long distance Coulombic behavior, and, for $r \rightarrow 0$, $F_v(r) \simeq \frac{2}{\sqrt{\pi}} \frac{r}{d_v}$; in this way, the Coulombic singularity is eliminated.

As for the scalar interaction, after trying different expressions, we take the potential function of Eq. (59) in the following form:

$$V_{(2)}^s(r) = \bar{V}_s \frac{1}{2} \left[\text{erf} \left(\frac{(r - r_s)}{d_s} \right) - 1 \right]. \quad (62)$$

Note that this potential represents a hole of depth approximately equal to $-\bar{V}_s$ at $r = 0$, while for $r \rightarrow \infty$, one has $V_s = 0$; the width of the hole is approximately r_s ; finally, the parameter d_s is related to the squareness of the hole.

As it will explained in Section 8, we shall use different numerical values for some parameters of the model in order to reproduce accurately the resonances above the open charm threshold.

The reduced interactions $\hat{W}_{(2)}^v$ and $\hat{W}_{(2)}^s$ for our equation are obtained reducing the expressions of Eqs. (58) and (59), respectively. To this aim, we use the two-body reduction equations of Appendix C, specifically: Eq. (C.3) for the product of the time components (t) of the vector interaction and for the scalar (s) interaction; Eq. (C.5) for the product of the spatial parts of the vector interaction.

7. Solution method

In order to solve Eq. (44) for the DLE and Eq. (56) for the MWE, we use a variational procedure, introduced in Ref. [40], that consists in diagonalizing the operators of the equations in a HO basis. The trial wave functions of this basis can be written in the coordinate space as

$$\Phi_{n;L,S,J}(\mathbf{r}) = \langle \mathbf{r} | n; L, S, J \rangle = R_{n,L}(r; \bar{r}) [Y_L(\hat{r}) \otimes \chi_S]_J. \quad (63)$$

In the previous equation, the trial radial function is represented by $R_{n,L}(r; \bar{r})$, being n the principal HO quantum number and $\bar{r}$ the variational parameter with the dimension of longitude; $Y_{L,M_L}(\hat{r})$ is the corresponding spherical harmonic and χ_{S,M_S} , with $S = 0, 1$ is the $c\bar{c}$ coupled spin function. The orbital angular momentum and the spin are standardly coupled to the total angular momentum J, M_J . For brevity, we do not write M_J because it is irrelevant for the calculations of rotationally scalar operators.

Furthermore, for simplicity reasons, we do not consider the possibility of mixing between states with different values of L , because these effects have been shown to be negligible in semirelativistic calculations.

The radial HO functions have the explicit form

$$R_{n,L}(r; \bar{r}) = \frac{1}{\bar{r}^{\frac{3}{2}}} \left[\frac{2(n!)}{\Gamma(n + L + \frac{3}{2})} \right]^{\frac{1}{2}} s^L \mathcal{L}_n^{L+\frac{1}{2}}(s^2) \exp\left(-\frac{s^2}{2}\right), \quad (64)$$

where $s = r/\bar{r}$ is the adimensional variable and $\mathcal{L}_n^{L+\frac{1}{2}}(s^2)$ are the generalized Laguerre polynomials.

The matrix elements of the operator $\hat{G}_{(2)D}^{-1}(E_T)$ of Eq. (43) can be calculated in the coordinate space, because in that operator there only appear finite powers of the momentum operator, that is $\mathbf{p}^{2q}$, with $0 \leq q \leq 2$. On the contrary, $\hat{G}_{(2)M}^{-1}(E_T)$ of Eq. (55), due to the factor $1/\varepsilon$, depends *nonlocally* on the momentum; in consequence, its matrix elements must be evaluated in the momentum space. To this aim, we use the standard analytic expression of the HO wave functions in the momentum space $\Phi_{n;L,S,J}(\mathbf{p}) = \langle \mathbf{p} | n; L, S, J \rangle$. We recall that in both DLE and MWE, $\hat{W}_{(2)}$ is a local operator whose matrix elements are calculated in the coordinate space. In particular, the $\boldsymbol{\sigma}_i \cdot \mathbf{p}_i$ operators are applied to the wave functions of Eq. (63). As explained in Appendix A, this procedure would not be possible if positive (and negative) energy projectors were used, requiring, in any case, an integral equation in the momentum space.

We note that our reduced Eqs. (44) and (56) do not represent standard eigenvalue equations. On the contrary, due to the reduction procedure, $\hat{G}_{(2)D}^{-1}(E_T)$ and $\hat{G}_{(2)M}^{-1}(E_T)$ (given in Eq. (43) and Eq. (55), respectively), and also $\hat{W}_{(2)}$, *depend* on the total energy E_T ; consequently, we have to solve for both models an energy-dependent equation.

To this aim, we make the following replacement for $\hat{G}_{(2)X}^{-1}(E_T)$:

$$-\hat{G}_{(2)X}^{-1}(E_T) = \hat{\mathcal{F}}_X(E_T) - E_T, \quad (65)$$

where the subscript $X = D, M$ stands for DLE or MWE.

In this way, the energy-dependent equation can be formally written as

$$\left[\hat{\mathcal{F}}_X(E_T) + \hat{W}_{(2)}(E_T) \right] |\Phi\rangle = E_T |\Phi\rangle. \quad (66)$$

We replace E_T in the l.h.s. with the auxiliary parameter E_V , obtaining the following fictitious eigenvalue equation:

$$\left[\hat{\mathcal{F}}_X(E_V) + \hat{W}_{(2)}(E_V) \right] |\Phi\rangle = E_T |\Phi\rangle. \quad (67)$$

We can solve *variationally* this equation (as explained above) for a given E_V and determine the corresponding value of E_T in the r.h.s. Then, we vary E_V until the value found for E_T is equal to E_V of the l.h.s. This value gives the solution of Eq. (66) and represents the energy of the system.

As for the variational procedure to solve the fictitious eigenvalue equation (67), we obtain good numerical convergence for E_T , taking the first ten trial wave functions of the basis for each state. In more detail, the 10×10 l.h.s. matrix is diagonalized and minimized by means of the standard variational approach [40].

8. Study of the charmonium spectrum

In this section, we apply the reduced DLE and MWE to study the charmonium spectrum with the interaction introduced in Section 6. The obtained theoretical results and the experimental data [41] are shown in Table I; the values of the parameters used for the calculation are given in Table II.

The present model, that takes into account a *fixed* number of degrees of freedom, is expected to work properly for the resonances below the open charm threshold. For higher resonances some mechanism that takes into account the creation of new particles should be implemented.

We consider here the very simple, purely phenomenological, strategy of taking different values of some parameters of the interaction above the open charm threshold. In more detail, we introduce three intervals for the values of the resonance mass M of the spectrum. These intervals, I_1 , I_2 and I_3 , are defined as:

- $I_1, M < M_a$,
- $I_2, M_a \leq M < M_b$,
- $I_3, M \geq M_b$,

where M_a corresponds to the open charm threshold and M_b has been fixed, after some trials, to obtain a good reproduction of the data. Their values are given in Table II. As shown in Table I, in the interval I_1 , we have considered all the eight experimentally observed resonances; in the intervals I_2 and I_3 , we have considered respectively five and three not controversial resonances. For a discussion about the phenomenological interpretation of the resonances in different models, the interested reader is referred to Ref. [33].

TABLE I

Comparison between the experimental values of the charmonium spectrum and the results of the model. The states of the spectrum are grouped in the three mass intervals I_1 , I_2 and I_3 defined in the text. The intervals are separated by a line. The quantum numbers n , L , S and J have been introduced in Eq. (63); they represent the principal quantum number, the orbital angular momentum, the spin and the total angular momentum, respectively. All the masses are in MeV. The results of the columns Theor. (A) and Theor. (B) refer to the fits A and B, as specified in the text.

Name	$n^{2S+1}L_J$	Theor. (A)	Theor. (B)	Experiment
η_c	1^1S_0	2984	2983	2983.9 ± 0.5
J/ψ	1^3S_1	3096	3096	3096.9 ± 0.006
χ_{c0}	1^3P_0	3420	3422	3414.71 ± 0.30
χ_{c1}	1^3P_1	3504	3506	3510.67 ± 0.05
h_c	1^1P_1	3519	3521	3525.38 ± 0.11
χ_{c2}	1^3P_2	3564	3567	3556.17 ± 0.07
η'_c	2^1S_0	3639	3638	3637.5 ± 1.1
ψ'	2^3S_1	3685	3680	3686.097 ± 0.025
$\psi(3770)$	1^3D_1	3776	3765	3773.13 ± 0.35
$\psi(3823)$	1^3D_2	3816	3813	3822.2 ± 1.2
$\chi_{c1}(3872)$	2^3P_1	3869	3877	3871.69 ± 0.17
$\chi_{c2}(3930)$	2^3P_2	3936	3936	3927.2 ± 2.6
$\psi(4040)$	3^3S_1	4035	4034	4039 ± 1
$\chi_{c1}(4140)$	3^3P_1	4148	4148	4146.8 ± 2.4
$\psi(4260)$	4^3S_1	4228	4228	4230 ± 8
$\chi_{c1}(4274)$	4^3P_1	4275	4275	4274 ± 7

In principle, the parameters of the model are the quark mass m_q and the interaction parameters, introduced in Section 6, that are: $\alpha_v(I_i)$, $\bar{V}_v(I_i)$, $d_v(I_i)$, $\bar{V}_s(I_i)$, $r_s(I_i)$ and $d_s(I_i)$ for the three intervals I_1 , I_2 and I_3 .

The quark mass m_q has not been considered as a free parameter but has been fixed at the *current mass* QCD value [41], as shown in Table II.

TABLE II

Numerical values of the parameters of the model; m_q is fixed; M_a and M_b define the mass intervals. For the other parameters, as explained in the text, DLE and MWE stand for the two reduced equations; A and B stand for the two fits that have been performed. The reported numerical values represent the results of the fits of the free parameters. The parameters not given in this table are vinculated as explained in the text.

					Units
m_q	1275				MeV
M_a	3700				MeV
M_b	4080				MeV
	DLE(A)	DLE(B)	MWE(A)	MWE(B)	
$\alpha_v(I_1)$	1.566	1.614	1.574	1.615	
$\bar{V}_v(I_1)$	1.807	1.803	1.806	1.802	GeV
$d_v(I_1)$	0.2405	0.2500	0.2428	0.2510	fm
$\bar{V}_s(I_1)$	0.8270	0.8187	0.8230	0.8177	GeV
$r_s(I_1)$	1.484	1.518	1.488	1.520	fm
$d_s(I_1)$	0.7059	0.8800	0.7149	0.8838	fm
$\alpha_v(I_2)$	1.956	1.879	1.962	1.883	
$\bar{V}_v(I_2)$	2.005	1.854	2.001	1.853	GeV
$r_s(I_2)$	1.905		1.905		fm

We have performed *two fits*, denoted as “A” and “B”, with the objective of obtaining an accurate theoretical reproduction of whole experimental spectrum with the smallest possible number of free parameters. To this aim, we have *vinculated* the numerical values of some parameters in the different intervals.

In more detail, as shown in Table II, in the interval I_1 , all the interaction parameters are free parameters of the fit.

In the interval I_2 , $\alpha_v(I_2)$, $\bar{V}_v(I_2)$ are free parameters; the vinculated parameters are: $d_v(I_2) = d_v(I_1)$, $\bar{V}_s(I_2) = \bar{V}_s(I_1)$, $d_s(I_2) = d_s(I_1)$; in the fit A, $r_s(I_2)$ is a free parameters, while in the fit B, it is vinculated: $r_s(I_2) = r_s(I_1)$.

In the interval I_3 , all the parameters (in both fits A and B) are vinculated as follows: $\alpha_v(I_3) = \alpha_v(I_2)$, $\bar{V}_v(I_3) = \bar{V}_v(I_1)$, $d_v(I_3) = d_v(I_1)$, $\bar{V}_s(I_3) = \bar{V}_s(I_1)$, $r_s(I_3) = r_s(I_1)$ and $d_s(I_3) = d_s(I_1)$.

As a result of the fit procedure, the *same theoretical masses* have been obtained by using the reduced DLE and MWE; these values, for the two fits, are shown in the columns Theor. (A) and Theor. (B) of Table I. The results of the fit A, with one more free parameter, are slightly better than the results of fit B.

The values of the free parameters of the fits present small differences for the two equations, as reported in the columns DLE(A), DLE(B) and MWE(A), MWE(B) of Table II.

Both the reduced DLE and MWE allow for an accurate reproduction of the spectrum, showing that for the charmonium case, there is no argument to prefer one of the two equations. Some more comments are given in Conclusions.

9. Conclusions

A local, energy-dependent reduction of the DLE has been derived. The same technique has been also applied to the MWE, obtaining in both cases a relativistic equation that can be solved with standard numerical techniques. Further investigation is needed to relate more strictly the reduced equation to the dynamics of the underlying field theory.

The reduced equations have been applied to the study of the charmonium spectrum obtaining accurate results. Both the DLE and the MWE give the same spectrum with small differences of the free parameters. This result can be related to the reduction procedure: the contributions of the $+-$, $-+$ and $--$ states, that are *different* for the DLE and MWE, are diminished by the reduction operators K_i while, in both equations, the more relevant contributions are given by the $++$ states that are the same for the two equations.

A deeper study of the interaction, possibly considering different Lorentz structures beyond the standard vector-scalar model, should be also undertaken.

The author thanks the group of “Gestión de Recursos de Computo Científico, Laboratorio de Biología Computacional, Facultad de Ciencias — Universidad Nacional de Colombia” for the access to the cluster that was used to perform the numerical calculations of this work.

Appendix A

Three-dimensional two-body wave equations

We give here some technical details about the TDRWEs for two-body bound systems related to the present work.

One-body case. We start from some relevant one-body quantities.

The spinors for a free Dirac particle of momentum $\mathbf{p}$ (omitting the two component spin factor), can be written in the form of

$$u_\lambda = u_\lambda(m; \mathbf{p}, \boldsymbol{\sigma}) = M \cdot U_\lambda, \quad (\text{A.1})$$

where $\lambda = \pm 1$ is the energy sign; the U_λ are written as

$$U_+ = U_+(m; \mathbf{p}, \boldsymbol{\sigma}) = \begin{pmatrix} 1 \\ +\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{\varepsilon + m} \end{pmatrix}, \quad U_- = U_-(m; \mathbf{p}, \boldsymbol{\sigma}) = \begin{pmatrix} -\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{\varepsilon + m} \\ 1 \end{pmatrix}, \quad (\text{A.2})$$

where

$$\varepsilon = \varepsilon(m; \mathbf{p}) = +\sqrt{\mathbf{p}^2 + m^2} \quad (\text{A.3})$$

is the on-shell positive energy of the particle. The factor

$$M = M(m; \mathbf{p}) = \sqrt{\frac{\varepsilon + m}{2\varepsilon}} \quad (\text{A.4})$$

normalizes to 1 the spinors, that is $u_\lambda^\dagger u_\xi = \delta_{\lambda\xi}$. Note that the U_λ , due to ε , depend *nonlocally* on the momentum $\mathbf{p}$.

The spinors of Eq. (A.1) obviously diagonalize the free Dirac Hamiltonian

$$u_\lambda^\dagger H^{\text{free}} u_\xi = \lambda \cdot \delta_{\lambda\xi} \cdot \varepsilon. \quad (\text{A.5})$$

We also introduce here the energy-sign operator

$$S = S(m; \mathbf{p}, \boldsymbol{\alpha}, \beta) = \frac{1}{\varepsilon} \cdot H^{\text{free}} \quad (\text{A.6})$$

that, applied to the free spinors, gives

$$S u_\lambda = \lambda u_\lambda. \quad (\text{A.7})$$

The operator S appears in the MWE and will be used in the following when discussing that three-dimensional relativistic equation.

We introduce the one-particle projection operators onto positive ($\lambda = +1$) and negative ($\lambda = -1$) energy states

$$\Lambda^\lambda = \Lambda^\lambda(m; \mathbf{p}, \boldsymbol{\alpha}, \beta) = \frac{1}{2\varepsilon} \left(\varepsilon + \lambda H^{\text{free}} \right) = \frac{1}{2} (1 + \lambda S) = \sum_\lambda u_\lambda u_\lambda^\dagger, \quad (\text{A.8})$$

where ε is the relativistic particle energy, defined in Eq. (A.3), and H^{free} is the free Dirac Hamiltonian of Eq. (2).

One can use the positive and negative energy Dirac spinors u_λ of Eq. (A.1) to rewrite the original Dirac equation (with interaction) as a coupled equation for positive and negative energy components.

A generic Dirac state, decomposed into the two spinors of Eq. (A.1), is written, in the *ket* notation, as

$$|\Psi\rangle = \sum_\xi u_\xi |\Phi_\xi\rangle. \quad (\text{A.9})$$

We recall that the two-component wave functions, in the momentum space, are standardly written as: $\Phi_\xi(\mathbf{p}) = \langle \mathbf{p} | \Phi_\xi \rangle$.

We consider the Dirac equation (1) for the state of Eq. (A.9) and multiply from the left by $u_\lambda^\dagger$. We also introduce the *projected interaction operator* $W_{(1)}^{\lambda,\xi} = u_\lambda^\dagger W_{(1)} u_\xi$. With standard calculations and using Eq. (A.5), one obtains the following coupled equations:

$$\sum_\xi \left[(\lambda \cdot \varepsilon - E) \delta_{\lambda\xi} + W_{(1)}^{\lambda,\xi} \right] |\Phi_\xi\rangle = 0. \quad (\text{A.10})$$

To solve this coupled equation, it is not possible to use the coordinate space. Even if a *local* interaction $W_{(1)} = W_{(1)}(\mathbf{r})$ were considered, the presence of the *nonlocal* $u_\lambda^\dagger$ and u_ξ in $W_{(1)}^{\lambda,\xi}$ requires, in any case, to use the momentum space. Defining

$$\mathcal{W}_{(1)}^{\lambda,\xi}(\mathbf{p}, \mathbf{p}') = \langle \mathbf{p} | u_\lambda^\dagger W_{(1)} u_\xi | \mathbf{p}' \rangle, \quad (\text{A.11})$$

one obtains the following coupled integral equations:

$$\sum_\xi \left[(\lambda \cdot \varepsilon - E) \delta_{\lambda\xi} \Phi_\xi(\mathbf{p}) + \int d^3 p' \mathcal{W}_{(1)}^{\lambda,\xi}(\mathbf{p}, \mathbf{p}') \Phi_\xi(\mathbf{p}') \right] = 0. \quad (\text{A.12})$$

If the off-diagonal matrix elements can be considered small, that is

$$\mathcal{W}_{(1)}^{\lambda,\xi}(\mathbf{p}, \mathbf{p}') \simeq 0 \quad \text{for} \quad \lambda \neq \xi \quad (\text{A.13})$$

and also $\Phi_-(\mathbf{p})$ is negligible, one obtains an *approximate* equation for $\Phi_+(\mathbf{p})$ in the form of

$$(\varepsilon - E) \Phi_+(\mathbf{p}) + \int d^3 p' \mathcal{W}_{(1)}^{+,+}(\mathbf{p}, \mathbf{p}') \Phi_+(\mathbf{p}') = 0. \quad (\text{A.14})$$

Equation (A.14) represents the projection of the Dirac equation onto the positive energy states only. It can be solved numerically or used to obtain a nonrelativistic reduction by means of an expansion in powers of p/m .

We introduce now the Green function for the one-body case; it can be written as

$$\Gamma_{(1)} = \frac{1}{E\gamma^0 - \mathbf{p} \cdot \boldsymbol{\gamma} - m} = \left[\frac{\Lambda^+}{E - \varepsilon} + \frac{\Lambda^-}{E + \varepsilon} \right] \beta = G_{(1)} \cdot \beta, \quad (\text{A.15})$$

where E represents the particle energy; in the propagator of the Feynman graphs, E is replaced by p^0 and the singularity of the denominator is avoided by means of the substitution $m \rightarrow m - i\eta$ ($\eta > 0$).

The inverse of the one-body Green function is straightforwardly obtained in the form of

$$\Gamma_{(1)}^{-1} = \beta [\Lambda^+ \cdot (E - \varepsilon) + \Lambda^- \cdot (E + \varepsilon)] = \beta G_{(1)}^{-1}. \quad (\text{A.16})$$

With standard algebra, one finds

$$D = -\beta \Gamma_{(1)}^{-1} = -G_{(1)}^{-1}, \quad (\text{A.17})$$

where D is the one-body Dirac operator defined in Eq. (3). In consequence, the Dirac equation for an interacting particle can be written in the following equivalent forms:

$$(D + W_{(1)}) |\Psi\rangle = 0, \quad |\Psi\rangle = G_{(1)} W_{(1)} |\Psi\rangle. \quad (\text{A.18})$$

The first form is the standard one, the second form has been obtained by means of Eq. (A.17); due to the nonlocal character of $G_{(1)}$, it must be transformed into an integral equation in the momentum space.

Two-body case. The three-dimensional two-body Green function can be written in a general form (for different models) by means of the projection operators of the two particles

$$\Gamma_{(2)X} = \sum_{\lambda, \xi} \Lambda_1^\lambda \Lambda_2^\xi g_X^{\lambda\xi} \cdot \beta_1 \beta_2 = G_{(2)X} \cdot \beta_1 \beta_2, \quad (\text{A.19})$$

where the subscript X denotes the selected model. Specifically, for the DLE one has

$$g_D^{\lambda\xi} = \frac{1}{E_1 + E_2 - \lambda\varepsilon_1 - \xi\varepsilon_2}, \quad (\text{A.20})$$

note that in the CM one has $\varepsilon_1 = \varepsilon_2 = \varepsilon$; consequently, for $(\xi, \lambda) = (+, -)$ and $(\xi, \lambda) = (-, +)$, one has unphysical poles in the Green function; for the SIE one has

$$\begin{aligned} g_S^{++} &= \frac{1}{E_1 + E_2 - \varepsilon_1 - \varepsilon_2}, \\ g_S^{+-} &= g_S^{-+} = 0, \\ g_S^{--} &= \frac{1}{-E_1 - E_2 - \varepsilon_1 - \varepsilon_2}, \end{aligned} \quad (\text{A.21})$$

finally, for the MWE, one has

$$g_M^{\lambda\xi} = \frac{\lambda\xi}{\lambda E_1 + \xi E_2 - \varepsilon_1 - \varepsilon_2}. \quad (\text{A.22})$$

Note that with respect to the SIE, in the MWE the crossed graphs are taken into account by means of the eikonal approximation and give nonvanishing values to the coefficients g_M^{+-} , g_M^{-+} , while $g_M^{++} = g_S^{++}$, $g_M^{--} = g_S^{--}$.

The equation for the wave function, in all the three cases discussed here, is formally written as

$$|\Psi\rangle = G_{(2)X} W_{(2)} |\Psi\rangle. \quad (\text{A.23})$$

Given that $G_{(2)X}$ is, in any case, a nonlocal operator, also for a local interaction $W_{(2)}$, Eq. (A.23) must be written as an integral equation in order to perform practical calculations.

The two-body Green function $G_{(2)X}$ is invertible if all the coefficients $g_X^{\lambda\xi}$ are *nonvanishing*. This is the case of the DLE and MWE, but not of the SIE. The inverse has the form of

$$G_{(2)X}^{-1} = \sum_{\lambda,\xi} A_1^\lambda A_2^\xi \cdot \frac{1}{g_X^{\lambda\xi}}. \quad (\text{A.24})$$

For the DLE and MWE, one can write the wave equation in the following general form:

$$\left[-G_{(2)X}^{-1} + W_{(2)} \right] |\Psi\rangle = 0. \quad (\text{A.25})$$

With standard calculation, one finds

$$D_1 + D_2 = -G_{(2)D}^{-1} \quad (\text{A.26})$$

for the DLE, and

$$D_1 S_2 + D_2 S_1 = -G_{(2)M}^{-1} \quad (\text{A.27})$$

for the MWE, with the energy sign operators S_i of Eq. (A.6). In this way, Eqs. (31) and (51) are obtained.

For the SIE, using the properties of the projectors A_1^λ , A_2^ξ , with some algebra, one can write

$$\begin{aligned} [D_1 + D_2 + (A_1^+ A_2^- - A_1^- A_2^+) W_{(2)}] |\Psi\rangle &= 0, \\ A_1^+ A_2^- |\Psi\rangle &= A_1^- A_2^+ |\Psi\rangle = 0. \end{aligned} \quad (\text{A.28})$$

Note that the interaction term is multiplied from the left by the *nonlocal* projection operators and that one has to require the second line conditions for the *ket* $|\Psi\rangle$. We observe that in the DLE, both the noninteracting and interaction term are of local form; in the MWE, the noninteracting term is nonlocal but the interaction term is local; the SIE is globally nonlocal due to the projection operators that multiply the interaction term.

Equation (A.25) can be also written as a coupled equation for the positive and negative energy components of $|\Psi\rangle$. These components are defined, analogously to the one-body case of Eq. (A.9), by means of the following equation:

$$|\Psi\rangle = \sum_{\lambda,\xi} u_{1,\lambda} u_{2,\xi} \cdot |\Phi_{\lambda\xi}\rangle. \quad (\text{A.29})$$

We also introduce the following projections for the interaction operator:

$$W_{(2)}^{\lambda\xi,\eta\rho} = u_{1,\lambda}^\dagger u_{2,\xi}^\dagger W_{(2)} u_{1,\eta} u_{2,\rho}. \quad (\text{A.30})$$

In this way, the wave equation can be written in the form of

$$\sum_{\eta,\rho} \left[-\frac{1}{g_X^{\lambda\xi}} \cdot \delta_{\lambda\eta} \delta_{\xi\rho} + W_{(2)}^{\lambda\xi,\eta\rho} \right] |\Phi_{\eta\rho}\rangle = 0. \quad (\text{A.31})$$

In the case of the SI equation, one has $|\Phi_{+-}\rangle = |\Phi_{-+}\rangle = 0$.

Note that g_X^{++} has the same form for all the models. In consequence, if one considers only the projections onto the $++$ states (disregarding all the other components of the wave function), the wave equation takes the form of

$$\left[\varepsilon_1 + \varepsilon_2 - E_1 - E_2 + W_{(2)}^{++,++} \right] |\Phi_{++}\rangle. \quad (\text{A.32})$$

In the last two Eqs. (A.31), (A.32), the interaction operator has a nonlocal form. For this reason, one has to transform these equations into integral equations. In the CM frame, one has to specify the relative variables of the bound system; for equal mass particles, using Eqs. (33)–(36), one defines

$$\mathcal{W}_{(2)}^{\lambda\xi,\eta\rho}(\mathbf{p}, \mathbf{p}') = \langle \mathbf{p} | W_{(2)}^{\lambda\xi,\eta\rho} | \mathbf{p}' \rangle. \quad (\text{A.33})$$

One also has $g_X^{\lambda\xi} = g_X^{\lambda\xi}(m, E_T; \mathbf{p})$; in this way, Eq. (A.31) can be transformed into the following coupled integral equations:

$$\sum_{\eta,\rho} \left[-\frac{1}{g_X^{\lambda\xi}} \delta_{\lambda\eta} \delta_{\xi\rho} \Phi_{\eta\rho}(\mathbf{p}) + \int d^3 p' \mathcal{W}_{(2)}^{\lambda\xi,\eta\rho}(\mathbf{p}, \mathbf{p}') \Phi_{\eta\rho}(\mathbf{p}') \right] = 0. \quad (\text{A.34})$$

The projection onto positive energy states $++$ of Eq. (A.32) becomes

$$[2\varepsilon(\mathbf{p}) - E_T] \Phi_{++}(\mathbf{p}) + \int d^3 p' \mathcal{W}_{(2)}^{++,++}(\mathbf{p}, \mathbf{p}') \Phi_{++}(\mathbf{p}') = 0. \quad (\text{A.35})$$

In this equation, denoted as PESE, one can include, without inconsistencies, some retardation contributions. Furthermore, this equation has been successfully used to study heavy quarkonium spectra in a relativistic model.

Comparison with the projection of our model. In our model, the projection operator K of Eq. (21) is a *local*, energy-dependent operator. Note that for an on-shell particle

$$K(m, E = \varepsilon; \mathbf{p}, \boldsymbol{\sigma}) = U_+(m; \mathbf{p}, \boldsymbol{\sigma}). \quad (\text{A.36})$$

The normalization factor of Eq. (28) also reduces to M of Eq. (A.4).

We observe that a correlated state of our model $|\Psi_{\text{corr}}\rangle$ contains positive and negative energy components. To analyze this point, we have previously introduced

$$\begin{aligned} U_+^\dagger \cdot K &= \frac{\varepsilon + E}{m + E}, \\ U_-^\dagger \cdot K &= \frac{(\varepsilon - E) \mathbf{p} \cdot \boldsymbol{\sigma}}{(\varepsilon + m)(m + E)}. \end{aligned} \quad (\text{A.37})$$

Then, the positive energy amplitude for a state $|\Psi_{\text{corr}}\rangle$ is given, in the momentum space, by the following equation:

$$\begin{aligned} \langle \mathbf{p}; + | \Psi_{\text{corr}} \rangle &= N_{(1)} \cdot M \cdot K^\dagger \cdot U_+ \cdot \Phi(\mathbf{p}) \\ &= N_{(1)} \cdot \left[\frac{\varepsilon + m}{2\varepsilon} \right]^{1/2} \cdot \frac{\varepsilon + E}{m + E} \cdot \Phi(\mathbf{p}). \end{aligned} \quad (\text{A.38})$$

For the negative energy amplitude, one has

$$\begin{aligned} \langle \mathbf{p}; - | \Psi_{\text{corr}} \rangle &= N_{(1)} \cdot M \cdot K^\dagger \cdot U_- \cdot \Phi(\mathbf{p}) \\ &= N_{(1)} \cdot \frac{(\varepsilon - E)}{(2\varepsilon)^{1/2}(\varepsilon + m)^{1/2}(m + E)} \mathbf{p} \cdot \boldsymbol{\sigma} \cdot \Phi(\mathbf{p}). \end{aligned} \quad (\text{A.39})$$

To obtain the previous equations, Eqs. (A.4), (A.37) and the implicit definition of N of Eq. (28) have been used; $\Phi(\mathbf{p})$ is the two component spinor in the momentum space.

We note that, in any case, our model introduces automatically some negative-energy component in the correlated Dirac wave function.

Again, for $E = \varepsilon$, we have $\langle \mathbf{p}; + | \Psi_{\text{corr}} \rangle = \Phi_+(\mathbf{p})$ and $\langle \mathbf{p}; - | \Psi_{\text{corr}} \rangle = 0$. In this limit, our model is equivalent to the standard projection onto positive energy states, represented by Eq. (A.14) for the one-body case and by Eqs. (A.32), (A.35), denoted as PESE, for the two-body case.

Appendix B

Reduction of the one-body interaction

We generalize here the procedure for calculating the one-body reduced interaction.

For the one-body scalar interaction, we have

$$W_{(1)}^s = \beta \cdot V_{(1)}^s(r). \quad (\text{B.1})$$

The reduced interaction is obtained by means of the reduction operator K of Eq. (21), that is

$$\hat{W}_{(1)}^s = K^\dagger W_{(1)}^s K = V_{(1)}^s(r) - \frac{1}{(m+E)^2} \boldsymbol{\sigma} \cdot \mathbf{p} V_{(1)}^s(r) \boldsymbol{\sigma} \cdot \mathbf{p}. \quad (\text{B.2})$$

In the case of a vector interaction, for the time component we have

$$W_{(1)}^0 = \mathcal{I} \cdot V_{(1)}^0(r). \quad (\text{B.3})$$

The reduced interaction is

$$\hat{W}_{(1)}^0 = K^\dagger W_{(1)}^0 K = V_{(1)}^0(r) + \frac{1}{(m+E)^2} \boldsymbol{\sigma} \cdot \mathbf{p} V_{(1)}^0(r) \boldsymbol{\sigma} \cdot \mathbf{p}. \quad (\text{B.4})$$

For the 3-vector part of the interaction, we take a vector function that depends, in general, on $\mathbf{r}$

$$W_{(1)}^v = \boldsymbol{\alpha} \cdot \mathbf{V}_{(1)}^v(\mathbf{r}). \quad (\text{B.5})$$

The corresponding reduced interaction is

$$\hat{W}_{(1)}^v = \frac{1}{(m+E)} \cdot \left[\left(\boldsymbol{\sigma} \cdot \mathbf{V}_{(1)}^v(\mathbf{r}) \right) (\boldsymbol{\sigma} \cdot \mathbf{p}) + (\boldsymbol{\sigma} \cdot \mathbf{p}) \left(\boldsymbol{\sigma} \cdot \mathbf{V}_{(1)}^v(\mathbf{r}) \right) \right]. \quad (\text{B.6})$$

With straightforward calculations, one obtains the following transformation equations that can be used to simplify the previous expressions that contains *two* Pauli matrices $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} \cdot \mathbf{p} V(r) \boldsymbol{\sigma} \cdot \mathbf{p} = \frac{1}{2} \{ \mathbf{p}^2, V(r) \} + \frac{1}{2} \nabla^2 V(r) + \mathbf{l} \cdot \boldsymbol{\sigma} \frac{1}{r} V'(r) \quad (\text{B.7})$$

with the orbital angular momentum $\mathbf{l} = \mathbf{r} \times \mathbf{p}$, and

$$(\boldsymbol{\sigma} \cdot \mathbf{V}(\mathbf{r}))(\boldsymbol{\sigma} \cdot \mathbf{p}) + (\boldsymbol{\sigma} \cdot \mathbf{p})(\boldsymbol{\sigma} \cdot \mathbf{V}(\mathbf{r})) = \mathbf{p} \cdot \mathbf{V}(\mathbf{r}) + \mathbf{V}(\mathbf{r}) \cdot \mathbf{p} + \boldsymbol{\sigma} \cdot \nabla \times \mathbf{V}(\mathbf{r}). \quad (\text{B.8})$$

Appendix C

Reduction of the two-body interaction

For the two-body scalar interaction, we have

$$W_{(2)}^s = \beta_1 \beta_2 \cdot V_{(2)}^s(r). \quad (\text{C.1})$$

For the product of the time components of the vector interaction, we have

$$W_{(2)}^t = \mathcal{I}_1 \mathcal{I}_2 \cdot V_{(2)}^t(r). \quad (\text{C.2})$$

In both cases, the reduced interaction is obtained by using (with a similar procedure) the operators K_1 and K_2 ; we summarize the results in the following way:

$$\begin{aligned} \hat{W}_{(2)}^c &= K_1^\dagger K_2^\dagger W_{(2)}^c K_2 K_1 \\ &= V_{(2)}^c(r) + \tau^c \cdot \left[\frac{1}{(m_1 + E_1)^2} \boldsymbol{\sigma}_1 \cdot \mathbf{p}_1 V_{(2)}^c(r) \boldsymbol{\sigma}_1 \cdot \mathbf{p}_1 \right. \\ &\quad \left. + \frac{1}{(m_2 + E_2)^2} \boldsymbol{\sigma}_2 \cdot \mathbf{p}_2 V_{(2)}^c(r) \boldsymbol{\sigma}_2 \cdot \mathbf{p}_2 \right] \\ &\quad + \frac{1}{(m_1 + E_1)^2 (m_2 + E_2)^2} (\boldsymbol{\sigma}_1 \cdot \mathbf{p}_1) (\boldsymbol{\sigma}_2 \cdot \mathbf{p}_2) V_{(2)}^c(r) (\boldsymbol{\sigma}_2 \cdot \mathbf{p}_2) (\boldsymbol{\sigma}_1 \cdot \mathbf{p}_1), \end{aligned} \quad (\text{C.3})$$

where the superscript c denotes the two interactions, that is $c : s, t$; we also introduced τ^c , with $\tau^s = -1$ and $\tau^t = +1$.

For the product of the spatial parts of the vector interaction, we have

$$W_{(2)}^v = \boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2 \cdot V_{(2)}^v(r). \quad (\text{C.4})$$

The reduction is obtained by means of the operators K_1 and K_2 , applying for the two particles the procedure used for deriving Eq. (B.6). The result is

$$\begin{aligned} \hat{W}_{(2)}^v &= K_1^\dagger K_2^\dagger W_{(2)}^v K_2 K_1 = \frac{1}{(m_1 + E_1)(m_2 + E_2)} \\ &\times \left[V_{(2)}^v(r) (\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_1) (\mathbf{p}_2 \cdot \boldsymbol{\sigma}_2) (\mathbf{p}_1 \cdot \boldsymbol{\sigma}_1) + (\mathbf{p}_1 \cdot \boldsymbol{\sigma}_1) V_{(2)}^v(r) (\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_1) (\mathbf{p}_2 \cdot \boldsymbol{\sigma}_2) \right. \\ &\quad \left. + (\mathbf{p}_2 \cdot \boldsymbol{\sigma}_2) (\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_1) V_{(2)}^v(r) (\mathbf{p}_1 \cdot \boldsymbol{\sigma}_1) + (\mathbf{p}_1 \cdot \boldsymbol{\sigma}_1) (\mathbf{p}_2 \cdot \boldsymbol{\sigma}_2) (\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_1) V_{(2)}^v(r) \right]. \end{aligned} \quad (\text{C.5})$$

Finally, we recall that, for the charmonium spectrum calculation, the momentum operators $\mathbf{p}_1, \mathbf{p}_2$ are given in Eq. (35) and $r = |\mathbf{r}|$, with $\mathbf{r}$ given in Eq. (36).

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