ON A HIGHER-DERIVATIVE SPINOR THEORY

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(Received January 25, 2021; accepted March 05, 2021)

We study a linear spinorial theory for a field ψ of rest-mass m in which the Dirac Lagrangian $L_0 = \bar{\psi}\mathcal{D}\psi$ is augmented by a higher-derivative term $L_1 = r_0\bar{\psi}\mathcal{D}^2\psi$, where $\mathcal{D} = i\gamma^k\partial_k - m$ and r_0 is a constant with the dimension of length. Defining operators through $L = \psi^+\hat{L}\psi$ and setting $p_\kappa = -i\partial/\partial x^\kappa$, the velocity of the charge cloud is $\alpha^\kappa \equiv \gamma^0\gamma^\kappa = -\partial\hat{L}_0/\partial p_\kappa \equiv \mathrm{d}x^\kappa/\mathrm{d}t$, which commutes with x^κ, as pointed out by Schrödinger. Therefore, α^κ can be regarded as a coordinate (independent of position x^κ), to which there corresponds a canonical momentum $\pi_\kappa = -\partial\hat{L}_1/\partial\dot{\alpha}^\kappa$ that anti-commutes with α^κ. We also discuss the high-energy limit valid at radii $r \lesssim r_0 \lesssim 10^{-16}$ cm $\ll 1/m$, where ψ is approximately massless, and in the static case obeys the Laplace equation $\Delta\psi \approx 0$. Expansion of ψ in spherical harmonics shows that a non-vanishing electric charge density $\mathcal{J}^0 = e\sqrt{-g}\psi^+\psi$ is only finite at the origin $r = 0$ if ψ is spherically symmetric, in agreement with experiment.

DOI:10.5506/APhysPolB.52.159

1. Introduction

Following earlier papers by Breit [1] and Fock [2] on the Dirac [3, 4] theory of the electron, it was pointed out by Schrödinger [5] that the velocity $\mathrm{d}x^\kappa/\mathrm{d}t$ of the charge cloud ($\kappa = 1, 2, 3$), interpreted as $\alpha^\kappa \equiv \gamma^0\gamma^\kappa$, commutes with both the position x^κ and its conjugate momentum $p_\kappa \equiv -i\hbar\partial/\partial x^\kappa$ (see footnote 2 on p. 421 of Ref. [5]). This seems to imply that the velocity should be regarded as a coordinate independent of the position, a viewpoint reminiscent of higher-derivative theories of gravity, especially in the mini-superspace idealization, where the radius function α and its corresponding velocity $\mathrm{d}\alpha/\mathrm{d}t$ are also independent coordinates that commute with one another, each possessing its own canonically conjugate momentum. This circumstance naturally leads one to a study of the electron theory augmented

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by the presence of a higher-derivative spinorial term. For the difference between $\mathrm{d}x^\kappa/\mathrm{d}t$ and the quantity p^κ/m (where m is the rest mass) emphasized in Ref. [2] (see the discussion starting from Eq. (19) on p. 132) and in Ref. [5] (see p. 418) can be understood by associating the former with the *microscopic* motion of the electron cloud, reflecting its formulation in terms of the Dirac field Lagrangian $L_0 = \bar{\psi}\mathcal{D}\psi$, while the latter defines the *macroscopic* motion of the electron cloud when considered as a point particle.

This approach is further motivated by analogy with the two-dimensional world sheet of the superstring theory, the Lagrangian $\mathcal{L}_2^{(1)}$ of which can be modified by the addition of a higher-derivative "rigidity" term $\mathcal{L}_2^{(2)}$ proportional to the square of the equation of motion derived from the unmodified $\mathcal{L}_2^{(1)}$, as shown by Curtright *et al.* [6, 7]. A modification of this type is a natural hypothesis in the Dirac theory, which is so contrived that the square of the equation of motion, namely the Klein [8]–Gordon [9] equation (see also Schrödinger [10]), is automatically satisfied by all solutions of the original Lagrangian. (See also Ref. [11] and references therein for further details and discussion.)

Another way of looking at this problem is summarized in the abstract of Ref. [2] (on p. 127), which states that: *To one and the same classical mechanical quantity — the velocity of the electron — correspond in the Dirac theory two different quantum-mechanical quantities, which one can designate as corpuscular and wave velocity of the electron.*

If the velocity α^κ of the electron is considered as a coordinate, one might therefore expect, on the basis of the analyses of Refs. [1, 2] and [5], that an extended form of the spinor Lagrangian, including higher-derivative terms, would imply a momentum π_κ canonically conjugate to α^κ, which is simply related to the momentum p_κ canonically conjugate to x^κ, and also commutes with p_κ. We shall now show that these expectations are both realized for the theory $L = L_0 + L_1$, where $L_1 = r_0\bar{\psi}\mathcal{D}^2\psi$, starting from the basic Lagrangian description in Section 2, then deriving π_κ and the commutation relations in Section 3. The high-energy limit is discussed in Section 4.

2. The Lagrangian analysis

We shall formulate the problem in Minkowski space-time, defined by the line element

$$\mathrm{d}s^2 = \eta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{1}$$

for which the metric in the rest frame of the electron is $\eta_{ij} = \mathrm{diag}(1, -1, -1, -1, -1)$, assuming Cartesian coordinates $x^i = \left(x^0, x^\kappa\right)$, and the associated gamma matrices γ_i obey the Clifford algebra

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij}\,. \tag{2}$$

It is also useful to introduce the alpha matrices [3, 4] $\alpha_i = (\alpha_o \equiv \beta, \alpha_\kappa \equiv \beta\gamma_\kappa)$, where $\beta = \gamma_0$, which satisfy the Euclidean algebra

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \ . \tag{3}$$

The Dirac Lagrangian for the spinor field ψ of rest-mass m can then be written in the alternative forms

$$L_0 = \bar{\psi}\left(i\gamma^k\partial_k - m\right)\psi = \psi^+\left(i\partial_0 + i\alpha^\kappa\partial_\kappa - \beta m\right)\psi\,, \tag{4}$$

where $\bar{\psi} \equiv \psi^+\gamma^0$ is the Hermitian adjoint spinor and $\psi^+ = \psi^{*\mathrm{T}}$. The Dirac equation of motion is then derived by regarding ψ and $\bar{\psi}$ as independent fields, so that

$$\partial L_0/\partial\bar{\psi} = \left(i\gamma^k\partial_k - m\right)\psi = 0\,. \tag{5}$$

Either by premultiplication of Eq. (5) by β or by setting $\partial L_0/\partial\psi^+ = 0$, we obtain the Dirac equation in Hamiltonian form,

$$H\psi \equiv i\partial_0\psi = \left(-i\alpha^\kappa\partial_\kappa + \beta m\right)\psi\,, \tag{6}$$

since $\gamma_0 = \gamma^0$, and hence $\beta^2 = 1$, in the rest frame, so that $\alpha^i = (\beta, \beta\gamma^\kappa)$.

On the basis of analogy with the standard Lagrangian theory, and since the operator momentum canonically conjugate to x^κ is $p_\kappa = -i\partial_\kappa$, the operator expression for the velocity $\dot{x}^\kappa$, where $\dot{} \equiv \mathrm{d}/\mathrm{d}x^0$, was identified in Ref. [1] from Eq. (4) as

$$\dot{x}^\kappa = \alpha^\kappa \tag{7}$$

(up to a minus sign which, from Eq. (3), is inconsequential). This identification was confirmed in Refs. [2, 5] (see also Ref. [12]) by application of the Born–Jordan equation [13] for the time derivative of an arbitrary operator A

$$\dot{A} = \frac{i}{\hbar}\left(HA - AH\right)\,. \tag{8}$$

As emphasized in Ref. [5], Eq. (8) presupposes that the Hamiltonian H is constant, for setting $A = H$ we have $\dot{H} = 0$. Setting instead $A = x^\kappa$ in Eq. (8), from Eq. (6) and the commutation relation

$$x^\kappa p_\lambda - p_\lambda x^\kappa = i\hbar\delta^\kappa_\lambda\,, \tag{9}$$

we obtain Eq. (7), which implies that the instantaneous velocity is the velocity of light [1, 2, 5], the x^κ not commuting with H.

A further interesting consequence of Eq. (8) is that the γ_i depend upon time, since they do not commute with H, although the η_{ij} remain constant and the Klein–Gordon operator is unchanged, provided that Eq. (4) is satisfied. From Eq. (115) of Ref. [12], we have

$$\dot{\beta} = -2\gamma^\kappa \partial_\kappa \,, \qquad \dot{\gamma}_\kappa = 2\beta \left(\partial_\kappa + im\gamma_\kappa\right) . \tag{10}$$

Now let us return to the Dirac Eq. (4). Premultiplication by the factor $-\left(i\gamma^l \partial_l + m\right)$ yields the equation

$$\left[\Box + \beta\left(\dot{\beta}\partial_0 + \dot{\gamma}^\kappa \partial_\kappa\right) + m^2\right]\psi = 0 \,, \tag{11}$$

assuming that $\gamma_i = \gamma_i(x^0)$. On-shell, the term proportional to β vanishes, leaving the Klein–Gordon equation

$$\left(\Box + m^2\right)\psi = 0 \,. \tag{12}$$

Off-shell, however, this term does not vanish in general and, therefore, it is meaningful to consider the hypothetical Lagrangian

$$L \equiv L_0 + L_1 = \bar{\psi}\left[\left(i\gamma^k \partial_k - m\right) + r_0 \left(i\gamma^l \partial_l + m\right)\left(i\gamma^k \partial_k - m\right)\right]\psi \,. \tag{13}$$

On-shell, the second term in Eq. (13) reduces to the Klein–Gordon Lagrangian after removal of a total divergence, when it can be written in the standard quadratic form

$$L_1 = -r_0 \bar{\psi}\left(\Box + m^2\right)\psi = r_0\left[\left(\partial_k \bar{\psi}\right)\left(\partial^k \psi\right) - m^2 \bar{\psi}\psi\right] . \tag{14}$$

Note, however, that the constant prefactor r_0, which would be dimensionless for a scalar field ϕ, instead has the dimension of length, since $[\phi] = l^{-1}$, while $[\psi] = l^{-3/2}$. (We do not discuss the "Elko" theory of a spinor field with dimensionality l^{-1}, since this does not satisfy the Dirac equation that is the basis of the present analysis.)

This is important, for it shows that the modified Lagrangian (13) naturally introduces an additional length scale r_0, which is *a priori* arbitrary, but evidently should not exceed the (reduced) Compton wavelength $\lambda_C \equiv \hbar/m = 3.86 \times 10^{-11}$ cm, in order that the Dirac Lagrangian L_0 hold at large radii $r \gg r_0$, while the Klein–Gordon Lagrangian L_1 predominates at small radii $r \ll r_0$. L_1 contains only second-order derivatives, and hence the causality problem associated with the fourth-order derivatives that occur generically in higher-derivative theories does not arise here.

A second important feature is that the field equation is linear in ψ. The question of whether the fundamental field equation for the wave function is linear or non-linear in ψ is intrinsically complicated. Here, we recall that the Einstein gravitational theory $L^{(1)} = -R/2\kappa^2 + L_{\mathrm{matter}}$, although classically non-linear, yields a *linear* equation for the cosmological wave function Ψ, even when a higher-derivative geometrical term $L^{(2)} = B\mathcal{R}^2$ is included [14], after application of the Wheeler–DeWitt [15, 16] quantization method, thus justifying consideration of a field theory that is linear in the spinor wave function ψ.

A theory of the electron that is linear in the potential ϕ was proposed by Bopp [17], who modified the Maxwell Lagrangian of electrodynamics $L_{\mathrm{M}} = -F_{ij}F^{ij}/4$ by the addition of a higher-derivative term of the form $\sim \lambda^2 \left(\partial_k F_{ij}\right)^2$, in such a way that the equation for a static, spherically-symmetric potential is changed from $\Delta\phi = 0$ to

$$\varDelta\left(\varDelta - \lambda^{-2}\right)\phi = 0\,. \tag{15}$$

By requiring ϕ to be properly behaved both at infinity and at the origin, we derive the solution (see p. 353 of Ref. [17])

$$\phi(r) = e\left(1 - \mathrm{e}^{-r/\lambda}\right)/r\,. \tag{16}$$

This theory assumes a point electron of charge e for which $\phi(r \to \infty) = e/r$, $\phi(r = 0) = e/\lambda$, implying a finite self-energy, the field equations being similar to the Proca–Yukawa equations of nuclear theory.

The general higher-derivative Lagrangian theory

$$\mathcal{L} = \mathcal{L}\left(\phi_a, \partial_j\phi_a, \partial_j\partial_\kappa\phi_a\right) \tag{17}$$

was analysed by Podolsky and Kikuchi [18], applying the Ostrogradski [19] method.

Interestingly, Stelle [20] found that the addition of a higher-derivative gravitational term $B\mathcal{R}^2 \equiv B\left(R^2 - \xi R_{ij}R^{ij}\right)$ to the Einstein–Hilbert Lagrangian $-R/2\kappa^2$ (where $\kappa^2 = 8\pi G_{\mathrm{N}}$, G_{N} being the Newton gravitational constant), produces the same improvement in the Newtonian gravitational potential ϕ_{g} of a point particle of rest-mass M as does the Lagrangian of Ref. [17] in the case of a point electron of charge e. Thus, in place of the electrostatic potential $\phi(r)$ defined by Eq. (16), from Eq. (3.7) of Ref. [20] we have

$$\phi_{\mathrm{g}}(r) = \frac{-2G_{\mathrm{N}}M}{r}\left(1 + \frac{1}{3}\mathrm{e}^{-M_0 r} - \frac{4}{3}\mathrm{e}^{-M_2 r}\right), \tag{18}$$

where the spin-0 and spin-2 higher-derivative masses are [20]

$$M_0 = \frac{1}{2\sqrt{(3-\xi)B\kappa}}\,, \qquad M_2 = \frac{1}{\sqrt{2\xi B\kappa}}\,, \tag{19}$$

and consequently $\phi_{\mathrm{g}}(r)$ is also properly behaved both at infinity and at the origin, since $\phi_{\mathrm{g}}(r \to \infty) = -2G_{\mathrm{N}}M/r$, $\phi_{\mathrm{g}}(r = 0) = -2G_{\mathrm{N}}M(4M_2 - M_0)/3$.

For the effective action of the heterotic superstring theory of Gross *et al.* [21–23], after reduction to four dimensions [14], the dimensionless parameters have the values $B =$ constant, $\xi = 1$, and therefore $M_2 = 2M_0$.

3. The commutation relations

The fundamental commutator relating the position x^κ and momentum p_κ of the electron is given by Eq. (9), which follows automatically from the operator substitution $p_\kappa \to -i\hbar\partial/\partial x^\kappa$, while the separate components commute with one another

$$x^\kappa x_\lambda - x_\lambda x^\kappa = 0\,, \qquad p^\kappa p_\lambda - p_\lambda p^\kappa = 0\,. \tag{20}$$

By contrast, the matrix operators $\dot{x}^\kappa \equiv \alpha^\kappa$ are unrelated to the p_κ, as emphasized in Ref. [5], and therefore commute with the x^κ, although their separate components do not commute with one another. Rather, they obey the anti-commutation law expressed by Eq. (3), and consequently the $\dot{\alpha}^\kappa$ and α^κ *anti-commute*, for differentiation of Eq. (3) with respect to x^0 yields the equation

$$\dot{\alpha}_\kappa \alpha_\lambda + \alpha_\lambda \dot{\alpha}_\kappa = 0\,, \tag{21}$$

after symmetrisation.

Since the matrices α^κ behave differently from the c-numbers x^κ, then the relationship between α_κ and its canonically conjugate momentum π_κ differs from Eq. (9). From the Lagrangian L_0 expressed in the Hamiltonian form of Eq. (4), we see that the operator H is related to the scalar Hamiltonian density $\mathcal{H}$ by the equation

$$\mathcal{H} = \psi^+ H \psi\,. \tag{22}$$

Thus, writing in general

$$L = \psi^+ \hat{L} \psi\,, \tag{23}$$

we can define the momentum canonically conjugate to x^κ via the Lagrangian operator $\hat{L}$ as

$$p_\kappa = -\partial \hat{L}_0/\partial \dot{x}^\kappa \equiv -\partial \hat{L}_0/\partial \alpha^\kappa\,. \tag{24}$$

Considering now the Lagrangian L defined by Eq. (13), after substitution from Eqs. (3) and (10) we find that, off-shell,

$$L_1 = -r_0 \bar{\psi}\left(\Box + m^2\right)\psi + r_0 \psi^+ \left[\left(\alpha^\kappa \dot{\beta} + \dot{\alpha}^\kappa \beta\right)\partial_\kappa - \dot{\beta}\partial_0\right]\psi\,, \tag{25}$$

and therefore if the α^κ is regarded as a coordinate, then its canonically conjugate momentum π_κ is defined by

$$\pi_\kappa = -\partial \hat{L}_1/\dot{\alpha}^\kappa = -\beta r_0 \partial_\kappa = -i\beta r_0 p_\kappa\,. \tag{26}$$

Equation (26) shows, as stated in Section 1, that π_κ and p_κ are simply related and commute with one another, since $\partial_\kappa \beta = 0$, assuming that $\beta = \beta(x^0)$. Further, in place of the commutator relating x^κ and p_λ, we find that α^κ and π_λ anti-commute, since

$$\alpha^\kappa \pi_\lambda + \pi_\lambda \alpha^\kappa = -r_0 \left(\alpha^\kappa \beta + \beta \alpha^\kappa\right) \partial_\lambda = 0\,. \tag{27}$$

4. Discussion

The theories of Refs. [17] and [20] briefly described in Section 2 above analyse the effect of higher-derivative corrections to the static, spherically-symmetric electromagnetic or gravitational potential generated by a point charge or mass, which inevitably leads to problems with the implicit singularity at the spatial origin, where the source is idealized as a delta function. Here, we recall that Einstein and Pauli [24] proved that there exists no stationary non-singular solution to the vacuum equations of general relativity which represents a field of non-vanishing total mass or charge. While the theory of Dirac [3, 4], on the other hand, deals with the probability four-amplitude density $\mathcal{P}^i = \sqrt{-g}\bar{\psi}\gamma^i\psi$, which, when multiplied by the electronic charge e, yields the *distribution* of electrical charge and current density in the form

$$\mathcal{J}^i = e\sqrt{-g}\bar{\psi}\gamma^i\psi\,, \tag{28}$$

where g is the determinant of the metric g_{ij}, and is therefore obviously relevant from a complementary standpoint.

For the sake of simplicity, let us consider a static, spherically-symmetric configuration, as a first approximation to the notion of a spatially extended electron. Close to the origin, at distances $r \ll 1/m$, the spinor is almost massless, the only solution to Eq. (5) being $\psi \approx \psi_0$, where ψ_0 is a constant, in which case the charge density is $\mathcal{J}^0(r) \approx e\psi_0^2 r^2$, which thus tends to zero at $r \to 0$ (and diverges as $r \to \infty$ if $m = 0$, unless $\psi_0 = 0$).

Consider then instead the modified theory of Eq. (13). Since the electron is known to exhibit no structure down to scales $\sim 10^{-16}$ cm, we now assume that $r_0 \lesssim 10^{-16}\,\mathrm{cm} \ll 1/m$, whereupon the field equation (11) derived from L_1 alone reduces to

$$\left[\Box + \beta\left(\dot{\beta}\partial_0 + \dot{\gamma}^\kappa\partial_\kappa\right)\right]\psi \approx 0\,. \tag{29}$$

After substitution from Eq. (10), assuming the static case $\partial_0\psi = 0$ and also that $\partial_\kappa \gg im\gamma_\kappa$, we obtain approximately the Laplace equation

$$\Delta\psi \approx 0\,. \tag{30}$$

In spherical polar coordinates $x^i = (t, r, \theta, \varphi)$, for which $g_{ij} = \mathrm{diag}(1, -1, -r^2, -r^2 \sin^2\theta)$ and $\sqrt{-g} = r^2 \sin\theta$, we have the familiar eigenfunction expansion for the wave function

$$\psi(r, \theta, \varphi) = \sum\sum \psi_{lm} R_l(r) P_l^m(\cos\theta)\, \mathrm{e}^{im\varphi}\,, \tag{31}$$

the radial part of which obeys the equation

$$\frac{\mathrm{d}^2 R_l}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R_l}{\mathrm{d}r} - \frac{k^2 R_l}{r^2} = 0\,, \tag{32}$$

while the angular part, defined by the associated Legendre polynomials $P_l^m(\cos\theta)$, obeys the equation

$$\frac{\mathrm{d}^2 P_l^m}{\mathrm{d}\theta^2} + \cot\theta \frac{\mathrm{d}P_l^m}{\mathrm{d}\theta} + \left(k^2 - m^2 \mathrm{cosec}^2\theta\right) P_l^m = 0\,, \tag{33}$$

setting the separation constant $k^2 = l(l+1)$, where in Eqs. (31) to (35), l and m denote the azimuthal and magnetic quantum numbers, respectively ($l \geq 0, -l \leq m \leq l$). In the axisymmetric case $m = 0$, Eq. (33) reduces to Legendre's equation for $P_l^0 \equiv P_l$, when written as a function of $x \equiv \cos\theta$.

The solutions to the radial Eq. (32) are

$$R_l(r) = r^l\,,\ r^{-(l+1)}\,, \tag{34}$$

substitution of which into Eq. (28) shows that the charge distribution density is

$$\mathcal{J}^0 = \mathrm{e}\sqrt{-g}\psi^+\psi \sim r^{2(l+1)}\,,\ r^{-2l}\,. \tag{35}$$

Since $l \geq 0$, for the first solution Eq. (35) $\mathcal{J}^0(r \to 0) \to 0$ for all l, while the second solution for $\mathcal{J}^0$ diverges at the origin for all $l \geq 1$. If we seek a solution for which the charge density is finite and non-divergent at $r = 0$, the only possibility is $l = 0$, which is the case of spherical symmetry, since then $m = 0$ and $P_0(x) = 1$.

Note that the first solution $R_0(r)$ of Eq. (34) is a constant, which is simply the solution $\psi \approx \psi_0$ to Eq. (5). While all solutions to the linear first-order Eq. (5) are necessarily also solutions to the second-order Eq. (11), the converse is evidently not true, and it is really the second solution (34) which is of interest to us here.

Thus, we find the important result, for this model, that the charge distribution has to be spherically symmetric if it is to be regular and non-zero at the origin. Although this quantity is beyond the reach of direct experimental detection for the electron, precise measurements have been made of its electric dipole moment d_e. Theoretically, the Lagrangian L defined by Eq. (13) is Hermitian, local and invariant under Lorentz transformations $\eta_{ij} \to \eta'_{ij} = \Lambda^2\eta_{ij}$, $x^i \to x'^i = \Lambda^{-1}x^i$, and consequently also invariant under the discrete symmetry TCP. In fact L_1 is invariant under T (remember that the *operator* Eq. (8) is invariant under T, since $t \to -t$ implies that $H \to -H$) and P separately, which leads to the prediction that $d_\mathrm{e} = 0$, consistent with the current upper limits $|d_\mathrm{e}| \le 10^{-29} - 10^{-28}$ cm listed by Zyla *et al.* [25] and references therein, and thus allowing a spherical charge distribution.

On the basis of a similarity between the Proca–Yukawa equations of nuclear theory and the equations of electromagnetism derived from the Maxwell Lagrangian modified by the addition of higher-derivative terms $\sim \lambda^2(\partial_k F_{ij})^2$, it was conjectured in Ref. [17] (see p. 345) that the Yukawa force is responsible not only for holding the nuclei together, but also for the cohesion of electrons. From the viewpoint of this analogy, it is therefore also interesting that the r.m.s. charge radius of the proton r_p can be determined experimentally. An early analysis of the scattering of high-energy electrons off protons showed that the charge distribution of the proton is relatively smooth, and that $r_p \approx 0.8 \times 10^{-13}$ cm — see Hofstadter [26] for a review. Subsequent experimental developments and elaboration of method have left this result virtually unchanged to the present day — from Ref. [25] and references therein we find the current average $r_p = 0.8409 \pm 0.0004 \times 10^{-13}$ cm. Also, the nuclear density profile is assumed to be spherically symmetric, depending upon radius as

$$\rho(r) = \rho_0 \Big/ \left[1 + \mathrm{e}^{(r-r_1)/a}\right] , \tag{36}$$

so that $\rho(r_1) = \rho_0/2$, where $r_1 = r_* A^{1/3}$, $r_* = 1.07 \times 10^{-13}$ cm, $A \equiv Z + N$ is the atomic weight, Z and N being the proton and neutron numbers, respectively, $\rho_Z(r)$ is proportional to $\rho_N(r)$, and a is a phenomenological constant.

Clearly, further investigation of this problem is necessary.

This paper was written at the University of Cambridge, Cambridge, England.

REFERENCES

[1] G. Breit, «An Interpretation of Dirac's Theory of the Electron», *Proc. Natl. Acad. Sci. USA* **14**, 553 (1928).

[2] V. Fock, «Über den Begriff der Geschwindigkeit in der Diracschen Theorie des Elektrons», *Z. Phys.* **55**, 127 (1929).

[3] P.A.M. Dirac, «The quantum theory of the electron», *Proc. R. Soc. London A* **117**, 610 (1928).

[4] P.A.M. Dirac, «The quantum theory of the Electron. Part II», *Proc. R. Soc. London A* **118**, 351 (1928).

[5] E. Schrödinger, «Über die kräftefreie Bewegung in der relativistischen Quantenmechanik», *Sitz. Preuss. Akad. Wiss. Berlin, Phys.-Math. Kl.* **24**, 418 (1930).

[6] T.L. Curtright, G.I. Ghandour, C.K. Zachos, «Classical dynamics of strings with rigidity», *Phys. Rev. D* **34**, 3811 (1986).

[7] T.L. Curtright, P. van Nieuwenhuizen, «Supersprings», *Nucl. Phys. B* **294**, 125 (1987).

[8] O. Klein, «Quantentheorie und fünfdimensionale Relativitätstheorie», *Z. Phys.* **37**, 895 (1926).

[9] W. Gordon, «Der Comptoneffekt nach der Schrödingerschen Theorie», *Z. Phys* **40**, 117 (1926).

[10] E. Schrödinger, «Quantisierung als Eigenwertproblem», *Ann. Phys.* **79**, 489 (1926).

[11] M.D. Pollock, «The Wheeler–DeWitt Equation for the Superstring World Sheet», *Int. J. Mod. Phys. D* **06**, 91 (1997).

[12] M.D. Pollock, «On the electron self-energy», *Eur. Phys. J. Plus* **133**, 132 (2018).

[13] M. Born, P. Jordan, «Zur Quantenmechanik», *Z. Phys.* **34**, 858 (1925).

[14] M.D. Pollock, «On the Derivation of the Wheeler–DeWitt Equation in the Heterotic Superstring Theory», *Int. J. Mod. Phys. A* **07**, 4149 (1992); *Erratum ibid.* **27**, 1292005 (2012).

[15] J.A. Wheeler, in: C.M. DeWitt, J.A. Wheeler (Eds.) «Battelle Rencontres: 1967 Lectures in Mathematics and Physics», *Benjamin*, New York 1968, p. 242.

[16] B.S. DeWitt, «Quantum Theory of Gravity. I. The Canonical Theory», *Phys. Rev.* **160**, 1113 (1967).

[17] F. Bopp, «Eine lineare Theorie des Elektrons», *Ann. Phys.* **38**, 345 (1940).

[18] B. Podolsky, C. Kikuchi, «A Generalized Electrodynamics Part II — Quantum», *Phys. Rev.* **65**, 228 (1944).

[19] M. Ostrogradski, «Les equations differentielles», *Mém. Acad. Imp. Sci. St.-Pétersbourg VI Sér. Sci. Math. Phys.* **IV**, 385 (1850).

[20] K.S. Stelle, «Classical gravity with higher derivatives», *Gen. Relat. Gravit.* **9**, 353 (1978).

[21] D.J. Gross, J.A. Harvey, E. Martinec, R. Rohm, «Heterotic String», *Phys. Rev. Lett.* **54**, 502 (1985).

[22] D.J. Gross, J.A. Harvey, E. Martinec, R. Rohm, «Heterotic string theory (I). The free heterotic string», *Nucl. Phys. B* **256**, 253 (1985).

[23] D.J. Gross, J.A. Harvey, E. Martinec, R. Rohm, «Heterotic string theory (II). The interacting heterotic string», *Nucl. Phys. B* **267**, 75 (1986).

[24] A. Einstein, W. Pauli, «On the Non-Existence of Regular Stationary Solutions of Relativistic Field Equations», *Ann. Math.* **44**, 131 (1943).

[25] P.A. Zyla *et al.*, «Review of Particle Physics», *Prog. Theor. Exp. Phys.* **2020**, (2020).

[26] R. Hofstadter, «Nuclear and Nucleon Scattering of High-Energy Electrons», *Annu. Rev. Nucl. Part. Sci.* **7**, 231 (1957).

[27] D.G. Ravenhall, «Electron Scattering and Nuclear Charge Distributions», *Rev. Mod. Phys.* **30**, 430 (1958).